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**MATRIX SUMMABILITY OF A CLASS OF DERIVED FOURIER
SERIES**

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Let f be L -integrable and periodic with period 2π , and let

$$(1.1) \quad \sum_{n=1}^{\infty} n(b_n \cos nx - a_n \sin nx)$$

be the derived Fourier series of the function f with partial sums $s'_n(x)$. We write

$$\begin{aligned} \psi_x(t) &= f(x+t) - f(x-t); \\ g_x(t) &= \frac{\psi_x(t)}{4 \sin t/2}. \end{aligned}$$

In this paper, the following theorems are established.

THEOREM 1. Let $A = (a_{mn})$ be a regular infinite matrix of real numbers. Then, for every $x \in [-\pi, \pi]$ for which $g_x(t)$ is of bounded variation on $[0, \pi]$,

$$(1.2) \quad \lim_{m \rightarrow \infty} \sum_{n=1}^{\infty} a_{mn} s'_n(x) = g_x(0+)$$

if and only if

$$(1.3) \quad \lim_{m \rightarrow \infty} \sum_{n=1}^{\infty} a_{mn} \sin(n+1/2)t = 0 \quad \text{for all } t \in [0, \pi].$$

THEOREM 2. Let $A = (a_{mn})$ be an almost regular infinite matrix of real numbers. Then, for each $x \in [-\pi, \pi]$ for which $g_x(t)$ is of bounded variation on $[0, \pi]$,

$$\lim_{p \rightarrow \infty} \frac{1}{p} \sum_{j=0}^{p-1} t'_{m+j}(x) = g_x(0+)$$

uniformly in m if and only if

$$\lim_{p \rightarrow \infty} \frac{1}{p} \sum_{j=0}^{p-1} \sum_{n=1}^{\infty} a_{m+j,n} \sin(n+1/2)t = 0 \quad \text{for all } t \in [0, \pi],$$

uniformly in m , where

$$t'_m(x) = \sum_{n=1}^{\infty} a_{mn} s'_n(x).$$

2. *Proof of Theorem 1.* We have

$$\begin{aligned} (2.1) \quad s'_n(x) &= \frac{1}{\pi} \int_0^\pi \psi_x(t) \left(\sum_{k=1}^n k \sin kt \right) dt \\ &= -\frac{1}{\pi} \int_0^\pi \psi_x(t) \frac{d}{dt} \left[\frac{\sin(n+1/2)t}{2 \sin t/2} \right] dt \\ &= I_n + \frac{2}{\pi} \int_0^\pi \sin(n+1/2)t dg_x(t), \end{aligned}$$

where

$$(2.2) \quad I_n = \frac{1}{\pi} \int_0^\pi g_x(t) \frac{\sin(n + 1/2)t}{\tan t/2} dt;$$

$$(2.3) \quad \sum_{n=1}^{\infty} a_{mn} s'_n(x) = \sum_{n=1}^{\infty} a_{mn} I_n + \frac{2}{\pi} \int_0^\pi L_m(t) dg_x(t),$$

where

$$(2.4) \quad L_m(t) = \sum_{n=1}^{\infty} a_{mn} \sin(n + 1/2)t.$$

Since $g_x(t)$ is of bounded variation on $[0, \pi]$ and tends to $g_x(0+)$ as $t \rightarrow 0$, $g_x(t) \cos t/2$ has the same properties; so, by Jordan's convergence criterion for Fourier series,

$$(2.5) \quad I_n \longrightarrow g_x(0+) \quad \text{as } n \longrightarrow \infty.$$

By the regularity of our method of summation, it follows that

$$(2.6) \quad \lim_{m \rightarrow \infty} \sum_{n=1}^{\infty} a_{mn} I_n = g_x(0+).$$

Hence we have to show that, if (1.3) holds, then

$$(2.7) \quad \lim_{m \rightarrow \infty} \int_0^\pi L_m(t) dg_x(t) = 0,$$

and conversely.

By a theorem on the weak convergence of sequences in the Banach space of all continuous functions defined on a finite closed interval (see Banach [1], pp. 134-135), it follows that (2.7) holds if and only if

$$(2.8) \quad |L_m(t)| \leq K \quad \text{for all } m \quad \text{and for all } t \in [0, \pi]$$

and (1.3) holds, where K is a constant.

Since (2.8) is automatically satisfied by one of the regularity conditions on A , it follows that (2.7) holds if and only if (1.3) holds. Thus the proof of the theorem is completed.

REMARKS. (a) We observe that, for each $g_0(t)$ of bounded variation on $[0, \pi]$, we have a corresponding odd function $f \in L[-\pi, \pi]$ given by

$$f(t) = 1/2 \gamma_0(t) = 2g_0(t) \cdot \sin t/2 \quad \text{on } [0, \pi].$$

(b) If $a_{mn} = 1/m$ for $n \leq m$ and zero for $n > m$, then the condition (1.3) is obviously satisfied.

3. **Note.** A bounded sequence $\{s_n\}$ is said to be almost convergent to s if

$$(3.1) \quad \lim_{p \rightarrow \infty} \frac{s_n + s_{n+1} + \dots + s_{n+p-1}}{p} = s$$

uniformly in n (see Lorentz [4]).

It is easy to see that a convergent sequence is almost convergent and the limits are the same.

Let $A = (a_{mn})$ be an infinite matrix of real numbers. A bounded sequence $\{s_n\}$ is said to be almost A -summable to s if the A -transform of $\{s_n\}$ is almost convergent to s , and the matrix A is said to be almost regular if $s_n \rightarrow s$ implies that the sequence $\{t_m\}$ of the A -transforms of $\{s_n\}$ is almost convergent to s .

Necessary and sufficient conditions for the matrix A to be almost regular are as follows (see King [3]):

$$(3.2) \quad \sup_{p \geq 1} \left(\sum_{n=1}^{\infty} \frac{1}{p} \left| \sum_{j=m}^{m+p-1} a_{jn} \right| \right) < M (m = 1, 2, \dots; M = a \text{ constant});$$

$$(3.3) \quad \lim_{p \rightarrow \infty} \frac{1}{p} \sum_{j=m}^{m+p-1} a_{jn} = 0 \text{ uniformly in } m \text{ } (n = 1, 2, \dots);$$

$$(3.4) \quad \lim_{p \rightarrow \infty} \frac{1}{p} \sum_{j=m}^{m+p-1} \sum_{n=1}^{\infty} a_{jn} = 1 \text{ uniformly in } m.$$

We establish the following

THEOREM 2. *Let $A = (a_{mn})$ be an almost regular infinite matrix of real numbers. Then, for every $x \in [-\pi, \pi]$ for which $g_x(t)$ is of bounded variation on $[0, \pi]$,*

$$(3.5) \quad \lim_{p \rightarrow \infty} \frac{1}{p} \sum_{j=0}^{p-1} t'_{m+j}(x) = g_x(0+)$$

uniformly in m if and only if

$$(3.6) \quad \lim_{p \rightarrow \infty} \frac{1}{p} \sum_{j=0}^{p-1} \sum_{n=1}^{\infty} a_{m+j,n} \sin(n + 1/2)t = 0 \text{ for all } t \in [0, \pi],$$

uniformly in m , where

$$t'_m(x) = \sum_{n=1}^{\infty} a_{mn} s'_n(x),$$

$s'_n(x)$ being the partial sum of the derived Fourier series (1.1) of f .

Proof. We have, by (2.1),

$$\begin{aligned}
 & \frac{1}{p} \sum_{j=0}^{p-1} t'_{m+j}(x) = \frac{1}{p} \sum_{j=0}^{p-1} \sum_{n=1}^{\infty} a_{m+j,n} s'_n(x) \\
 (3.7) \quad & = \frac{1}{p} \sum_{j=0}^{p-1} \sum_{n=1}^{\infty} a_{m+j,n} I_n + \frac{2}{\pi} \int_0^{\pi} \left[\frac{1}{p} \sum_{j=0}^{p-1} \sum_{n=1}^{\infty} a_{m+j,n} \sin(n+1/2)t \right] dg_x(t) \\
 & = J_1 + J_2,
 \end{aligned}$$

say.

By (2.5), A being almost regular,

$$(3.8) \quad J_1 \longrightarrow g_x(0+) \text{ uniformly in } m \text{ as } p \longrightarrow \infty.$$

So we have to show that (3.6) holds if and only if

$$J_2 \longrightarrow 0 \text{ uniformly in } m \text{ as } p \longrightarrow \infty.$$

Now,

$$\begin{aligned}
 & \left| \frac{1}{p} \sum_{j=0}^{p-1} \sum_{n=1}^{\infty} a_{m+j,n} \sin(n+1/2)t \right| \\
 (3.9) \quad & = \left| \frac{1}{p} \sum_{n=1}^{\infty} \sin(n+1/2)t \sum_{j=0}^{p-1} a_{m+j,n} \right| \\
 & \leq \frac{1}{p} \sum_{n=1}^{\infty} \left| \sin(n+1/2)t \right| \cdot \left| \sum_{j=0}^{p-1} a_{m+j,n} \right| \\
 & \leq \sum_{n=1}^{\infty} \frac{1}{p} \left| \sum_{j=0}^{p-1} a_{m+j,n} \right| < M \text{ for all } p \text{ and } m, \text{ by (3.2)}.
 \end{aligned}$$

Hence the remainder of the proof is similar to that of Theorem 1.

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