SHAPE GROUPS AND PRODUCTS

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In recent papers S. Mardešić and J. Segal have used ANR-systems to obtain an alternate approach to Borsuk's shape theory. At the same time, they have extended the theory to include all compact Hausdorff spaces. In this paper some of the results of Mardešić and Segal are used to obtain Borsuk's fundamental groups, and to extend some of Borsuk's results that relate shapes and products to arbitrary products. A result relating (direct) products and shape groups is also obtained.

The reader is referred to Chapter 1 of [11] for all categorical definitions; e.g., inverse system, terminal object, inverse limit.

If $X = \{X_a, p_{aa}, A\}$ and $Y = \{Y_b, q_{bb}, B\}$ are inverse systems in a category $\mathscr{A}$, a morphism of inverse systems $f: X \rightarrow Y$ consists of an increasing function $f: B \rightarrow A$ and a collection of $\mathscr{A}$-morphisms $f_b: X_{f(b)} \rightarrow Y_b$ such that if $b \leq b'$ then $f_b p_{f(b)f(b')} = q_{bb'} f_{b'}$. If $X_\infty = \lim X$ and $Y_\infty = \lim Y$ exist and $f: X \rightarrow Y$ is a morphism of inverse systems then for each $b \in B$ the composition $f_b p_{f(b)}: X_\infty \rightarrow Y_b$ satisfies if $b \leq b'$ then

$$f_b p_{f(b)} = f_b p_{f(b)f(b')} p_{f(b')} = q_{bb'} f_{b'} p_{f(b')}.$$ 

By the universal mapping property of $Y_\infty$ there is a unique $\mathscr{A}$-morphism $f_\infty: X_\infty \rightarrow Y_\infty$ such that if $b \in B$ then $q_b f_\infty = f_b p_{f(b)}$. The $\mathscr{A}$-morphism $f_\infty$ is said to be induced by $f$.

Under the usual definitions of composition and identities, there is a category, denoted $\text{inv}(\mathscr{A})$, whose objects are inverse systems in $\mathscr{A}$ and whose morphisms are morphisms of inverse systems. The reader will note that this category is not the category used by Mardešić and Segal [6]. It is not difficult to show that if $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ are morphisms of inverse systems and if $X_\infty$, $Y_\infty$ and $Z_\infty = \lim Z$ exist then $(gf)_\infty = g_\infty f_\infty$.

An ANR-system is an inverse system $X = \{(X_a, x_a), P_{aa}, A\}$ in the category of pointed topological spaces where $A$ is closure-finite (i.e., for every $a \in A$, the set of predecessors of $a$ is finite) and each $X_a$ is a compact ANR for normal spaces. This definition differs from that usually given in that the authors in [6] required each $X_a$ to be compact ANR for metric spaces. This condition is easily relaxed; as was done in [5]. As in [5], we use notions and results from [6] in this setting without specific citations.
If $X$ is a compact Hausdorff space, $x_0 \in X \subseteq M$, an inclusion ANR-system in $M$ associated with $(X, x_0)$ is an ANR-system $X = \{(X_a, x_0), i_{aa'}, A\}$ associated with $(X, x_0)$ where

1. each $X_a$ is a neighborhood of $X$ in $M$
2. $X = \bigcap_{a \in A} X_a$
3. if $a \leq a'$ then $i_{aa'}: (X_{a'}, x_0) \to (X_a, x_0)$ is an inclusion map.

If $A = N$ (the set of natural numbers) then $X$ is said to be an inclusion ANR-sequence and is denoted $X = \{(X_k, x_0), i_{kk'}, A\}$. If $X$ is contained in a parallelootope $P = \prod_{\omega \in \Omega} I_\omega$, $\Omega = \Lambda$ then [5] $(X, x_0)$ has an associated inclusion ANR-system (sequence if $\Omega$ is countable).

Another useful category is the category of ANR-systems, developed by Mardešić and Segal in [6]. The objects of this category are ANR-systems $X = \{(X_a, x_a), P_{aa'}, A\}$ (recall our definition differs somewhat from that used in [6]). A morphism in this category $F: X \to Y = \{(Y_b, y_b), q_{bb'}, B\}$, called a map of systems, consists of an increasing function $f: B \to A$ and a collection of maps (i.e., continuous functions) $f_b: X_{f(b)} \to Y_b$ such that if $b \leq b'$ then $f_b P_{f(b), f(b')} \equiv q_{bb'} f_{b'}$; i.e., the diagram

$$
\begin{array}{ccc}
X_{f(b')} & \xrightarrow{p} & X_{f(b)} \\
\downarrow{f_{b'}} & & \downarrow{f_b} \\
Y_{b'} & \xrightarrow{q} & Y_b
\end{array}
$$

commutes up to homotopy.

2. The shape groups. In [6], Mardešić and Segal define the concept of homotopy between two maps of systems. To be more precise, two maps of systems $f, g: X \to Y$ are said to be homotopic, written $f \equiv g$, provided that for every $b \in B$ there is an $a \in A, a \geq f(b), g(b)$, such that $f_b P_{f(b), a} \equiv g_b P_{g(b), a}$. Noting the similarities between the category $\text{inv} (\mathcal{X})$ and the category of ANR-systems one can define a similar relation in $\text{inv} (\mathcal{X})$.

Let $\mathcal{X}$ be a category. Two morphisms $f, g: X \to Y$ of inverse systems in $\mathcal{X}$ are $\sim$-related ($f \sim g$) if for each $b \in B$ there is an index $a \in A, a \geq f(b), g(b)$ such that $f_b P_{f(b), a} = g_b P_{g(b), a}$.

**Theorem 2.1.** The relation $\sim$ is an equivalence relation.

**Proof.** The proof is as in Theorem 2 of [6].

**Theorem 2.2.** Let $f, f': X \to Y$ and $g, g': Y \to Z$. If $f \sim f'$ and $g \sim g'$ then $gf \sim g'f'$. 
A morphism \( f : X \to Y \) is a \( \sim \)-equivalence provided there is a morphism \( g : Y \to X \) (called the \( \sim \)-inverse of \( f \)) such that \( gf \sim 1_X \) and \( fg \sim 1_Y \). In this case, \( X \) and \( Y \) are said to be \( \sim \)-equivalent (\( X \sim Y \)).

**Theorem 2.3.** The relation \( \sim \) is an equivalence relation of inverse systems in \( \mathcal{A} \).

**Proof.** See Theorem 4 of [6].

**Theorem 2.4.** If \( f, g : X \to Y \) are \( \sim \)-related morphisms and \( X_\infty \) and \( Y_\infty \) both exist then \( f_\infty = g_\infty \).

**Proof.** By definition, \( f_\infty : X_\infty \to Y_\infty \) is the unique \( \mathcal{A} \)-morphism satisfying for all \( b \in B \), \( q_b f_\infty = f_b p_{f(b)} \). Similarly, \( g_\infty : X_\infty \to Y_\infty \) is the unique \( \mathcal{A} \)-morphism satisfying for all \( b \in B \), \( q_b g_\infty = g_b p_{g(b)} \). Choose \( a \geq f(b) \), \( g(b) \) such that \( f_b p_{f(b)} = g_b p_{g(b)} \). Now \( p_{f(b)} = p_{f(b)} \circ p_a \) and \( p_{g(b)} = p_{g(b)} \circ p_a \) so that

\[
q_b g_\infty = g_b p_{g(b)} = g_b p_{g(b)} \circ p_a = f_b p_{f(b)} \circ p_a = f_b p_{f(b)}.
\]

By the uniqueness, \( f_\infty = g_\infty \).

**Corollary 2.5.** If \( X \sim Y \) and \( X_\infty \), \( Y_\infty \) both exist then \( X_\infty \) and \( Y_\infty \) are \( \mathcal{A} \)-equivalent objects.

**Proof.** If \( f : X \to Y \) and \( g : Y \to X \) are such that \( gf \sim 1_X \) and \( fg \sim 1_Y \) then \( g_\infty f_\infty = (gf)_\infty = 1_X \) and \( f_\infty g_\infty = (fg)_\infty = 1_Y \).

If \( X = \{(X_a, \alpha_a), p_{\alpha a}, A\} \) is an ANR-system, let \( \pi_n(X) = \{\pi_n(X_a, \alpha_a), \rho_{aa'}, \lambda\} \) denote the inverse system of groups where \( \pi_n(X_a, \alpha_a) \) is the \( n \)th homotopy group of \( (X_a, \alpha_a) \) and if \( a \leq a' \) then \( \rho_{aa'} : \pi_n(X_{a'}, \alpha_{a'}) \to \pi_n(X_a, \alpha_a) \) is the homomorphism induced by \( p_{aa'} \); i.e., if \( [\xi] \in \pi_n(X_{a'}, \alpha_{a'}) \) then \( \rho_{aa'}(\xi) = [p_{aa'} \alpha]\).

If \( f : X \to Y \) is a map of systems, \( f \) induces a morphism of inverse systems \( f_\pi : \pi_n(X) \to \pi_n(Y) \) where \( f_\pi = f : B \to A \) and \( (f_b)_{\alpha} : \pi_n(X_{f(b)}), \alpha_{f(b)} \to \pi_n(Y_{b}, \alpha_{b}) \) is the homomorphism induced by \( f_b \). This gives a covariant functor \( \pi_n \) between the category of ANR-systems and the category of inverse systems of groups.

**Theorem 2.6.** If \( f, g : X \to Y \) are homotopic maps of systems (\( f \cong g \)) then the induced morphisms \( f_\pi, g_\pi : \pi_n(X) \to \pi_n(Y) \) are \( \sim \)-related (\( f_\pi \sim g_\pi \)).
Proof. For each $b \in B$, choose $a \in A$ such that $a \geq f(b)$, $g(b)$ and $f_b p_{f(b)a} \simeq g_b p_{g(b)a}$. Then if $[\xi] \in \pi_n(X_a, x_a)$,

$$(f_b)_* \rho_{f(b)a} [\xi] = [f_b p_{f(b)a} \xi] = [g_b p_{g(b)a} \xi] = (g_b)_* \rho_{g(b)a} [\xi].$$

Corollary 2.7. If $X \cong Y$ then $\pi_n(X) \sim \pi_n(Y)$.

Mardešić and Segal have shown in [6] that if $X$ and $Y$ are ANR-systems associated with $(X, x_Q)$ and $(Y, y_0)$, respectively, then any map $i: X \rightarrow Y$ is ANR-systems associated with $(X, x_0)$ then any map of systems $i: X \rightarrow X'$ associated with the identity $1_{x,0}: (X, x_0) \rightarrow (X, x_0)$ is a homotopy equivalence. By Corollary 2.7 and Corollary 2.5, $i_\omega: \lim \pi_n(X) \rightarrow \lim \pi_n(X')$ is an isomorphism. Suppose $f: (X, x_0) \rightarrow (Y, y_0)$ is a map, $X, X'$ are ANR-systems associated with $(X, x_0)$ and $Y, Y'$ are ANR-systems associated with $(Y, y_0)$. Let $i: X \rightarrow X'$ and $j: Y \rightarrow Y'$ be homotopy equivalences associated with $1_{x,0}$ and $1_{y,0}$, respectively. Let $f: X \rightarrow Y$ and $f': X' \rightarrow Y'$ be maps of systems associated with $f$. It follows [5] that $jf \cong f' i: X \rightarrow Y'$. By Corollary 2.7 and Corollary 2.5, one has that $f^\omega f_\infty = f'_\infty j_\omega: \lim \pi_n(X) \rightarrow \lim \pi_n(Y')$.

If $(X, x_0)$ is a pointed compact Hausdorff space and $X$ is any ANR-system associated with $(X, x_0)$ then the $n$th shape group of $(X, x_0)$ is given by $\pi_n(X, x_0) = \lim \pi_n(X)$. If $f: (X, x_0) \rightarrow (Y, y_0)$ then the homomorphism $f^\omega: \pi_n(X, x_0) \rightarrow \pi_n(Y, y_0)$ is said to be induced by $f$. It is easy to show that $(1_{x,0})^\omega = 1_{x,0}$ and $(fg)^\omega = f^\omega g^\omega$. Corollary 2.7 also shows that the $n$th shape group is a shape invariant. It is shown in §3 that this definition of $\pi_n$ extends that given by Borsuk in [1].

Theorem 2.8. There is a homomorphism $p: \pi_n(X, x_0) \rightarrow \pi_n(X, x_0)$ such that for all $a \in A$, $(p_a)_* = \rho_a p$ where $(p_a)_*: \pi_n(X, x_0) \rightarrow \pi_n(X_a, x_a)$ is the homomorphism induced by $p_a: (X, x_0) \rightarrow (X_a, x_a)$.

Proof. The collection of maps $p_a: (X, x_0) \rightarrow (X_a, x_a)$ induces homomorphisms $(p_a)_*: \pi_n(X, x_0) \rightarrow \pi_n(X_a, x_a)$ such that if $a \leq a'$ then $(p_a)_* = \rho_{a'a'}(p_{a'})_*$. By the universal mapping property of $\pi_n(X, x_0)$ there is a unique homomorphism $p: \pi_n(X, x_0) \rightarrow \pi_n(X, x_0)$ such that for all $a \in A$, $(p_a)_* = \rho_a p$.

Theorem 2.9. If $X \in ANR$ then $\pi_n(X, x_0) \approx \pi_n(X, x_0)$.

Proof. Since $X \in ANR$, there is a special ANR-system $X = \ldots$
\{(X, x_0), 1_{X,x_0}\} \text{ associated with } (X, x_0). \text{ Then } \pi_n(X) = \{\pi_n(X, x_0), 1_{\pi_n(X,x_0)}\} \text{ has as inverse limit the group } \pi_n(X, x_0).

**Theorem 2.10.** If X is a compact Hausdorff space, \(x_0 \in X\) and \(X_0\) is the component of X containing \(x_0\) then \(\pi_n(X, x_0) = \pi_n(X_0, x_0)\).

**Proof.** Assume \(X \subset I^0\) and \(X = \{(X_a, x_0), i_{aa'}, A\}\) is an inclusion ANR-system associated with \((X, x_0)\). For each \(a \in A\) let \(X_{a_0}\) denote the component of \(X_a\) containing \(x_0\). Since a compact ANR is locally contractible, it is locally path connected. It follows that each \(X_{a_0}\) is a compact path connected ANR.

**Claim.** \(X_0 = \{(X_{a_0}, x_0), i_{aa'}, A\}\) is an inclusion ANR-system associated with \((X_0, x_0)\). It suffices to show that \(X_0 = \bigcap_{a \in A} X_{a_0}\). Certainly \(X_0 \subset \bigcap_{a \in A} X_{a_0}\) since \(X_0\) a compact connected subset of \(I^0\) implies that if \(N\) is any neighborhood of \(X_0\) there is a path connected neighborhood \(U\) of \(X_0\) such that \(U \subset N\). Let \(x \in \bigcap_{a \in A} X_{a_0} - X_0\). Then \(x \in X - X_0\) so let \(X_0\) denote the component of \(X\) to which \(x\) belongs. Then there are disjoint open sets \(U_0, U_1\) such that \(U_i \cap X = X_i\ (i = 0, 1)\). Since \(I^0\) is normal, there are open sets \(V_0, V_1\) such that \(X_i \subset V_i \subset \bar{V}_i \subset U_i\ (i = 0, 1)\). Since \(V = V_0 \cup V_1 \cup [I^0 - (\bar{V}_0 \cup \bar{V}_1)]\) is a neighborhood of \(X\) in \(I^0\) there is an \(a \in A\) such that \(X_a \subset V\). Now \(X_{a_0} \subset V_0\) and \(x \in V_1\), a contradiction since \(V_0 \cap V_1 = \emptyset\). Thus \(X_0 = \bigcap_{a \in A} X_{a_0}\) and the claim is proven. By a well-known theorem, \(\pi_n(X_a, x_0) = \pi_n(X_{a_0}, x_0)\) so that \(\pi_n(X) = \pi_n(X_0)\).

It follows then that \(\pi_n(X, x_0) = \pi_n(X_0, x_0)\).

If \(x_0, x_1 \in X\) and \(\omega: I \to X\) is a path in \(X\) connecting \(x_0\) and \(x_1\) then for each \(a \in A\), \(\omega\) induces an isomorphism \(\omega_a: \pi_n(X_a, x_0) \to \pi_n(X_a, x_1)\). If \(a \leq a'\) then \(i_{aa'}, \omega_a = \omega_{a'}\) and it is not hard to show that \(\pi_n(X, x_0) \cong \pi_n(X, x_1)\). Thus we have the following theorem.

**Theorem 2.11.** If \(x_0\) and \(x_1\) are in the same path component of \(X\) then \(\pi_n(X, x_0) \cong \pi_n(X, x_1)\).

**Question.** Is Theorem 2.11 valid if one replaces path component with component? Using Theorem 4.1 of [4], one can easily show the following is true.

**Theorem 2.12.** If \(X\) is a movable compact metric space and if \(x_0\) and \(x_1\) are in the same component of \(X\) then \(\pi_n(X, x_0) \cong \pi_n(X, x_1)\).

3. Equivalence of the inverse limit and Borsuk’s definition of \(\pi_n\). Let \(X\) be a compact metric space and \(x_0 \in X\). Assume that \(X\) is embedded in \(Q\) (Hilbert cube). Let \((S, a)\) denote the pointed \(n\)-dimensional sphere. An approximative map of \((S, a)\) toward \((X, x_0),\)
$\xi = \{ \xi_k, (S, a) \rightarrow (X, x_0) \}$ is a sequence of maps $\xi_k: (S, a) \rightarrow (Q, x_0)$ such that for every neighborhood $(V, x_0)$ of $(X, x_0)$ in $Q$ there is an index $k_0$ such that if $k \geq k_0$ then $\xi_k \equiv \xi_{k+1}$ in $(V, x_0)$. Two approximative maps $\xi$ and $\eta = \{ \eta_k, (S, a) \rightarrow (X, x_0) \}$ are said to be homotopic, $\xi \equiv \eta$, if for each neighborhood $(V, x_0)$ of $(X, x_0)$ in $Q$ there is an index $k_0$ such that if $k \geq k_0$ then $\xi_k \equiv \eta_k$ in $(V, x_0)$. The homotopy class of an approximative map $\xi$ is denoted by $[\xi]$.

If $\xi, \eta: (S, a) \rightarrow (Q, x_0)$ are maps, their join, $\xi \ast \eta: (S, a) \rightarrow (Q, x_0)$ is defined as follows. Let $P$ and $P'$ be $n$-dimensional balls on $S$ such that $a \in S - P, a \in S - P'$ and $P' \subset S - P$. Let $\alpha, \beta: (S, a) \times I \rightarrow (S, a)$ be homotopies such that $\alpha(x, 0) = \beta(x, 0) = x$ for all $x \in S$ and $\alpha(S - P, 1) = a = \beta(S - P', 1)$. Define

$$\left(\xi \ast \eta\right)(x) = \begin{cases} \xi \alpha(x, 1) & \text{if } x \in S - P' \\ \eta \beta(x, 1) & \text{if } x \in S - P \end{cases}$$

Note: if $[\xi], [\eta] \in \pi_n(X, x_0)$ where $(X, x_0) \subset (Q, x_0)$ then $[\xi][\eta] = [\xi \ast \eta]$ is the group operation in $\pi_n(X, x_0)$. Let $\pi_n^0(X, x_0)$ denote the group of fundamental classes of approximative maps of $(S, a)$ toward $(X, x_0)$ with operation $\ast$ defined as follows. If $[\xi], [\eta] \in \pi_n^0(X, x_0)$ then

$$[\xi][\eta] = \left[\left(\xi_k \ast \eta_k, (S, a) \rightarrow (X, x_0)\right)\right].$$

Then $\pi_n^0(X, x_0)$ is the $n$th fundamental group defined by Borsuk in [1].

**Theorem 3.1.** If $X = \{(X_k, x_0), i_{kk}\}$ is an inclusion ANR-sequence in $Q$ associated with $(X, x_0) \subset (Q, x_0)$ then $\pi_n^0(X, x_0) \approx \lim \pi_n(X) = \pi_n(X, x_0)$.

**Proof.** Let $\lambda_k: \pi_n^0(X, x_0) \rightarrow \pi_n(X_k, x_0)$ be given as follows. If $[\xi] \in \pi_n^0(X, x_0)$ then since $(X_k, x_0)$ is a neighborhood of $(X, x_0)$ in $Q$ there is an index $m_k$ such that if $m \geq m_k$ then $\xi_m \equiv \xi_{m_k}$ in $(X_k, x_0)$. Define $\lambda_k([\xi]) = [\xi_m] \in \pi_n(X_k, x_0)$. If $[\xi] = [\eta]$ then there is an index $m_0$ such that if $m \geq m_0$ then $\xi_m \equiv \eta_m$ in $(X_k, x_0)$ so that $\lambda_k$ is a well-defined function. If $[\xi], [\eta] \in \pi_n^0(X, x_0)$ and $m_0$ is “large enough” then

$$\lambda_k([\xi][\eta]) = \lambda_k([\xi_m \ast \eta_m, (S, a) \rightarrow (X, x_0)])$$

$$= [\xi_m \ast \eta_m]$$

$$= [\xi_m][\eta_m]$$

$$= \lambda_k([\xi])[\lambda_k([\eta]).$$

Thus each $\lambda_k$ is a group homomorphism.

**Note.** If $\lambda_k([\xi]) = \lambda_k([\eta]$ for all $k$, then $[\xi] = [\eta]$. Let $(V, x_0)$ be a
neighborhood of \((X, x_0)\) in \(Q\). Choose \(k\) so that \((X_k, x_0) \subset (V, x_0)\). Then \(\lambda_k[\xi] = \lambda_k[\eta]\) implies there is an \(m_0\) such that if \(m \geq m_0\) then \(\xi_m \equiv \eta_m\) in \((X_k, x_0) \subset (V, x_0)\).

We will now show that \(\pi^Q_n(X, x_0, \{\lambda_k\})\) is a terminal object in the category \(\text{inv}(\pi_n(X))\) [9], from which it will follow by uniqueness of inverse limit that \(\pi^Q_n(X, x_0) \approx \pi_n(X, x_0)\). To show \((\pi^Q_n(X, x_0, \{\lambda_k\})\) is in the category \(\text{inv}(\pi_n(X))\), one must show that if \(k \leq k'\) then \(\lambda_k = \rho_{kk'} \lambda_{k'}\), where \(\rho_{kk'}: \pi_n(X_k, x_0) \to \pi_n(X_{k'}, x_0)\) is the homomorphism induced by \(\iota_{kk'}: (X_{k'}, x_0) \to (X_k, x_0)\). Choose \(m_0 \geq m_k, m_{k'}\). Then \(\lambda_k[\xi] = [\xi_{m_0}] = \rho_{kk'} \lambda_{k'}[\xi]\).

It remains to show \((\pi^Q_n(X, x_0, \{\lambda_k\})\) is a terminal object; i.e., if \(G\) is any group and \(\sigma_k: G \to \pi_n(X_k, x_0)\) are group homomorphisms such that if \(k \leq k'\) then \(\sigma_k = \rho_{kk'} \sigma_{k'}\), then there is a unique group homomorphism \(\sigma: G \to \pi^Q_n(X, x_0)\) such that \(\sigma_k = \lambda_k \sigma\) for all \(k\). The uniqueness follows immediately from the above note.

**Existence.** Let \(g \in G\). Define \(\sigma(g) = \left[\{\hat{\xi}_k, (S, \alpha) \to (X, x_0)\}\right]\) where \(\hat{\xi}_k: (S, \alpha) \to (Q, x_0)\) satisfies \(\hat{\xi}_k \in \sigma_k(g) \in \pi_n(X_k, x_0)\). First, \(\{\hat{\xi}_k, (S, \alpha) \to (X, x_0)\}\) is an approximative map of \((S, \alpha)\) toward \((X, x_0)\). If \((U, x_0)\) is any neighborhood of \((X, x_0)\) in \(Q\) choose \(k_0\) such that \(k \geq k_0\) implies that \((X_k, x_0) \subset (U, x_0)\). Then \(\sigma_k(g) = \rho_{k_{k+1}} \sigma_{k+1}(g)\) so that \(\hat{\xi}_k \equiv \hat{\xi}_{k+1}\) in \((X_k, x_0) \subset (U, x_0)\). Next, \(\sigma\) is a well-defined function for if \(\hat{\xi} = \left[\{\hat{\xi}_k, (S, \alpha) \to (X, x_0)\}\right]\) and \(\hat{\xi}' = \left[\{\hat{\xi}_k, (S, \alpha) \to (X, x_0)\}\right]\) are such that \(\hat{\xi}_k, \hat{\xi}_k' \in \sigma_k(g)\) for each \(k\), then if \((U, x_0)\) is any neighborhood of \((X, x_0)\) in \(Q\) choose \(k_0\) such that if \(k \geq k_0\) then \((X_k, x_0) \subset (U, x_0)\). Then \(\hat{\xi}_k \equiv \hat{\xi}'_k\) in \((U, x_0)\) and hence \([\xi] = [\xi']\). Also, \(\sigma\) is a group homomorphism. Each \(\sigma_k\) is a homomorphism so that \(\sigma_k(g,g_2) = \sigma_k(g_1) \ast \sigma_k(g_2)\). Thus if \(\hat{\xi}_k \in \sigma_k(g), \hat{\eta}_k \in \sigma_k(g_2)\) then \(\hat{\xi}_k \ast \hat{\eta}_k \in \sigma_k(g_1) \ast \sigma_k(g_2)\). That is, \(\sigma(g,g_2) = \{[\xi_k \equiv \eta_k, (S, \alpha) \to (X, x_0)]\}\). But \(\sigma(g_1) \ast \sigma(g_2) = \{[\xi_k \ast \eta_k, (S, \alpha) \to (X, x_0)]\}\) so that \(\sigma\) is group homomorphism.

Finally, \(\sigma_k = \lambda_k \sigma\) for each \(k\). Since \(\lambda_k \sigma(g) = [\xi_{m_k}] \in \pi_n(X_k, x_0)\), it suffices to show \(\xi_{m_k} \equiv \hat{\xi}_k\) in \((X_k, x_0)\). If \(k \geq m_k\) then by the definition of \(m_k\), \(\xi_k \equiv \hat{\xi}_{m_k}\) in \((X_k, x_0)\). If \(m_k \geq k\) then \(\sigma_k(g) = \rho_{kk'} \sigma_{m_k}(g)\) so that \(\xi_{m_k} \equiv \hat{\xi}_k\) in \((X_k, x_0)\).

This completes the proof of the theorem.

4. The product of a family of inverse systems. Let \(\Omega\) be an index set. For each \(\omega \in \Omega\), let \(X^\omega = \{X_\omega, p_\omega^\alpha, A^\omega\}\) be an inverse system of topological spaces (a similar construction can be made for groups, R-modules, etc.). Let \(I = \{(F, \sigma): F\) is a finite nonempty subset of \(\Omega\) and \(\sigma: F \to \bigcup_{\omega \in A} A^\omega\) is a function such that \(\sigma(\omega) \in A^\omega\) for all \(\omega \in F\}\). Order \(I\) by \((F, \sigma) \leq (F', \sigma')\) iff \(F \subset F'\) and \(\sigma(\omega) \leq \sigma'(\omega)\) for all \(\omega \in F\). For \((F, \sigma) \in I\) let \(X_{(F, \sigma)} = \prod_{\omega \in F} X^{\sigma(\omega)}\). If \((F, \sigma) \leq (F', \sigma')\) then let \(\sigma_{(F, \sigma)}(F', \sigma') : X_{(F', \sigma')} \to X_{(F, \sigma)}\) be the composition of the natural projection \(\gamma:\prod_{\omega \in F'} X^{\sigma'(\omega)} \to \prod_{\omega \in F} X^{\sigma(\omega)}\).
and the product map $\prod p_{\alpha'(\omega)\sigma'(\omega)}: \prod_{\omega \in F} X_{\alpha'(\omega)}' \to \prod_{\omega \in F} X_{\sigma'(\omega)}$. It is not difficult to show that $X = \{X(F, \sigma), p(F, \sigma)(F', \sigma'), \Gamma\}$ is an inverse system. The inverse system $X$ is called the product of the family $\{X^\alpha: \alpha \in \Omega\}$ and is denoted $X = \prod_{\omega \in \Omega} X^\omega$. It can be shown that $\prod_{\omega \in \Omega} X^\omega$ is the categorical product of the family $\{X^\omega: \omega \in \Omega\}$.

**Example.** If each $A^\omega$ is a singleton, each $X^\omega = I^\omega = I$ is the unit interval and $X^\omega = \{J^\omega, l^\omega\}$ where $l^\omega: I^\omega \to I^\omega$ is the identity map then the above construction gives the usual representation of $1^\Omega = \prod_{\omega \in \Omega} I^\omega$ as the inverse limit of $\{I^\omega, p_{\alpha\alpha'}, F(\Omega)\}$ where $F(\Omega)$ is the set of all nonempty finite subsets of $\Omega$ ordered by inclusion and $p_{\alpha\alpha'}: I^\alpha = \prod_{\omega \in \alpha'} I^\omega \to I^\alpha$, is the natural projection (see [6]).

**Theorem 4.1.** $\lim_{\omega \in \Omega} X^\omega = \prod_{\omega \in \Omega} \lim X^\omega$.

**Proof.** Let $X^\omega = \lim_{\omega \in \Omega} X^\omega$. We show $\prod_{\omega \in \Omega} X^\omega$ is a terminal object in the category $\text{inv}(\prod_{\omega \in \Omega} X^\omega)$. For $(F, \sigma) \in \Gamma$, let $p_{(F, \sigma)}: \prod_{\omega \in \Omega} X^\omega \to X_{(F, \sigma)}$ be the composition of the natural projection $\eta: \prod_{\omega \in \Omega} X^\omega \to \prod_{\omega \in F} X^\omega$, and the product map $\prod_{\omega \in F} X^\omega \to \prod_{\omega \in F} X_{(\omega', \sigma')}$. It is not hard to show that if $(F, \sigma) \leq (F', \sigma')$ then $p_{(F', \sigma')} p_{(F, \sigma)} = p_{(F, \sigma)}$. Thus $(\prod_{\omega \in \Omega} X^\omega, (p_{(F, \sigma)}))$ is in the category $\text{inv}(\prod X^\omega)$.

It remains to show that $\prod_{\omega \in \Omega} X^\omega$ is a terminal object. That is, if $Y$ is any space and $f_{(F, \sigma)}: Y \to X_{(F, \sigma)}$ is a family of maps such that if $(F, \sigma) \leq (F', \sigma')$ then $p_{(F, \sigma)} f_{(F, \sigma)} = f_{(F', \sigma')}$, then there is a unique map $f: Y \to \prod_{\omega \in \Omega} X^\omega$ such that for all $(F, \sigma) \in \Gamma$, $p_{(F, \sigma)} f = f_{(F, \sigma)}$. If $\omega \in \Omega$ and $a \in A^\omega$ let $\sigma_a: \{\omega\} \to A^\omega$ be the function $\sigma_a(\omega) = a$. Then $((\omega), \sigma_a) \in \Gamma$ and $f^\omega_a = f_{((\omega), \sigma_a)}: Y \to X^\omega_a$ is a family of maps such that if $a \leq a'$ then $((\omega), \sigma_a) \leq ((\omega), \sigma_a')$, so that $p^\omega_{a'} f^\omega_a = f^\omega_a$. By the universal mapping property of $X^\omega_a$, there is a unique $f^\omega_a: Y \to X^\omega_a$ such that $p^\omega_{a'} f^\omega_a = f^\omega_a$ for all $a \in A^\omega$. Let $f: Y \to \prod_{\omega \in \Omega} X^\omega_a$ be the unique map thus defined. Then $f$ satisfies $p_{(F, \sigma)} f = f_{(F, \sigma)}$. Furthermore, if $g: Y \to \prod_{\omega \in \Omega} X^\omega_a$ is any map that satisfies $p_{(F, \sigma)} g = f_{(F, \sigma)}$ then $p_{((\omega), \sigma_a)} g = f_{((\omega), \sigma_a)} = f^\omega_a$. It follows then that $f = g$.

**Corollary 4.2.** If $X^\omega = \{X^\alpha, p^\alpha_{\alpha'}, A^\alpha\}, \omega \in \Omega$, is a family of ANR-systems where $X^\omega$ is associated with $X^\omega$, then $\prod_{\omega \in \Omega} X^\omega$ is an ANR-system associated with $\prod_{\omega \in \Omega} X^\omega$.

**Proof.** It suffices to note that if each $A^\omega$ is closure-finite then so is $\Gamma'$ and that the product of a finite number of ANR's is an ANR.

Suppose $X^\omega = \{X^\alpha, p^\alpha_{\alpha'}, A^\alpha\}, \omega \in \Omega; \Omega' = \{Y^\lambda, q^\lambda_{\beta\beta'}, B^\lambda\}, \lambda \in \Lambda$, are inverse systems (or ANR-systems) and $\theta: \Lambda \to \Omega$ is a one-to-one function such that for each $\lambda \in \Lambda$ there is a map $f^\lambda: X^{\theta(\lambda)} \to Y^\lambda$. Recall, a map
\[ f^i : X^{0(i)} \to Y^i \] consists of an increasing function \( f^i : B^i \to A^{\theta(i)} \) together with a family of maps \( f^i_b : X^{0(i)}_{f^i(b)} \to Y^i_b, b \in B^i \), such that if \( b \leq b' \) then \( q_{b,b'} f^i_b = f^i_{b'} \) (in the ANR-system case, \( q_{b,b'} f^i_b \equiv f^i_{b'} \)). Define \( f: \Gamma_Y \to \Gamma_X \) by \( f(F, \sigma) = (\theta(F), f, \sigma) \) where \( f, \sigma : \theta(F) \to \bigcup_{\omega \in \Omega} A^\omega \) is given by \( f_\omega(\theta(\lambda)) = f^i(\sigma(\lambda)) \in A^{\theta(i)} \). Then \( X_{f,\sigma} = \prod_{\omega \in \theta(F)} X^\omega_{f,(\omega)} = \prod_{\omega \in F} X^{0(i)}_{f,\sigma(\lambda)} \) so define \( f_{(F,\sigma)} : X_{f,\sigma} \to Y_{(F,\sigma)} \) as the product map \( \prod f^i_b : \prod_{\lambda \in F} X^{0(i)}_{f,\sigma(\lambda)} \to \prod_{\lambda \in F} Y^i_{\lambda} \). One then checks that if \( (F, \sigma) \leq (F', \sigma') \) then \( f(F, \sigma) \leq f(F', \sigma') \) and \( q_i f_{(F,\sigma)} = f_{(F',\sigma')} \) (in the ANR-system case, \( q_i f_{(F,\sigma)} = f_{(F,\sigma)} \)). Thus there is a map \( \bar{f} : \prod_{\omega \in \Omega} X^\omega \to \prod_{\lambda \in F} Y^i \).

If \( Z = \{Z_\epsilon, r_{cc}^c, C^\epsilon\}, \tau \in T, \) is another family of inverse systems and \( \phi: \Sigma \to T \) is a one-to-one function such that for all \( \omega \in \Omega \) there is a \( g^\omega : Z^{0(\omega)} \to X^\omega \) then there is a "natural composition" given by \( \phi \theta: \Lambda \to T \) and \( \bar{f} \bar{g}^{\theta(i)} : Z^{0(\theta)} \to Y^\theta(i) \). It is left to the reader to verify that the map determined by the composition is the same as the composition of the respective determined maps.

There is a "natural identity", \( \theta: \Omega \to \Omega \) the identity function and each \( \text{id} : X^\omega \to X^\omega \) the identity map. It is left to the reader to verify that the identity \( \text{id} : \prod_{\omega \in \Omega} X^\omega \to \prod_{\omega \in \Omega} X^\omega \) is determined by the natural identity.

We now restrict our attention to the ANR-system case when \( \Omega = \Lambda \) and \( \theta \) is the identity.

**Theorem 4.3.** If \( f^\omega, g^\omega : X^\omega \to Y^\omega \) are families of maps of systems such that \( f^\omega \equiv g^\omega \) for all \( \omega \in \Omega \) then \( f \equiv g : \prod_{\omega \in \Omega} X^\omega \to \prod_{\omega \in \Omega} Y^\omega \).

**Proof.** For each \( b \in B^\omega \) there is an \( a_b \in A^\omega, a_b \geq f^\omega(b), g^\omega(b) \) such that \( \tau_\omega \in A^\omega \) be an increasing function such that \( \tau_\omega(b) \geq a_b \) for all \( b \in B^\omega \). If \( (F, \sigma) \in \Gamma_Y \), consider \( (F, \sigma) \in \Gamma_X \) where \( \tau : F \to \bigcup_{\omega \in \Omega} A^\omega \) is given by \( \tau(\omega) = \tau_\omega(\sigma(\omega)) \). First, \( (F, \tau) \geq f(F, \sigma), g(F, \sigma) \). Since \( \theta \) is the identity, \( f(F, \sigma) = (F, f, \sigma) \) where \( f_\omega(\sigma) = f^\omega(\sigma(\omega)) \). Then \( (F, \tau) \geq (F, f, \sigma) \) since \( \tau(\omega) = \tau_\omega(\sigma(\omega)) \geq a_\sigma(\omega) \geq f^\omega(\sigma(\omega)) = f_\omega(\sigma(\omega)) \).

Similarly, \( (F, \sigma) \geq g(F, \sigma) \). Furthermore, \( f^\omega \equiv g^\omega \) implies \( f(F, \sigma) \equiv g(F, \sigma) \).

**Corollary 4.4.** If \( Sh(X^\omega) = Sh(Y^\omega) \) for all \( \omega \in \Omega \) then
Corollary 4.4 allows one to define the product of shapes as follows: \[ Sh(\prod_{\omega \in \Omega} X^\omega) = Sh(\prod_{\omega \in \Omega} Y^\omega). \]

In [5] Mardešić gives the notion of a shape retraction. For our purposes we use the following definition: if \( j: X \to Y \) is an embedding then a map of systems \( r: Y \to X \) is a shape retraction iff \( r_j \cong 1_X \) where \( j: X \to Y \) is a map of systems associated with \( j \). It is routine to verify this definition is equivalent to the one given by Mardešić. If there is an embedding \( j: X \to Y \) and a shape retraction \( r: Y \to X \) then \( X \) is said to be a shape retract of \( Y \).

**Corollary 4.5.** If \( r^\omega: Y^\omega \to X^\omega \) is a shape retraction for all \( \omega \) then \( r: \prod_{\omega \in \Omega} Y^\omega \to \prod_{\omega \in \Omega} X^\omega \) is also a shape retraction.

**Proof.** Let \( Y^\omega, X^\omega \) be associated with \( Y^\omega, X^\omega \), respectively, and \( j^\omega: X^\omega \to Y^\omega \) the required embeddings. Let \( j: \prod_{\omega \in \Omega} X^\omega \to \prod_{\omega \in \Omega} Y^\omega \) be the embedding determined by the family \( \{j^\omega: \omega \in \Omega\} \). It is routine to verify that the map determined by the family \( \{j^\omega: X^\omega \to Y^\omega\} \) is associated with \( j \). We have that \( r^\omega j^\omega \cong 1_{X^\omega} \) where \( 1_{X^\omega}: X^\omega \to X^\omega \) is the map associated with the identity. By the above theorem, \( r_j \cong 1_{\prod_{\omega \in \Omega} X^\omega} \).

5. Products of ASR and ANSR-sets. In [5] Mardešić gives definitions for absolute shape retract (ASR) and absolute neighborhood shape retracts (ANSR). These correspond to Borsuk’s FAR and FANR-sets, respectively, in the metric case. We will use the following characterizations: A compact Hausdorff space \( X \) is an ASR (respectively, ANSR) if there is a compact AR (respectively, ANR) \( Y \) and an embedding \( j: X \to Y \) such that \( X \) is a shape retract of \( Y \) (see [10] and [5]).

**Theorem 5.1.** If \( X = \prod_{\omega \in \Omega} X^\omega \) then \( X \in \text{ASR} \) iff \( X^\omega \in \text{ASR} \) for all \( \omega \in \Omega \).

**Proof.** If \( X \in \text{ASR} \) there is a \( Y \in \text{AR} \), an embedding \( j: X \to Y \) and a shape retraction \( r: Y \to X \). Since each natural projection \( p^\omega: X \to X^\omega \) is a retraction, the associated maps of systems \( p^\omega: X \to X^\omega \) are shape retractions. It follows [5] that \( p^\omega r^\omega: Y \to X^\omega \) is a shape retraction. Thus, each \( X^\omega \) is an ASR.

Conversely, if \( X^\omega \in \text{ASR} \) for all \( \omega \in \Omega \), then for each \( \omega \in \Omega \) there is an AR-set \( Y^\omega \) such that \( X^\omega \) is a shape retract of \( Y^\omega \). Since the product of any family of AR-sets is an AR-set, we have by Corollary 4.5 that \( X \in \text{ASR} \).
**Theorem 5.2.** If $X = \prod_{\omega \in \Omega} X^\omega$ then $X \in \text{ANSR}$ iff $X^\omega \in \text{ANSR}$ for all $\omega$ and $X^\omega \in \text{ASR}$ for all but a finite number of $\omega$.

*Proof.* If $X^\omega \in \text{ANSR}$ for all $\omega$ and $X^\omega \in \text{ASR}$ for all but finitely many $\omega$, say $\omega_1, \omega_2, \cdots, \omega_n$, then for all $\omega$ there is an ANR-set $Y^\omega$ and a shape retraction $r^\omega$: $Y^\omega \to X^\omega$ such that $Y^\omega \in \text{AR}$ if $\omega \neq \omega_k$ $(k = 1, 2, \cdots, n)$. Then $\prod_{\omega \in \Omega} Y^\omega \in \text{ANR}$ and there is a shape retraction $r: \prod_{\omega \in \Omega} Y^\omega \to \prod_{\omega \in \Omega} X^\omega$ so that $\prod_{\omega \in \Omega} X^\omega \in \text{ANSR}$.

Conversely, if $X \in \text{ANSR}$ then as in the proof of Theorem 5.1, each $X^\omega \in \text{ANSR}$. We may assume without loss that $X^\omega \subset I^\omega = \prod_{1 \leq i \leq \omega} I$ and $X = \prod_{\omega \in \Omega} I^\omega$. By Theorem IV. 2.10 of [10], there is a closed neighborhood $W$ of $X$ in $I^\omega$ and a shape retraction $r: W \to X$. There is a finite subset of $\Omega$, $\{\omega_1, \omega_2, \cdots, \omega_n\}$ and neighborhoods $U_i$ of $X^\omega_i$ in $I^{\omega_i}$ $(i = 1, 2, \cdots, n)$ such that $X = \prod_{\omega \in \Omega} X^\omega \subset \prod_{i=1}^n U_i \times \prod_{\omega \neq \omega_i} I^\omega \subset W$.

Let $i: X \to W$, $j_\omega: X^\omega \to I^\omega$ denote the inclusion maps and let $p_\omega: X \to X^\omega$ be the natural projections. Choose inclusion maps $j'_\omega: I^\omega \to W$ for $\omega \neq \omega_i$ $(i = 1, 2, \cdots, n)$ and $i_\omega: X^\omega \to X$ such that $j'_\omega j_\omega = i_\omega$ and $p_\omega i_\omega = 1_{X^\omega}$. Then $r i = 1_X$ so that for $\omega \neq \omega_i$ $(i = 1, 2, \cdots, n)$,

$$p_\omega r j'_\omega j_\omega \simeq p_\omega r i_\omega \simeq p_\omega i_\omega \simeq 1_{X^\omega} \omega.$$ 

Hence $p_\omega r j'_\omega: I^\omega \to X^\omega$ is a shape retraction for $\omega \neq \omega_i$ $(i = 1, 2, \cdots, n)$. Thus $X^\omega, \omega \neq \omega_i$ $(i = 1, 2, \cdots, n)$, is an ASR-set.

6. Products and shape groups. An inspection of Theorem 4.1 shows that the proof does not involve the fact that each $X^\omega$ is a topological space. It remains valid, for example, whenever the objects are groups. This fact together with the fact that the (usual) homotopy group of a product is the direct product of the (usual) homotopy groups of its factors, [11] Exercise B.5, p. 419, gives the following theorem.

**Theorem 6.1.** If $(X, x_0) = \prod_{\omega \in \Omega} (X^\omega, x_0^\omega)$ then $\pi_n(X, x_0) = \prod_{\omega \in \Omega} \pi_n \left( (X^\omega, x_0^\omega) \right)$.

*Proof.* For each $\omega$ let $X^\omega = \{ (X^\omega, x_0^\omega), p_{\omega 0}, A^\omega \}$ be an ANR-system associated with $(X^\omega, x_0^\omega)$. Then

$$\pi_n(X, x_0) = \pi_n \left( \prod_{\omega \in \Omega} (X^\omega, x_0^\omega) \right) = \lim_{\omega} \pi_n \left( \prod_{\omega \in \Omega} X^\omega \right)$$
\[
= \lim \pi_n(X_{(F,a)}, p_{(F,a)}(F',a'), \Gamma)
\]
\[
= \lim \{\pi_n(\prod_{\omega \in F} (X^\omega_{\sigma(a)}, a^\omega_{\sigma(a)}), \rho_{(F,a)}(F',a'), \Gamma) \}
\]
\[
= \lim \{\prod_{\omega \in F} \pi_n(X^\omega_{\sigma(a)}, a^\omega_{\sigma(a)}), \prod_{\omega \in F} \rho_{\sigma(a)}(a', A^\omega) \} 
\]
\[
= \prod_{\omega \in D} \{\pi_n(X^\omega_{\sigma(a)}, a^\omega_{\sigma(a)}), \rho_{\sigma(a)}(a', A^\omega) \} 
\]

Note. In recent correspondence M. Moszyńska indicated that she has defined the concept of "limit homotopy groups" which correspond to our definition of the shape groups. Her approach, to appear in [8] and [9], is more categorical than ours. For completeness we have included our definition and proof that the shape are isomorphic to Borsuk's fundamental groups. The approach to the latter (Theorem 3.1) is somewhat different than the approach she used (see [9]).

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