AN ERGODIC PROPERTY OF LOCALLY COMPACT AMENABLE SEMIGROUPS

JAMES CHIN-SZE WONG
Let $M(S)$ be the Banach algebra of all bounded regular Borel measures on a locally compact semigroup $S$ with variation norm and convolution as multiplication and $M_0(S)$ the probability measures in $M(S)$. We obtain necessary and sufficient conditions for the semigroup $S$ to have the (ergodic) property that for each $\nu \in M(S)$, $|\nu(S)| = \inf \{||\nu*\mu||: \mu \in M_0(S)\}$, an extension of a known result for locally compact groups.

1. Notations and terminologies. We shall follow Hewitt and Ross [9] for basic notations and terminologies concerning integration on locally compact spaces. Let $S$ be a locally compact semigroup with jointly continuous multiplication and $M(S)$ the Banach algebra of all bounded regular Borel measures on $S$ with total variation norm and convolution $\mu*\nu$, $\mu, \nu \in M(S)$ as multiplication where

$$\int f \, d\mu*\nu = \int \int f(xy) \, d\mu(x) \, d\nu(y) = \int \int f(xy) \, d\nu(y) \, d\mu(x)$$

for $f \in C_0(S)$ the space of all continuous functions on $S$ which vanish at infinity. (See for example [1], [6], or [18].) Let $M_0(S) = \{\mu \in M(S): \mu \geq 0 \text{ and } ||\mu|| = 1\}$ be the set of all probability measures in $M(S)$. Consider the continuous dual $M(S)^*$ of $M(S)$. Denote by $1$ the element in $M(S)^*$ such that $1(\mu) = \int d\mu = \mu(S)$, $\mu \in M(S)$. Clearly $||1|| = 1$.

2. Convolution of functionals and measures, means. Let $F \in M(S)^*$, $\mu \in M(S)$, we define a linear functional $l_\mu F = \mu \odot F$ on $M(S)$ by $\mu \odot F(\nu) = F(\mu*\nu)$, $\nu \in M(S)$. Clearly $\mu \odot F \in M(S)^*$. In fact $||\mu \odot F|| \leq ||\mu|| \cdot ||F||$. Similarly we define $F \odot \mu = r_\mu F$.

A linear functional $M \in M(S)^*$ is called a mean if $M(F) \geq 0$ if $F \geq 0$ and $M(1) = 1$. Here $F \geq 0$ means $F(\mu) \geq 0$ for all $\mu \geq 0$ in $M(S)$. An equivalent definition is

$$\inf \{F(\mu): \mu \in M_0(S)\} \leq M(F) \leq \sup \{F(\mu): \mu \in M_0(S)\}$$

for any $F \in M(S)^*$.

Consequently $||M|| = M(1) = 1$ for any mean $M$ on $M(S)^*$. It follows that the set of means is weak* compact and convex. Each probability measure $\mu \in M_0(S)$ is a mean if we put $\mu(F) = F(\mu)$, $F \in$
M(S)*. An application of Hahn-Banach Separation Theorem shows that $M_d(S)$ is weak* dense in the set of means on $M(S)^*$.

A mean $M$ is topological left invariant if $M(\mu \odot F) = M(F)\nu\mu \in M_d(S)$ and $F \in M(S)^*$ (see Greenleaf [7] for the case of locally compact groups).

3. Topological right stationarity and ergodic property. Following Wong [16], we say that $S$ is topological right stationary if for each $F \in M(S)^*$, there is a net $\mu_\alpha \in M_d(S)$ and some scalar $\beta$ such that $F \odot \mu_\alpha \rightarrow \beta \cdot 1$ weak* in $M(S)^*$.

**Theorem 3.1.** Let $S$ be a locally compact semigroup, the following statements are equivalent:

1. $S$ is topological right stationary.
2. For each $\nu \in M(S)$, $|\nu(S)| = \inf \{||\nu*\mu||: \mu \in M_d(S)\}$.
3. There is a net $\mu_\alpha \in M_d(S)$ such that $||\mu*\mu_\alpha - \mu_\alpha|| \rightarrow 0$ for any $\mu \in M_d(S)$.
4. $M(S)^*$ has a topological left invariant mean.

**Proof.**

(1) implies (2).

Assume that $S$ is topological right stationary, we modify the arguments in Glicksberg [5, Lemma 2.1] to show that $S$ has (ergodic) property (2). Observe that

$$||\nu*\mu|| = |\nu*\mu|(S) \geq |\nu*\mu(S)| = |\nu(S)\mu(S)| = |\nu(S)|$$

for any $\mu \in M_d(S)$, $\nu \in M(S)$. Hence $|\nu(S)| \leq \inf \{||\nu*\mu||: \mu \in M_d(S)\}$. Now let $c = \inf \{||\nu*\mu||: \mu \in M_d(S)\} > 0$. By Hahn-Banach Extension Theorem, there is some $F \in M(S)^*$ such that $||F|| = 1$ and

$$c \leq |(F, \sigma)| \text{ for any } \sigma \in C_\nu,$$

the norm closure of the convex set $\{\nu*\mu: \mu \in M_d(S)\}$ in $M(S)$. Let $C_\nu$ be the weak* closure of the convex set $\{F \odot \mu: \mu \in M_d(S)\}$ in $M(S)^*$. Since $(F, \sigma*\mu) = (F \odot \mu, \sigma)$, it follows that

$$c \leq |(G, \sigma)| \quad \forall \sigma \in C_\nu \quad \text{and} \quad G \in C_\nu.$$

But $S$ is topological right stationary, there is some $\beta$ such that $\beta \cdot 1 \in C_\nu$ (here we depart from Glicksberg’s proof in [5, Lemma 2.1], see remarks below). Now $\beta \cdot 1$ is constant on $C$, since

$$(\beta \cdot 1, \nu*\mu) = \beta \cdot (\nu*\mu)(S) = \beta \cdot \nu(S) \cdot \mu(S)$$

$$= \beta \cdot \nu(S) = (\beta \cdot 1, \nu)$$

for any $\mu \in M_d(S)$. Moreover,
\[ c \leq |(\beta \cdot 1, \nu)| = \inf \{|(\beta \cdot 1, \nu * \mu)|: \mu \in M_0(S)\} \]
\[ \leq |\beta| \cdot \inf \{|\nu * \mu|: \mu \in M_0(S)\} \]
\[ = |\beta| \cdot c. \]

Consequently \(|\beta| = 1\) and \(c = |(\beta \cdot 1, \nu)| = |\beta| \cdot |\nu(S)| = |\nu(S)|.\)

(2) implies (3).

Except that we work with measures instead of functions this is practically the same as in the locally compact group case (Greenleaf [7, Theorem 3.7.3]). Let \(\mu \in M_0(S)\) be fixed. Consider the directed system \(J = \{\alpha\}\) where \(\alpha = (\mu, \mu_2, \ldots, \mu_n; \varepsilon), \mu_i \in M_0(S), \varepsilon > 0, n \) finite. \(\alpha \geq \alpha'\) means \(\{\mu_i\} \supset \{\mu'_i\}\) and \(\varepsilon \leq \varepsilon'.\) For each \(\alpha \in J,\) we always have \((1, \mu * \mu - \mu) = 0 \forall i = 1, 2, \ldots, n.\) By assumption, there exist \(\sigma_1, \sigma_2, \ldots, \sigma_n \in M_0(S)\) such that

\[ \|((\mu_i * \mu - \mu) * \sigma_i)\| < \varepsilon \]
\[ \|((\mu_n * \mu - \mu) * \sigma_i * \sigma_2 * \cdots * \sigma_n)\| < \varepsilon \]

\[ \|((\mu_n * \mu - \mu) * \sigma_i * \sigma_2 * \cdots * \sigma_n)\| < \varepsilon. \]

(Note \((1, \nu) = 0\) implies \((1, \nu * \sigma_i) = 0.\) Put \(\sigma_\alpha = \sigma_1 * \sigma_2 * \cdots * \sigma_n,\) then

\[ \|((\mu_k * \mu - \mu) * \sigma_\alpha)\| \]
\[ \leq \|((\mu_k * \mu - \mu) * \sigma_1 * \cdots * \sigma_k)\| \|\sigma_{k+1} * \cdots * \sigma_n\| \]
\[ = \|((\mu_n * \mu - \mu) * \sigma_1 * \cdots * \sigma_n)\| < \varepsilon \]

\(\forall k = 1, 2, \ldots, n.\) Finally define \(\mu_\alpha = \mu * \sigma_\alpha \in M_0(S)\) for \(\alpha \in J.\) Then \(|\nu * \mu_\alpha - \mu_\alpha| \rightarrow 0\) for any \(\nu \in M_0(S).\)

(3) implies (4) and (4) implies (1).

These are the same as in the locally compact group case and we omit the details. The reader may consult Greenleaf [7] and Wong [16].

4. Remarks. Equivalence of (2) and (4) is an analogue of a result of H. Reiter on ergodic property of locally compact amenable groups (see Greenleaf [7, Theorem 3.7.3 p. 77]). Equivalence of (1) and (4) is an extension in a slightly different form of a result in Wong [16].

In the proof of [7, Theorem 3.7.3], Greenleaf used Rickert’s fixed point theorem [7, Theorem 3.3.1]. If we were to employ the same arguments in proving that (1) implies (2), we would have to invoke an analogous fixed point theorem (see Wong [17, Theorem 3.3] which has a natural extension to locally compact semigroups) for the compact convex set \(C_{fr}\) (referring to the proof of (1) implies (2)) to produce a
fixed point $G \in C_p$ of norm 1 such that $G \bullet \mu = G \forall \mu \in M_0(S)$. The question is whether $G = \beta \cdot 1$ for some scalar $\beta$? If $S$ is a locally compact group, such a $G$ is necessarily "constant" on $M_0(S) = L_1(S)$ (the absolutely continuous measures) and, hence on $M(S)$. For general $S$, Greenleaf's proof may not carry over.

Finally, it is easy to see that our definitions are consistent with previous ones given in Greenleaf [7] and Wong [16] for locally compact groups.

5. Continuous functions in $M(S)^*$. Let $CB(S)$ be the space of all bounded continuous on $S$ with supreumum norm. If $\mu \in M(S)$, $f \in CB(S)$, we can define $l_{\mu}f = \mu \bullet f$ and $r_{\mu}f = f \bullet \mu$ (both in $CB(S)$ again) by putting

$$
\mu \bullet f(s) = \int f(ts)d\mu(t)
$$

$$
f \bullet \mu(s) = \int f(st)d\mu(t), \ s \in S
$$

(see Williamson [15]). Invariant means on $CB(S)$ are defined in the obvious and usual way. Each function $f \in CB(S)$ can be regarded as a functional $Tf \in M(S)^*$ such that

$$
Tf(\mu) = \int f d\mu, \ \mu \in M(S).
$$

The map $T: CB(S) \to M(S)^*$ is a linear isometry (into) which commutes with convolution operators $l_{\mu}$ (and also $r_{\mu}$). Moreover $T \geq 0$ and $T(1) = 1$. Therefore, the two definitions of invariant mean on $CB(S)$ and its image under $T$ agree. However, unlike the group case, it is not known if $M(S)^*$ has a topological left invariant mean when $CB(S)$ does.

REFERENCES


Received July 17, 1972 and in revised form March 27, 1973. Research supported by NRC of Canada Research Grant No. A8227.

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Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION
Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan

* C. R. DePrima California Institute of Technology, Pasadena, CA 91109, will replace J. Dugundji until August 1974.

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Mir Maswood Ali, *Content of the frustum of a simplex* ........................................ 313
Mieczyslaw Altman, *Contractors, approximate identities and factorization in Banach algebras* ................................................................. 323
Charles Francis Amelin, *A numerical range for two linear operators* ............ 335
John Robert Baxter and Rafael Van Severen Chacon, *Nonlinear functionals on $C([0, 1] \times [0, 1])$* ................................................................. 347
Stephen Dale Bronn, *Cotorsion theories* ........................................................... 355
Peter A. Fowler, *Capacity theory in Banach spaces* ........................................ 365
Jerome A. Goldstein, *Groups of isometries on Orlicz spaces* ......................... 387
Kenneth R. Goodearl, *Idealizers and nonsingular rings* ................................ 395
Robert L. Griess, Jr., *Automorphisms of extra special groups and nonvanishing degree 2 cohomology* ......................................................... 403
Paul M. Krajkiewicz, *The Picard theorem for multianalytic functions* .......... 423
Peter A. McCoy, *Value distribution of linear combinations of axisymmetric harmonic polynomials and their derivatives* ................................. 441
A. P. Morse and Donald Chesley Pfaff, *Separative relations for measures* ................................................................. 451
Albert David Polimeni, *Groups in which Aut(G) is transitive on the isomorphism classes of G* ................................................................. 473
Aribindi Satyanarayan Rao, *Matrix summability of a class of derived Fourier series* ................................................................. 481
Thomas Jay Sanders, *Shape groups and products* ........................................... 485
Ruth Silverman, *Decomposition of plane convex sets. II. Sets associated with a width function* ................................................................. 497
Richard Snay, *Decompositions of $E^3$ into points and countably many flexible dendrites* ................................................................. 503
John Griggs Thompson, *Nonsolvable finite groups all of whose local subgroups are solvable, IV* ................................................................. 511
Robert E. Waterman, *Invariant subspaces, similarity and isometric equivalence of certain commuting operators in $L_p$* ........................................ 593
James Chin-Sze Wong, *An ergodic property of locally compact amenable semigroups* ................................................................. 615
Julius Martin Zelmanowitz, *Orders in simple Artinian rings are strongly equivalent to matrix rings* ................................................................. 621