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**SUBNORMAL OPERATORS IN STRICTLY CYCLIC OPERATOR
ALGEBRAS**

RICHARD BOLSTEIN AND WARREN R. WOGEN

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It is shown that a subnormal operator cannot belong to a strictly cyclic and separated operator algebra unless it is normal and has finite spectrum. Further, a subnormal operator not of this type cannot have a strictly cyclic commutant.

1. Let \mathcal{H} be a complex Hilbert space, and let \mathcal{A} be a subset of the algebra $\mathcal{B}(\mathcal{H})$ of all bounded linear operators on \mathcal{H} . A vector $x \in \mathcal{H}$ with the property that $\mathcal{A}x = \{Ax: A \in \mathcal{A}\}$ is the full Hilbert space is said to be a *strictly cyclic vector* for \mathcal{A} , and \mathcal{A} is said to be *strictly cyclic* if such a vector exists. A vector x is called a *separating vector* for \mathcal{A} if no two distinct operators in \mathcal{A} agree at x . The set \mathcal{A} is said to be *strictly cyclic and separated* if there is a vector x which is both strictly cyclic and separating for \mathcal{A} .

Strictly cyclic operator algebras have recently been investigated by Mary Embry [2] and Alan Lambert [3]. Let \mathcal{A}' denote the *commutant* of the set \mathcal{A} , that is, \mathcal{A}' is the set of all bounded linear operators which commute with every operator in \mathcal{A} . Note that if x is a *cyclic vector* for \mathcal{A} (meaning $\mathcal{A}x$ is dense in \mathcal{H}), then x is separating for \mathcal{A}' .

LEMMA 1. *Let \mathcal{A} be a strictly cyclic subset of $\mathcal{B}(\mathcal{H})$. If \mathcal{A} is abelian, then it is maximal abelian, $\mathcal{A} = \mathcal{A}'$. Thus, a strictly cyclic abelian subset is automatically a weakly closed algebra.*

This lemma, which indicates the severity of the condition of strict cyclicity, is a sharper form of a result of Lambert [3].

Proof. Let x be strictly cyclic for \mathcal{A} , and let $B \in \mathcal{A}'$. Then there exists $A \in \mathcal{A}$ such that $Ax = Bx$. But $\mathcal{A} \subset \mathcal{A}'$ by hypothesis, so $A \in \mathcal{A}'$. Since x is separating for \mathcal{A}' , we have $B = A \in \mathcal{A}$, and the proof is complete.

If \mathcal{A} is strictly cyclic and abelian, then it is strictly cyclic and separated by Lemma 1. Mary Embry [2] showed that the converse holds if \mathcal{A} is the commutant of a single operator. Thus, if A is normal and $\{A\}'$ is strictly cyclic and separated, then $\{A\}'$ consists of normal operators by Fuglede's theorem. In a private communication to the authors, Mary Embry asked if "normal" could be replaced by "subnormal" in this statement. An operator is called *subnormal* if

it is the restriction of a normal operator to an invariant subspace. To this end, we show that if A is subnormal then strict cyclicity of $\{A\}'$ already forces A to be normal, and, moreover, its spectrum is a finite set. Thus, the commutant of a subnormal operator cannot be strictly cyclic and separated unless the underlying Hilbert space is finite-dimensional (since the commutant is then abelian and hence the operator, which is normal, must have simple spectrum). More generally, it is shown that a uniformly closed subalgebra \mathcal{A} of $\mathcal{B}(\mathcal{H})$ which has a separating vector x with the property that $\mathcal{A}x$ is a closed subspace of \mathcal{H} (this is the case if x is also strictly cyclic) contains no subnormal operators except possibly for normal operators with finite spectrum.

2. Let μ be a finite positive Borel measure in the plane with compact support X , let $H^2(\mu)$ be the closure of the polynomials in $L^2(\mu)$, and put $H^\infty(\mu) = H^2(\mu) \cap L^\infty(\mu)$. The next theorem, which is used to derive the main result, may be of independent interest.

THEOREM 1. $H^\infty(\mu) = H^2(\mu)$ if, and only if, X is finite.

Proof. The sufficiency is trivial. Assume now that X is infinite. Note that the inclusion map of $H^\infty(\mu)$ into $H^2(\mu)$ is continuous. We will show that the inverse map is not continuous, and hence, by the Open Mapping Theorem, that $H^\infty(\mu) \neq H^2(\mu)$.

Since X is compact and infinite, its set X' of accumulation points is compact and nonempty. Choose $\lambda_0 \in X'$ such that $|\lambda_0| = \max\{|\lambda| : \lambda \in X'\}$, and let $D_1 = \{\lambda : |\lambda| \leq |\lambda_0|\}$. By the choice of λ , $X \setminus D_1$ is a countable set. Therefore, we can choose a closed disk D_2 which contains D_1 and is tangent to D_1 at λ_0 , in such a way that the boundary of D_2 intersects X only at λ_0 . Now note that we may as well assume that D_2 is the closed unit disc Δ , and that $\lambda_0 = 1$.

Now $X \setminus \Delta$ is a countable set $\{y_1, y_2, \dots\}$, and if this set infinite, we must have $\lim y_n = 1$. Let $K = \Delta \cup (X \setminus \Delta)$. Then K is a compact set which does not separate the plane. Define a sequence of functions $\{f_n\}$ on K by

$$f_n(z) = \begin{cases} z^n: & z \in \Delta \\ 0: & z = y_i, 1 \leq i \leq n \\ 1: & z = y_i, i > n. \end{cases}$$

Then, for each n , f_n is continuous on K and analytic in its interior. By Mergelyan's theorem, each f_n is the uniform limit on K of a sequence of polynomials. Hence each $f_n \in H^\infty(\mu)$.

Let χ denote the function which has the value 1 at the point 1

and the value zero elsewhere. Clearly, $f_n \rightarrow \chi$ pointwise, and hence in the metric of $L^2(\mu)$ by dominated convergence. In particular, $\chi \in H^\infty(\mu)$. However, the point 1 is an accumulation point of the support of μ , and hence $\|f_n - \chi\|_\infty = 1$ for every n . Thus, $\{f_n\}$ converges to χ in $H^2(\mu)$ but not in $H^\infty(\mu)$.

THEOREM 2. *Let S be a subnormal operator on the Hilbert space \mathcal{H} , let \mathcal{A} be the uniformly closed algebra generated by S . If \mathcal{A} has a separating vector x such that $\mathcal{A}x$ is a closed subspace of \mathcal{H} , then the spectrum of S is a finite set, and hence S is normal.*

Proof. Let \mathcal{B} be the uniformly closed algebra generated by S and the identity operator I . Since $\mathcal{B}x$ is the sum of $\mathcal{A}x$ and the one-dimensional space spanned by x , and since we assume that $\mathcal{A}x$ is closed, we also have that $\mathcal{B}x$ is a closed subspace of \mathcal{H} .

Now $\mathcal{B}x$ is invariant under S and the restriction operator $S_0 = S|_{\mathcal{B}x}$ is subnormal. Since the uniformly closed algebra \mathcal{B}_0 generated by S_0 and I contains $\mathcal{B}|_{\mathcal{B}x}$, it follows that x is a strictly cyclic vector for \mathcal{B}_0 , that is, $\mathcal{B}_0x = \mathcal{B}x$. By the representation theorem for subnormal operators with a cyclic vector, Bram [1], S_0 is unitarily equivalent to the operator of multiplication by the identity function on some $H^2(\mu)$ space. Furthermore, the unitary equivalence can be constructed so that x corresponds to the constant function 1.

Now \mathcal{B}_0 corresponds via the unitary equivalence to the algebra of multiplication operators $M_\phi: f \rightarrow \phi f$ on $H^2(\mu)$, where ϕ belongs to the $L^\infty(\mu)$ -closure of the polynomials. Since any such function ϕ belongs to $H^\infty(\mu)$, it follows that the constant function 1 is a strictly cyclic vector for $\{M_\phi: \phi \in H^\infty(\mu)\}$, and hence that $H^\infty(\mu) = H^2(\mu)$. By Theorem 1, $H^2(\mu)$ is finite-dimensional.

It follows that $\mathcal{B}x$ is finite-dimensional, and, since $\mathcal{A} \subset \mathcal{B}$, so is $\mathcal{A}x$. Since x separates \mathcal{A} , it follows that \mathcal{A} is finite-dimensional. So there is a polynomial p such that $p(S) = 0$. Since $p(\sigma(S)) = \sigma(p(S)) = \{0\}$, $\sigma(S)$ is finite and hence S is normal.

COROLLARY 1. *Let \mathcal{A} be a uniformly closed subalgebra of $\mathcal{B}(\mathcal{H})$ which has a separating vector x such that $\mathcal{A}x$ is a closed subspace of \mathcal{H} . (This is the case if \mathcal{A} is strictly cyclic and separated.) Then \mathcal{A} contains no subnormal operator with infinite spectrum.*

Proof. Suppose $S \in \mathcal{A}$ is subnormal, and let $\mathcal{A}(S)$ be the uniformly closed algebra generated by S . Since $\mathcal{A}(S) \subset \mathcal{A}$, x separates $\mathcal{A}(S)$. Since the linear transformation $A \rightarrow Ax$ of \mathcal{A} onto $\mathcal{A}x$ is continuous and one-to-one, and since $\mathcal{A}x$ is closed by hypothesis, the transformation has a continuous inverse by the Open Mapping Theorem.

Therefore, $\mathcal{A}(S)x$ is closed, and the result follows from Theorem 2.

COROLLARY 2. *The commutant of a subnormal operator S is strictly cyclic if, and only if, S is normal and has finite spectrum.*

Proof. Suppose $\{S\}'$ has a strictly cyclic vector x . Then x separates $\{S\}''$, and it follows from [2, Lemma 2.1 (i)] that $\{S\}''x$ is a closed subspace. Thus, by Corollary 1, S has finite spectrum and hence is normal.

Conversely, if $\sigma(S) = \{\lambda_1, \dots, \lambda_n\}$, then each λ_j is an eigenvalue and \mathcal{H} is the direct sum of the corresponding eigensubspaces \mathcal{H}_j . It follows that $\{S\}' = \mathcal{B}(\mathcal{H}_1) \oplus \dots \oplus \mathcal{B}(\mathcal{H}_n)$. Hence any vector $x = x_1 + \dots + x_n$ where $0 \neq x_j \in \mathcal{H}_j$, $j = 1, \dots, n$, is strictly cyclic for $\{S\}'$.

COROLLARY 3. *Let S be a subnormal operator on a Hilbert space \mathcal{H} . If $\{S\}'$ is strictly cyclic and separated, then \mathcal{H} is finite-dimensional.*

Proof. By Corollary 2, S is normal, its spectrum is finite, and $\{S\}' = \mathcal{B}(\mathcal{H}_1) \oplus \dots \oplus \mathcal{B}(\mathcal{H}_n)$ with notation as in the proof of that corollary. If x is strictly cyclic for $\{S\}'$, then $x = x_1 + \dots + x_n$ where $0 \neq x_j \in \mathcal{H}_j$, all j . If some \mathcal{H}_j has dimension greater than 1, then there is a nonzero operator B_j on \mathcal{H}_j which annihilates x_j , and hence there is a nonzero $B \in \{S\}'$ such that $Bx = 0$. Therefore, if $\{S\}'$ is strictly cyclic and separated, each \mathcal{H}_j is one-dimensional and hence $\mathcal{H} = \mathcal{H}_1 \oplus \dots \oplus \mathcal{H}_n$ is finite-dimensional.

COROLLARY 4. *Let S be a subnormal operator on a Hilbert space \mathcal{H} . If $\{S\}''$ is strictly cyclic, then \mathcal{H} is finite-dimensional.*

Proof. If x is strictly cyclic for $\{S\}'' \subset \{S\}'$, then it is strictly cyclic and separating for $\{S\}'$ and the result follows from Corollary 3.

An operator A is said to be *strictly cyclic* if the weakly closed algebra generated by A and I has this property. Since this algebra is contained in the second commutant of A , it follows that the second commutant of a strictly cyclic operator is strictly cyclic. In view of Corollary 4, we have

COROLLARY 5. *There exist no strictly cyclic subnormal operators on an infinite-dimensional Hilbert space.*

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