SUBNORMAL OPERATORS IN STRICTLY CYCLIC OPERATOR ALGEBRAS

RICHARD BOLSTEIN AND WARREN R. WOGEN
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It is shown that a subnormal operator cannot belong to a strictly cyclic and separated operator algebra unless it is normal and has finite spectrum. Further, a subnormal operator not of this type cannot have a strictly cyclic commutant.

1. Let $\mathcal{H}$ be a complex Hilbert space, and let $\mathcal{A}$ be a subset of the algebra $\mathcal{B}(\mathcal{H})$ of all bounded linear operators on $\mathcal{H}$. A vector $x \in \mathcal{H}$ with the property that $\mathcal{A}x = \{Ax: A \in \mathcal{A}\}$ is the full Hilbert space is said to be a strictly cyclic vector for $\mathcal{A}$, and $\mathcal{A}$ is said to be strictly cyclic if such a vector exists. A vector $x$ is called a separating vector for $\mathcal{A}$ if no two distinct operators in $\mathcal{A}$ agree at $x$. The set $\mathcal{A}$ is said to be strictly cyclic and separated if there is a vector $x$ which is both strictly cyclic and separating for $\mathcal{A}$.

Strictly cyclic operator algebras have recently been investigated by Mary Embry [2] and Alan Lambert [3]. Let $\mathcal{A}'$ denote the commutant of the set $\mathcal{A}$, that is, $\mathcal{A}'$ is the set of all bounded linear operators which commute with every operator in $\mathcal{A}$. Note that if $x$ is a cyclic vector for $\mathcal{H}$ (meaning $\mathcal{A}x$ is dense in $\mathcal{H}$), then $x$ is separating for $\mathcal{A}'$.

LEMMA 1. Let $\mathcal{A}$ be a strictly cyclic subset of $\mathcal{B}(\mathcal{H})$. If $\mathcal{A}$ is abelian, then it is maximal abelian, $\mathcal{A} = \mathcal{A}'$. Thus, a strictly cyclic abelian subset is automatically a weakly closed algebra.

This lemma, which indicates the severity of the condition of strict cyclicity, is a sharper form of a result of Lambert [3].

Proof. Let $x$ be strictly cyclic for $\mathcal{A}$, and let $B \in \mathcal{A}'$. Then there exists $A \in \mathcal{A}$ such that $Ax = Bx$. But $\mathcal{A} \subseteq \mathcal{A}'$ by hypothesis, so $A \in \mathcal{A}'$. Since $x$ is separating for $\mathcal{A}'$, we have $B = A \in \mathcal{A}$, and the proof is complete.

If $\mathcal{A}$ is strictly cyclic and abelian, then it is strictly cyclic and separated by Lemma 1. Mary Embry [2] showed that the converse holds if $\mathcal{A}$ is the commutant of a single operator. Thus, if $A$ is normal and $\{A\}'$ is strictly cyclic and separated, then $\{A\}'$ consists of normal operators by Fuglede's theorem. In a private communication to the authors, Mary Embry asked if "normal" could be replaced by "subnormal" in this statement. An operator is called subnormal if
it is the restriction of a normal operator to an invariant subspace. To this end, we show that if $A$ is subnormal then strict cyclicity of $\{A\}'$ already forces $A$ to be normal, and, moreover, its spectrum is a finite set. Thus, the commutant of a subnormal operator cannot be strictly cyclic and separated unless the underlying Hilbert space is finite-dimensional (since the commutant is then abelian and hence the operator, which is normal, must have simple spectrum). More generally, it is shown that a uniformly closed subalgebra $\mathcal{A}$ of $B(H)$ which has a separating vector $x$ with the property that $\mathcal{A}x$ is a closed subspace of $H$ (this is the case if $x$ is also strictly cyclic) contains no subnormal operators except possibly for normal operators with finite spectrum.

2. Let $\mu$ be a finite positive Borel measure in the plane with compact support $X$, let $H^2(\mu)$ be the closure of the polynomials in $L^2(\mu)$, and put $H^\infty(\mu) = H^2(\mu) \cap L^\infty(\mu)$. The next theorem, which is used to derive the main result, may be of independent interest.

**Theorem 1.** $H^\infty(\mu) = H^2(\mu)$ if, and only if, $X$ is finite.

**Proof.** The sufficiency is trivial. Assume now that $X$ is infinite. Note that the inclusion map of $H^\infty(\mu)$ into $H^2(\mu)$ is continuous. We will show that the inverse map is not continuous, and hence, by the Open Mapping Theorem, that $H^\infty(\mu) \neq H^2(\mu)$.

Since $X$ is compact and infinite, its set $X'$ of accumulation points is compact and nonempty. Choose $\lambda_0 \in X'$ such that $|\lambda_0| = \max \{|\lambda|: \lambda \in X'\}$, and let $D_1 = \{\lambda: |\lambda| \leq |\lambda_0|\}$. By the choice of $\lambda$, $X \setminus D_1$ is a countable set. Therefore, we can choose a closed disk $D_2$ which contains $D_1$ and is tangent to $D_1$ at $\lambda_0$, in such a way that the boundary of $D_2$ intersects $X$ only at $\lambda_0$. Now note that we may as well assume that $D_2$ is the closed unit disc $\mathbb{D}$, and that $\lambda_0 = 1$.

Now $X \setminus \mathbb{D}$ is a countable set $\{y_1, y_2, \ldots\}$, and if this set infinite, we must have $\lim y_n = 1$. Let $K = \mathbb{D} \cup (X \setminus \mathbb{D})$. Then $K$ is a compact set which does not separate the plane. Define a sequence of functions $\{f_n\}$ on $K$ by

$$f_n(z) = \begin{cases} 
  z^n: & z \in \mathbb{D} \\
  0: & z = y_i, \; 1 \leq i \leq n \\
  1: & z = y_i, \; i > n.
\end{cases}$$

Then, for each $n$, $f_n$ is continuous on $K$ and analytic in its interior. By Mergelyan's theorem, each $f_n$ is the uniform limit on $K$ of a sequence of polynomials. Hence each $f_n \in H^\infty(\mu)$.

Let $\chi$ denote the function which has the value 1 at the point 1
and the value zero elsewhere. Clearly, \( f_n \to \chi \) pointwise, and hence in the metric of \( L^2(\mu) \) by dominated convergence. In particular, \( \chi \in H^\infty(\mu) \). However, the point 1 is an accumulation point of the support of \( \mu \), and hence \( \| f_n - \chi \|_\infty = 1 \) for every \( n \). Thus, \( \{ f_n \} \) converges to \( \chi \) in \( H^1(\mu) \) but not in \( H^\infty(\mu) \).

**Theorem 2.** Let \( S \) be a subnormal operator on the Hilbert space \( \mathcal{H} \), let \( \mathcal{A} \) be the uniformly closed algebra generated by \( S \). If \( \mathcal{A} \) has a separating vector \( x \) such that \( \mathcal{A}x \) is a closed subspace of \( \mathcal{H} \), then the spectrum of \( S \) is a finite set, and hence \( S \) is normal.

**Proof.** Let \( \mathcal{B} \) be the uniformly closed algebra generated by \( S \) and the identity operator \( I \). Since \( \mathcal{B}x \) is the sum of \( \mathcal{A}x \) and the one-dimensional space spanned by \( x \), and since we assume that \( \mathcal{A}x \) is closed, we also have that \( \mathcal{B}x \) is a closed subspace of \( \mathcal{H} \).

Now \( \mathcal{B}x \) is invariant under \( S \) and the restriction operator \( S_0 = S|_{\mathcal{B}x} \) is subnormal. Since the uniformly closed algebra \( \mathcal{B}_0 \) generated by \( S_0 \) and \( I \) contains \( \mathcal{B}|_{\mathcal{B}x} \), it follows that \( x \) is a strictly cyclic vector for \( \mathcal{B}_0 \), that is, \( \mathcal{B}_0x = \mathcal{B}x \). By the representation theorem for subnormal operators with a cyclic vector, Bram [1], \( S_0 \) is unitarily equivalent to the operator of multiplication by the identity function on some \( H^2(\mu) \) space. Furthermore, the unitary equivalence can be constructed so that \( x \) corresponds to the constant function 1.

Now \( \mathcal{B}_0 \) corresponds via the unitary equivalence to the algebra of multiplication operators \( M_\phi : f \to \phi f \) on \( H^2(\mu) \), where \( \phi \) belongs to the \( L^\infty(\mu) \)-closure of the polynomials. Since any such function \( \phi \) belongs to \( H^\infty(\mu) \), it follows that the constant function 1 is a strictly cyclic vector for \( \{ M_\phi : \phi \in H^\infty(\mu) \} \), and hence that \( H^\infty(\mu) = H^2(\mu) \). By Theorem 1, \( H^2(\mu) \) is finite-dimensional.

It follows that \( \mathcal{B}x \) is finite-dimensional, and, since \( \mathcal{A} \subset \mathcal{B} \), so is \( \mathcal{A}x \). Since \( x \) separates \( \mathcal{A} \), it follows that \( \mathcal{A} \) is finite-dimensional. So there is a polynomial \( p \) such that \( p(S) = 0 \). Since \( p(\sigma(S)) = \sigma(p(S)) = \{0\} \), \( \sigma(S) \) in finite and hence \( S \) is normal.

**Corollary 1.** Let \( \mathcal{A} \) be a uniformly closed subalgebra of \( \mathcal{B}(\mathcal{H}) \) which has a separating vector \( x \) such that \( \mathcal{A}x \) is a closed subspace of \( \mathcal{H} \). (This is the case if \( \mathcal{A} \) is strictly cyclic and separated.) Then \( \mathcal{A} \) contains no subnormal operator with infinite spectrum.

**Proof.** Suppose \( S \in \mathcal{A} \) is subnormal, and let \( \mathcal{A}(S) \) be the uniformly closed algebra generated by \( S \). Since \( \mathcal{A}(S) \subset \mathcal{A} \), \( x \) separates \( \mathcal{A}(S) \). Since the linear transformation \( A \to Ax \) of \( \mathcal{A} \) onto \( \mathcal{A}x \) is continuous and one-to-one, and since \( \mathcal{A}x \) is closed by hypothesis, the transformation has a continuous inverse by the Open Mapping Theorem.
Therefore, $\mathcal{A}(S)x$ is closed, and the result follows from Theorem 2.

**COROLLARY 2.** The commutant of a subnormal operator $S$ is strictly cyclic if, and only if, $S$ is normal and has finite spectrum.

**Proof.** Suppose $\{S\}'$ has a strictly cyclic vector $x$. Then $x$ separates $\{S\}'$, and it follows from [2, Lemma 2.1 (i)] that $\{S\}'x$ is a closed subspace. Thus, by Corollary 1, $S$ has finite spectrum and hence is normal.

Conversely, if $\sigma(S) = \{\lambda_1, \ldots, \lambda_n\}$, then each $\lambda_j$ is an eigenvalue and $\mathcal{H}$ is the direct sum of the corresponding eigensubspaces $\mathcal{H}_j$. It follows that $\{S\}' = \mathcal{B}(\mathcal{H}_1) \oplus \cdots \oplus \mathcal{B}(\mathcal{H}_n)$. Hence any vector $x = x_1 + \cdots + x_n$ where $0 \neq x_j \in \mathcal{H}_j$, $j = 1, \ldots, n$, is strictly cyclic for $\{S\}'$.

**COROLLARY 3.** Let $S$ be a subnormal operator on a Hilbert space $\mathcal{H}$. If $\{S\}'$ is strictly cyclic and separated, then $\mathcal{H}$ is finite-dimensional.

**Proof.** By Corollary 2, $S$ is normal, its spectrum is finite, and $\{S\}' = \mathcal{B}(\mathcal{H}_1) \oplus \cdots \oplus \mathcal{B}(\mathcal{H}_n)$ with notation as in the proof of that corollary. If $x$ is strictly cyclic for $\{S\}'$, then $x = x_1 + \cdots + x_n$ where $0 \neq x_j \in \mathcal{H}_j$, all $j$. If some $\mathcal{H}_j$ has dimension greater than 1, then there is a nonzero operator $B_j$ on $\mathcal{H}_j$ which annihilates $x_j$, and hence there is a nonzero $B \in \{S\}'$ such that $Bx = 0$. Therefore, if $\{S\}'$ is strictly cyclic and separated, each $\mathcal{H}_j$ is one-dimensional and hence $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$ is finite-dimensional.

**COROLLARY 4.** Let $S$ be a subnormal operator on a Hilbert space $\mathcal{H}$. If $\{S\}''$ is strictly cyclic, then $\mathcal{H}$ is finite-dimensional.

**Proof.** If $x$ is strictly cyclic for $\{S\}'' \subset \{S\}'$, then it is strictly cyclic and separating for $\{S\}'$ and the result follows from Corollary 3.

An operator $A$ is said to be strictly cyclic if the weakly closed algebra generated by $A$ and $I$ has this property. Since this algebra is contained in the second commutant of $A$, it follows that the second commutant of a strictly cyclic operator is strictly cyclic. In view of Corollary 4, we have

**COROLLARY 5.** There exist no strictly cyclic subnormal operators on an infinite-dimensional Hilbert space.
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