MAXIMAL INVARIANT SUBSPACES OF STRICTLY CYCLIC OPERATOR ALGEBRAS

MARY RODRIGUEZ EMBRY
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A strictly cyclic operator algebra $\mathcal{A}$ on a complex Banach space $X$ (dim $X \geq 2$) is a uniformly closed subalgebra of $\mathcal{L}(X)$ such that $\mathcal{A}x = X$ for some $x$ in $X$. In this paper it is shown that (i) if $\mathcal{A}$ is strictly cyclic and intransitive, then $\mathcal{A}$ has a maximal (proper, closed) invariant subspace and (ii) if $A \in \mathcal{L}(X)$, $A \neq zI$ and $\{A\}'$ (the commutant of $A$) is strictly cyclic, then $A$ has a maximal hyperinvariant subspace.

1. Notation and terminology. Throughout the paper $X$ is a complex Banach space of dimension greater than one and $\mathcal{L}(X)$ is the algebra of continuous linear operators on $X$. $\mathcal{A}$ will denote a uniformly closed subalgebra of $\mathcal{L}(X)$ which is strictly cyclic and $x_0$ will be a strictly cyclic vector for $\mathcal{A}$: that is, $\mathcal{A}x_0 = X$. We do not insist that the identity element $I$ of $\mathcal{L}(X)$ be an element of $\mathcal{A}$.

If $\mathcal{B} \subset \mathcal{L}(X)$, then the commutant of $\mathcal{B}$ is $\mathcal{B}' = \{E : E \in \mathcal{L}(X)$ and $EB = BE$ for all $B$ in $\mathcal{B}\}$. We shall use the terminology of “invariant” and “transitive” as follows: if $M \subset X$ and $\mathcal{B} \subset \mathcal{L}(X)$, then (i) $M$ is invariant under $\mathcal{B}$ if $\mathcal{B}M = \{Bm : B \in \mathcal{B}$ and $m \in M\} \subset M$, (ii) $M$ is an invariant subspace for $\mathcal{B}$ if $M$ is invariant under $\mathcal{B}$ and $M$ is a closed, nontrivial ($\neq \{0\}, X$) linear subspace of $X$, (iii) $\mathcal{B}$ is transitive if $\mathcal{B}$ has no invariant subspace and intransitive if $\mathcal{B}$ has an invariant subspace. Further, if $A \in \mathcal{L}(X)$ and $\{A\}'$ is intransitive, then each invariant subspace of $\{A\}'$ is called a hyperinvariant subspace of $A$. Finally an invariant subspace of $\mathcal{B}$ is maximal if it is not properly contained in another invariant subspace of $\mathcal{B}$.

2. Introduction. Strictly cyclic operator algebras have been studied by A. Lambert, D. A. Herrero, and the author of this paper. (See for example [2]-[6].) One of the major results in [2, Theorem 3.8], [3, Theorem 2], and [6, Theorem 4.5] is that a transitive subalgebra of $\mathcal{L}(X)$ containing a strictly cyclic algebra is necessarily strongly dense in $\mathcal{L}(X)$. In each of three developments the following is a key lemma: The only dense linear manifold invariant under a strictly cyclic subalgebra of $\mathcal{L}(X)$ is $X$. In Lemma 1 we shall present a generalization of this lemma which will be useful in the study of maximal invariant subspaces and noncyclic vectors of a strictly cyclic algebra $\mathcal{A}$.

**Lemma 1.** If $M$ is invariant under $\mathcal{A}$ and $x_0 \in M$, then $M = X$.  

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It should be noted that we do not require $M$ to be linear nor do we require, as was done in Lemma 3.4 of [2], that $I \in \mathcal{A}$. The proof given here is a slight modification of that given in [2].

**Proof.** We shall show that $\mathcal{A}x_0 \subset M$ and thus $X = \mathcal{A}x_0 \subset M$. Let $\{x_n\}$ be a sequence in $M$ such that $\lim_{n \to \infty} x_n = x_0$. By [2, Lemma 3.1 (ii)] there exists a sequence $\{A_n\}$ in $\mathcal{A}$ such that $A_n x_0 = x_0 - x_n$ and $\lim_{n \to \infty} \|A_n\| = 0$. Thus for $n$ sufficiently large, $\|A_n\| < 1$ and $(I - A_n)^{-1} = \sum_{k=0}^{\infty} (A_n)^k$. Consequently, $\mathcal{A}(I - A_n)^{-1} \subset \mathcal{A}$ and since $x_0 = (I - A_n)^{-1} x_n$, we have $\mathcal{A}x_0 = \mathcal{A}(I - A_n)^{-1} x_n \subset \mathcal{A}x_n \subset M$, as desired.

For the sake of future reference we restate and reprove the transitivity theorem.

**THEOREM 1.** If $\mathcal{A}$ is a strictly cyclic transitive subalgebra of $\mathcal{L}(X)$, then $\mathcal{A}$ is strongly dense in $\mathcal{L}(X)$.

**Proof.** Using Lemma 1 we can show (as in [2, Lemma 3.5]) that each densely defined linear transformation commuting with $\mathcal{A}$ is everywhere defined and continuous. Further, again using Lemma 1, we can show that if $E \in \mathcal{A}$ and $z \in \sigma(E)$, then either $zI - E$ is not one-to-one or does not have dense range. Thus if $\mathcal{A}$ is transitive, necessarily $E = zI$. Consequently, it follows from [1, p. 636 and Cor. 2.5, p. 641] that $\mathcal{A}$ is strongly dense in $\mathcal{L}(X)$.

3. Maximal invariant subspaces. In [2, Theorem 3.1] it is shown that every strictly cyclic, separated operator algebra $\mathcal{A}$ has a maximal invariant subspace. ($\mathcal{A}$ is separated by $x_0$ if $A = 0$ whenever $A \in \mathcal{A}$ and $Ax_0 = 0$.) Theorem 2 allows us to obtain the same result without the hypothesis that $\mathcal{A}$ be separated, provided $\mathcal{A}$ is intransitive.

**THEOREM 2.** An intransitive, strictly cyclic subalgebra $\mathcal{A}$ of $\mathcal{L}(X)$ has a maximal invariant subspace.

**Proof.** Let $\mathcal{M} = \{M: M$ is an invariant subspace of $\mathcal{A}\}$. By hypothesis $\mathcal{M} \neq \emptyset$. We shall order $\mathcal{M}$ by set inclusion and show that each linearly ordered subset of $\mathcal{M}$ has an upper bound in $\mathcal{M}$. To this end we let $\{M_\alpha\}$ be a linearly ordered subset of $\mathcal{M}$. Then $\bigcup_\alpha M_\alpha$ is invariant under $\mathcal{A}$. By Lemma 1, if $\bigcup_\alpha M_\alpha = X$, then $\bigcup_\alpha M_\alpha = X$ and consequently $x_0 \in M_\alpha$ for some value of $\alpha$. Since this last implies that $X = \mathcal{A}x_0 \subset \mathcal{A}M_\alpha \subset M_\alpha$ and contradicts the fact that $M_\alpha$ is a proper closed linear subspace of $X$, we see that $\bigcup_\alpha M_\alpha$ is not...
dense in $X$. Thus $\bigcup_a M_a$ is an element of $\mathcal{M}$ and is an upper bound for $\{M_a\}$. By the Maximality Principle $\mathcal{M}$ has a maximal element.

Lemma 1 and the Maximality Principle can be combined to arrive at other similar results. For example, (i) if $\mathcal{A}$ is intransitive and strictly cyclic, then $\mathcal{A}$ has a proper maximal invariant subset (this will be discussed further in §4) and (ii) if $X$ is a Hilbert space and $\mathcal{A}$ has a reducing subspace (that is, an invariant subspace of $\mathcal{A}$ which is also invariant under $\mathcal{A}^* = \{A^* : A \in \mathcal{A}\}$), then $\mathcal{A}$ has a maximal reducing subspace.

In [2, Theorem 3.7] it is shown that if $A$ is not a scalar multiple of $I$ and $\{A\}'$ is strictly cyclic, then $A$ has a hyperinvariant subspace. This result combined with Theorem 2 yields the following:

**Corollary 1.** If $A$ is not a scalar multiple of $I$ and $\{A\}'$ is strictly cyclic, then $A$ has a maximal hyperinvariant subspace.

We shall now turn our attention to intransitive, strictly cyclic operator algebras on a Hilbert space $X$. If $M$ is a closed linear subspace of $X, P_M$ will denote the orthogonal projection of $X$ onto $M$ and $M^\perp$ the orthogonal complement of $M: M^\perp = \{y: \langle y, m \rangle = 0 \text{ for all } m \text{ in } M\}$. Furthermore, $\mathcal{A}^* = \{A^*: A \in \mathcal{A}\}$.

In the Hilbert space situation we are able to conclude that $\mathcal{A}^*/M$ is strongly dense in $L(M^\perp)$ when $M$ is a maximal invariant subspace for $\mathcal{A}$. This remains an open question if $X$ is an arbitrary Banach space and is a particularly interesting one if $X$ is reflexive. For in that case if $M$ is a maximal invariant subspace of $\mathcal{A}$, then $M^\perp = \{x^*: x^*(M) = 0\}$ is a minimal invariant subspace of $\mathcal{A}^*$.

**Theorem 3.** Let $\mathcal{A}$ be a strictly cyclic operator algebra on a Hilbert space $X$. If $M$ is a maximal invariant subspace of $\mathcal{A}$, then

(i) $(I - P_M)\mathcal{A}(I - P_M)x_0 = M^\perp$ and (ii) $\mathcal{A}^*(I - P_M)$ is strongly dense in $L(M^\perp)$.

**Proof.** Note first that $(I - P_M)\mathcal{A}(I - P_M) = (I - P_M)\mathcal{A}$, so that (i) is immediate. Since $M$ is a maximal invariant subspace for $\mathcal{A}, M^\perp$ is a minimal invariant subspace for $\mathcal{A}^*$. Thus each of $\mathcal{A}^*(I - P_M)$ and $(I - P_M)\mathcal{A}(I - P_M)$ is transitive on $M^\perp$. Thus the uniform closure of $(I - P_M)\mathcal{A}(I - P_M)$ in $L(M^\perp)$ is transitive and by (i) is strictly cyclic; hence by Theorem 1 $(I - P_M)\mathcal{A}(I - P_M)$ is strongly dense in $L(M)$, which concludes our proof of (ii).

**Theorem 4.** Let $X$ be a Hilbert space, $A \in L(X)$ and $\{A\}'$ strictly cyclic. If $M$ is a maximal invariant subspace for $\{A\}'$, then there exists a multiplicative linear functional $f$ on $\{A\}''$ such
that for each \( E \) in \( \{A\}'' \), \((E - f(E)I)(X) \subseteq M\).

\textbf{Proof.} As we noted in the proof of Theorem 3,
\[
\mathcal{B} = (I - P_M)\{A\}'(I - P_M)
\]
is strongly dense in \( \mathcal{L}(M^\perp) \) and thus its commutant consists of the scalar multiples of the identity operator on \( M^\perp \). Since \( \{A\}'' \subseteq \{A\}' \) and \( M \) is invariant under \( \{A\}' \), we know that \((I - P_M)\{A\}''(I - P_M) \) is contained in the commutant of \( \mathcal{B} \) on \( M^\perp \) and hence \((I - P_M)\{A\}''(I - P_M) \subseteq \{z(I - P_M)\} \). Thus for \( E \) in \( \{A\}'' \), there exists a complex number \( z \) such that \((I - P_M)E(I - P_M) = z(I - P_M) \). Therefore, \((I - P_M)(E - zI) = 0 \) since \( M \) is invariant under \( \{A\}'' \); or equivalently \((E - zI)(X) \subseteq M \). Since \( M \) is a proper subset of \( X \), it is now obvious that the number \( z \) for which \((E - zI)(X) \subseteq M \) is unique. Define \( f(E) = z \).

That \( f \) is linear follows immediately from the fact that \( f(E) \) is the unique number for which \((E - f(E)I)(X) \subseteq M \). Furthermore, since \( M \) is invariant under \( \{A\}'' \), \((FE - f(E)F)(X) \subseteq M \) for all \( E, F \) in \( \{A\}'' \). Consequently (by uniqueness again), \( 0 = f(FE - f(E)F) = f(\underbrace{FE - f(E)F}) \) and thus we see that \( f \) is multiplicative.

\textbf{Corollary 2.} Let \( A \in \mathcal{L}(X) \) where \( X \) is a Hilbert space. If the range of \( A - zI \) is dense in \( X \) for each complex \( z \), then \( \{A\}' \) is not strictly cyclic.

\textbf{Proof.} Except for one minor technicality, Corollary 2 follows immediately from Theorem 4. For, if \( \{A\}' \) is strictly cyclic and intransitive, by Theorem 4 there exists a complex number \( f(A) \) such that the range of \( A - f(A)I \) is contained in a proper subspace of \( X \). By Corollary 1 the only other way in which \( \{A\}' \) can be strictly cyclic is when \( A = zI \) for some complex \( z \), in which case the range of \( A - zI \) is certainly not dense in \( X \).

In [2, Lemma 3.6] and [3, Proposition 2], it is shown that if \( E \in \mathcal{A}' \), where \( \mathcal{A} \) is strictly cyclic and \( z \in \sigma(E) \), then either \( zI - E \) is not one-to-one or \( zI - E \) does not have dense range. Corollary 2 now adds to our knowledge of \( \sigma(A) \) where \( \{A\}' \) is strictly cyclic: in this case we know that for at least one value of \( z \), the range of \( A - zI \) is nondense. Indeed we have the stronger result:

\textbf{Corollary 3.} Let \( A \in \mathcal{L}(X) \) where \( X \) is a Hilbert space. If \( \{A\}' \) is strictly cyclic, then there exists a common eigenvector for \( \{A^*\}'' \).

\textbf{Proof.} The case in which \( \{A\}' = \mathcal{L}(X) \) is trivial. Thus we assume \( A \neq zI \). By Theorem 4 if \( E \in \{A\}'' \), there exists a complex number \( f(E) \) such that \((E - f(E)I)(X) \subseteq M \) where \( M \) is a maximal
invariant subspace of \(\{A\}'\). Therefore, \(E^*(I - P_M)x_0 = f(E)^*(I - P_M)x_0\) and \((I - P_M)x_0 \neq 0\) since \(x_0\) is cyclic for \(\{A\}'\) and \(M\) is a proper invariant subspace for \(\{A\}'\).

4. Noncyclic vectors of \(\mathcal{A}\). In this last section of this paper we shall discuss briefly several properties of the set of noncyclic vectors of a strictly cyclic operator algebra \(\mathcal{A}\). A vector \(x\) is noncyclic for \(\mathcal{A}\) if \(\mathcal{A}x\) is not dense in \(X\). These results are summarized in Theorem 5. Parts (i) and (iii) of Theorem 5 also are found in [5, Theorem 2].

**Theorem 5.** Let \(N\) be the set of noncyclic vectors of a strictly cyclic operator algebra \(\mathcal{A}\),

(i) if \(x \not\in N\), then \(x\) is a strictly cyclic vector for \(\mathcal{A}\),

(ii) \(N\) is invariant under \(\mathcal{A}\),

(iii) \(N\) is closed in \(X\),

(iv) \(N\) is the unique proper maximal invariant subset of \(\mathcal{A}\),

(v) if \(N\) is not linear, then \(N + N = X\), where \(N + N = \{x + y : x, y \in N\}\).

**Proof.** (i) If \(x \not\in N\), then \(\mathcal{A}x = X\) and thus by Lemma 1 since \(\mathcal{A}x\) is invariant under \(\mathcal{A}\), we have \(\mathcal{A}x = X\) and \(x\) is strictly cyclic.

(ii) Assume that \(x \in N\) and \(A \in \mathcal{A}\). Then \(\mathcal{A}Ax \subset \mathcal{A}x\) and consequently \(\mathcal{A}Ax = X\). That is, \(Ax \in N\) for each \(A\) in \(\mathcal{A}\) which proves (ii).

(iii) By (ii) \(\mathcal{A}N \subset N\). Since \(\mathcal{A}\) has a strictly cyclic vector, we know by Lemma 1 that \(\bar{N}\) contains no strictly cyclic vector for \(\mathcal{A}\). Thus by (i) \(\bar{N}\) contains only noncyclic vectors for \(\mathcal{A}\), which says that \(N\) is closed.

(iv) By (ii) \(N\) is invariant under \(\mathcal{A}\). By hypothesis \(\mathcal{A}\) has a strictly cyclic vector so that \(N \neq X\). These two observations essentially prove (iv) since an element \(x\) of a proper invariant subset of \(\mathcal{A}\) is necessarily an element of \(N\). (v) If \(N\) is nonlinear, then since \(N\) is homogeneous, we know that \(N \neq N + N\). Therefore, since \(N + N\) is invariant under \(\mathcal{A}\) (by (ii) we know that \(N + N = X\) by (iv)).

To see that there exist strictly cyclic operator algebras for which \(N\) is linear and those for which \(N\) is nonlinear let us reconsider Example 1 of [2].

**Example.** Let \(X\) be a Banach space, \(\dim X \geq 2\) and let \(x_0 \in X\), \(x_0 \neq 0\). Let each of \(\hat{x}\) and \(\hat{y}\) be a continuous linear functional on \(X\) such that \(\hat{x}(x_0) = \hat{y}(x_0) = 1\). For each \(x\) in \(X\) define \(A_x\) by

\[ A_x y = \hat{x}(x)[y - \hat{y}(y)x_0] + \hat{y}(y)x \]
and let $\mathcal{A} = \{A_x : x \in X\}$.

It was observed in [2] that $\mathcal{A}$ is a strictly cyclic operator algebra with strictly cyclic, separating vector $x_0$.

A simple argument shows that a vector $y_0$ of $X$ is cyclic (and hence by Theorem 5 strictly cyclic) if and only if $y^*(y_0) \neq 0$ and $x^*(y_0) \neq 0$. Thus the set $N$ of noncyclic vectors coincides with $\ker y^* \cup \ker x^*$. Consequently, $N$ is linear if $x^*$ and $y^*$ are dependent and nonlinear otherwise.

**References**


Received July 14, 1972 and in revised form August 25, 1972.

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Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan
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