

# Pacific Journal of Mathematics

## **MAXIMAL INVARIANT SUBSPACES OF STRICTLY CYCLIC OPERATOR ALGEBRAS**

MARY RODRIGUEZ EMBRY

# MAXIMAL INVARIANT SUBSPACES OF STRICTLY CYCLIC OPERATOR ALGEBRAS

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A *strictly cyclic operator algebra*  $\mathcal{A}$  on a complex Banach space  $X$  ( $\dim X \geq 2$ ) is a uniformly closed subalgebra of  $\mathcal{L}(X)$  such that  $\mathcal{A}x = X$  for some  $x$  in  $X$ . In this paper it is shown that (i) if  $\mathcal{A}$  is strictly cyclic and intransitive, then  $\mathcal{A}$  has a maximal (proper, closed) invariant subspace and (ii) if  $A \in \mathcal{L}(X)$ ,  $A \neq zI$  and  $\{A\}'$  (the commutant of  $A$ ) is strictly cyclic, then  $A$  has a maximal hyperinvariant subspace.

1. Notation and terminology. Throughout the paper  $X$  is a complex Banach space of dimension greater than one and  $\mathcal{L}(X)$  is the algebra of continuous linear operators on  $X$ .  $\mathcal{A}$  will denote a uniformly closed subalgebra of  $\mathcal{L}(X)$  which is *strictly cyclic* and  $x_0$  will be a *strictly cyclic vector* for  $\mathcal{A}$ : that is,  $\mathcal{A}x_0 = X$ . We do not insist that the identity element  $I$  of  $\mathcal{L}(X)$  be an element of  $\mathcal{A}$ .

If  $\mathcal{B} \subset \mathcal{L}(X)$ , then the *commutant* of  $\mathcal{B}$  is  $\mathcal{B}' = \{E: E \in \mathcal{L}(X) \text{ and } EB = BE \text{ for all } B \text{ in } \mathcal{B}\}$ . We shall use the terminology of "invariant" and "transitive" as follows: if  $M \subset X$  and  $\mathcal{B} \subset \mathcal{L}(X)$ , then (i)  $M$  is *invariant* under  $\mathcal{B}$  if  $\mathcal{B}M = \{Bm: B \in \mathcal{B} \text{ and } m \in M\} \subset M$ , (ii)  $M$  is an *invariant subspace* for  $\mathcal{B}$  if  $M$  is invariant under  $\mathcal{B}$  and  $M$  is a closed, nontrivial ( $\neq \{0\}, X$ ) linear subspace of  $X$ , (iii)  $\mathcal{B}$  is *transitive* if  $\mathcal{B}$  has no invariant subspace and *intransitive* if  $\mathcal{B}$  has an invariant subspace. Further, if  $A \in \mathcal{L}(X)$  and  $\{A\}'$  is intransitive, then each invariant subspace of  $\{A\}'$  is called a *hyperinvariant subspace* of  $A$ . Finally an invariant subspace of  $\mathcal{B}$  is *maximal* if it is not properly contained in another invariant subspace of  $\mathcal{B}$ .

2. Introduction. Strictly cyclic operator algebras have been studied by A. Lambert, D. A. Herrero, and the author of this paper. (See for example [2]–[6].) One of the major results in [2, Theorem 3.8], [3, Theorem 2], and [6, Theorem 4.5] is that a transitive subalgebra of  $\mathcal{L}(X)$  containing a strictly cyclic algebra is necessarily strongly dense in  $\mathcal{L}(X)$ . In each of three developments the following is a key lemma: The only dense linear manifold invariant under a strictly cyclic subalgebra of  $\mathcal{L}(X)$  is  $X$ . In Lemma 1 we shall present a generalization of this lemma which will be useful in the study of maximal invariant subspaces and noncyclic vectors of a strictly cyclic algebra  $\mathcal{A}$ .

LEMMA 1. *If  $M$  is invariant under  $\mathcal{A}$  and  $x_0 \in \bar{M}$ , then  $M = X$ .*

(It should be noted that we do not require  $M$  to be linear nor do we require, as was done in Lemma 3.4 of [2], that  $I \in \mathcal{A}$ . The proof given here is a slight modification of that given in [2].)

*Proof.* We shall show that  $\mathcal{A}x_0 \subset M$  and thus  $X = \mathcal{A}x_0 \subset M$ . Let  $\{x_n\}$  be a sequence in  $M$  such that  $\lim_{n \rightarrow \infty} x_n = x_0$ . By [2, Lemma 3.1 (ii)] there exists a sequence  $\{A_n\}$  in  $\mathcal{A}$  such that  $A_n x_0 = x_0 - x_n$  and  $\lim_{n \rightarrow \infty} \|A_n\| = 0$ . Thus for  $n$  sufficiently large,  $\|A_n\| < 1$  and  $(I - A_n)^{-1} = \sum_{k=0}^{\infty} (A_n)^k$ . Consequently,  $\mathcal{A}(I - A_n)^{-1} \subset \mathcal{A}$  and since  $x_0 = (I - A_n)^{-1}x_n$ , we have  $\mathcal{A}x_0 = \mathcal{A}(I - A_n)^{-1}x_n \subset \mathcal{A}x_n \subset M$ , as desired.

For the sake of future reference we restate and reprove the transitivity theorem.

**THEOREM 1.** *If  $\mathcal{A}$  is a strictly cyclic transitive subalgebra of  $\mathcal{L}(X)$ , then  $\mathcal{A}$  is strongly dense in  $\mathcal{L}(X)$ .*

*Proof.* Using Lemma 1 we can show (as in [2, Lemma 3.5]) that each densely defined linear transformation commuting with  $\mathcal{A}$  is everywhere defined and continuous. Further, again using Lemma 1, we can show that if  $E \in \mathcal{A}$  and  $z \in \sigma(E)$ , then either  $zI - E$  is not one-to-one or does not have dense range. Thus if  $\mathcal{A}$  is transitive, necessarily  $E = zI$ . Consequently, it follows from [1, p. 636 and Cor. 2.5, p. 641] that  $\mathcal{A}$  is strongly dense in  $\mathcal{L}(X)$ .

3. Maximal invariant subspaces. In [2, Theorem 3.1] it is shown that every strictly cyclic, separated operator algebra  $\mathcal{A}$  has a maximal invariant subspace. ( $\mathcal{A}$  is separated by  $x_0$  if  $A = 0$  whenever  $A \in \mathcal{A}$  and  $Ax_0 = 0$ .) Theorem 2 allows us to obtain the same result without the hypothesis that  $\mathcal{A}$  be separated, provided  $\mathcal{A}$  is intransitive.

**THEOREM 2.** *An intransitive, strictly cyclic subalgebra  $\mathcal{A}$  of  $\mathcal{L}(X)$  has a maximal invariant subspace.*

*Proof.* Let  $\mathcal{M} = \{M : M \text{ is an invariant subspace of } \mathcal{A}\}$ . By hypothesis  $\mathcal{M} \neq \emptyset$ . We shall order  $\mathcal{M}$  by set inclusion and show that each linearly ordered subset of  $\mathcal{M}$  has an upper bound in  $\mathcal{M}$ . To this end we let  $\{M_\alpha\}$  be a linearly ordered subset of  $\mathcal{M}$ . Then  $\bigcup_\alpha M_\alpha$  is invariant under  $\mathcal{A}$ . By Lemma 1, if  $\overline{\bigcup_\alpha M_\alpha} = X$ , then  $\bigcup_\alpha M_\alpha = X$  and consequently  $x_0 \in M_\alpha$  for some value of  $\alpha$ . Since this last implies that  $X = \mathcal{A}x_0 \subset \mathcal{A}M_\alpha \subset M_\alpha$  and contradicts the fact that  $M_\alpha$  is a proper closed linear subspace of  $X$ , we see that  $\bigcup_\alpha M_\alpha$  is not

dense in  $X$ . Thus  $\overline{\bigcup_{\alpha} M_{\alpha}}$  is an element of  $\mathcal{M}$  and is an upper bound for  $\{M_{\alpha}\}$ . By the Maximality Principle  $\mathcal{M}$  has a maximal element.

Lemma 1 and the Maximality Principle can be combined to arrive at other similar results. For example, (i) if  $\mathcal{A}$  is intransitive and strictly cyclic, then  $\mathcal{A}$  has a proper maximal invariant subset (this will be discussed further in §4) and (ii) if  $X$  is a Hilbert space and  $\mathcal{A}$  has a reducing subspace (that is, an invariant subspace of  $\mathcal{A}$  which is also invariant under  $\mathcal{A}^* = \{A^*: A \in \mathcal{A}\}$ ), then  $\mathcal{A}$  has a maximal reducing subspace.

In [2, Theorem 3.7] it is shown that if  $A$  is not a scalar multiple of  $I$  and  $\{A\}'$  is strictly cyclic, then  $A$  has a hyperinvariant subspace. This result combined with Theorem 2 yields the following:

**COROLLARY 1.** *If  $A$  is not a scalar multiple of  $I$  and  $\{A\}'$  is strictly cyclic, then  $A$  has a maximal hyperinvariant subspace.*

We shall now turn our attention to intransitive, strictly cyclic operator algebras on a Hilbert space  $X$ . If  $M$  is a closed linear subspace of  $X$ ,  $P_M$  will denote the orthogonal projection of  $X$  onto  $M$  and  $M^{\perp}$  the orthogonal complement of  $M$ :  $M^{\perp} = \{y: \langle y, m \rangle = 0 \text{ for all } m \text{ in } M\}$ . Furthermore,  $\mathcal{A}^* = \{A^*: A \in \mathcal{A}\}$ .

In the Hilbert space situation we are able to conclude that  $\mathcal{A}^*/M$  is strongly dense in  $\mathcal{L}(M^{\perp})$  when  $M$  is a maximal invariant subspace for  $\mathcal{A}$ . This remains an open question if  $X$  is an arbitrary Banach space and is a particularly interesting one if  $X$  is reflexive. For in that case if  $M$  is a maximal invariant subspace of  $\mathcal{A}$ , then  $M^{\perp} = \{x^*: x^*(M) = 0\}$  is a minimal invariant subspace of  $\mathcal{A}^*$ .

**THEOREM 3.** *Let  $\mathcal{A}$  be a strictly cyclic operator algebra on a Hilbert space  $X$ . If  $M$  is a maximal invariant subspace of  $\mathcal{A}$ , then*

(i)  $(I - P_M)\mathcal{A}(I - P_M)x_0 = M^{\perp}$  and (ii)  $\mathcal{A}^*(I - P_M)$  is strongly dense in  $\mathcal{L}(M^{\perp})$ .

*Proof.* Note first that  $(I - P_M)\mathcal{A}(I - P_M) = (I - P_M)\mathcal{A}$ , so that (i) is immediate. Since  $M$  is a maximal invariant subspace for  $\mathcal{A}$ ,  $M^{\perp}$  is a minimal invariant subspace for  $\mathcal{A}^*$ . Thus each of  $\mathcal{A}^*(I - P_M)$  and  $(I - P_M)\mathcal{A}(I - P_M)$  is transitive on  $M^{\perp}$ . Thus the uniform closure of  $(I - P_M)\mathcal{A}(I - P_M)$  in  $\mathcal{L}(M^{\perp})$  is transitive and by (i) is strictly cyclic; hence by Theorem 1  $(I - P_M)\mathcal{A}(I - P_M)$  is strongly dense in  $\mathcal{L}(M^{\perp})$ , which concludes our proof of (ii).

**THEOREM 4.** *Let  $X$  be a Hilbert space,  $A \in \mathcal{L}(X)$  and  $\{A\}'$  strictly cyclic. If  $M$  is a maximal invariant subspace for  $\{A\}'$ , then there exists a multiplicative linear functional  $f$  on  $\{A\}''$  such*

that for each  $E$  in  $\{A\}''$ ,  $(E - f(E)I)(X) \subset M$ .

*Proof.* As we noted in the proof of Theorem 3,

$$\mathcal{B} = (I - P_M)\{A\}'(I - P_M)$$

is strongly dense in  $\mathcal{L}(M^\perp)$  and thus its commutant consists of the scalar multiples of the identity operator on  $M^\perp$ . Since  $\{A\}'' \subset \{A\}'$  and  $M$  is invariant under  $\{A\}'$ , we know that  $(I - P_M)\{A\}''(I - P_M)$  is contained in the commutant of  $\mathcal{B}$  on  $M^\perp$  and hence  $(I - P_M)\{A\}''(I - P_M) \subset \{z(I - P_M)\}$ . Thus for  $E$  in  $\{A\}''$ , there exists a complex number  $z$  such that  $(I - P_M)E(I - P_M) = z(I - P_M)$ . Therefore,  $(I - P_M)(E - zI) = 0$  since  $M$  is invariant under  $\{A\}''$ ; or equivalently  $(E - zI)(X) \subset M$ . Since  $M$  is a proper subset of  $X$ , it is now obvious that the number  $z$  for which  $(E - zI)(X) \subset M$  is unique. Define  $f(E) = z$ .

That  $f$  is linear follows immediately from the fact that  $f(E)$  is the unique number for which  $(E - f(E)I)(X) \subset M$ . Furthermore, since  $M$  is invariant under  $\{A\}''$ ,  $(FE - f(E)F)(X) \subset M$  for all  $E, F \in \{A\}''$ . Consequently (by uniqueness again),  $0 = f(FE - f(E)F) = f(FE) - f(E)f(F)$  and thus we see that  $f$  is multiplicative.

**COROLLARY 2.** *Let  $A \in \mathcal{L}(X)$  where  $X$  is a Hilbert space. If the range of  $A - zI$  is dense in  $X$  for each complex  $z$ , then  $\{A\}'$  is not strictly cyclic.*

*Proof.* Except for one minor technicality, Corollary 2 follows immediately from Theorem 4. For, if  $\{A\}'$  is strictly cyclic and intransitive, by Theorem 4 there exists a complex number  $f(A)$  such that the range of  $A - f(A)I$  is contained in a proper subspace of  $X$ . By Corollary 1 the only other way in which  $\{A\}'$  can be strictly cyclic is when  $A = zI$  for some complex  $z$ , in which case the range of  $A - zI$  is certainly not dense in  $X$ .

In [2, Lemma 3.6] and [3, Proposition 2], it is shown that if  $E \in \mathcal{N}'$  where  $\mathcal{N}$  is strictly cyclic and  $z \in \sigma(E)$ , then either  $zI - E$  is not one-to-one or  $zI - E$  does not have dense range. Corollary 2 now adds to our knowledge of  $\sigma(A)$  where  $\{A\}'$  is strictly cyclic: in this case we know that for at least one value of  $z$ , the range of  $A - zI$  is nondense. Indeed we have the stronger result:

**COROLLARY 3.** *Let  $A \in \mathcal{L}(X)$  where  $X$  is a Hilbert space. If  $\{A\}'$  is strictly cyclic, then there exists a common eigenvector for  $\{A^*\}''$ .*

*Proof.* The case in which  $\{A\}' = \mathcal{L}(X)$  is trivial. Thus we assume  $A \neq zI$ . By Theorem 4 if  $E \in \{A\}''$ , there exists a complex number  $f(E)$  such that  $(E - f(E)I)(X) \subset M$  where  $M$  is a maximal

invariant subspace of  $\{A\}'$ . Therefore,  $E^*(I - P_M)x_0 = f(E)^*(I - P_M)x_0$  and  $(I - P_M)x_0 \neq 0$  since  $x_0$  is cyclic for  $\{A\}'$  and  $M$  is a proper invariant subspace for  $\{A\}'$ .

4. **Noncyclic vectors of  $\mathcal{A}$ .** In this last section of this paper we shall discuss briefly several properties of the set of noncyclic vectors of a strictly cyclic operator algebra  $\mathcal{A}$ . A vector  $x$  is noncyclic for  $\mathcal{A}$  if  $\mathcal{A}x$  is not dense in  $X$ . These results are summarized in Theorem 5. Parts (i) and (iii) of Theorem 5 also are found in [5, Theorem 2].

**THEOREM 5.** *Let  $N$  be the set of noncyclic vectors of a strictly cyclic operator algebra  $\mathcal{A}$ ,*

- (i) *if  $x \notin N$ , then  $x$  is a strictly cyclic vector for  $\mathcal{A}$ ,*
- (ii)  *$N$  is invariant under  $\mathcal{A}$ ,*
- (iii)  *$N$  is closed in  $X$ ,*
- (iv)  *$N$  is the unique proper maximal invariant subset of  $\mathcal{A}$ ,*
- (v) *if  $N$  is not linear, then  $N + N = X$ , where  $N + N = \{x + y: x, y \in N\}$ .*

*Proof.* (i) If  $x \notin N$ , then  $\overline{\mathcal{A}x} = X$  and thus by Lemma 1 since  $\mathcal{A}x$  is invariant under  $\mathcal{A}$ , we have  $\mathcal{A}x = X$  and  $x$  is strictly cyclic. (ii) Assume that  $x \in N$  and  $A \in \mathcal{A}$ . Then  $\mathcal{A}Ax \subset \mathcal{A}x$  and consequently  $\mathcal{A}Ax \neq X$ . That is,  $Ax \in N$  for each  $A$  in  $\mathcal{A}$  which proves (ii). (iii) By (ii)  $\mathcal{A}N \subset N$ . Since  $\mathcal{A}$  has a strictly cyclic vector, we know by Lemma 1 that  $\bar{N}$  contains no strictly cyclic vector for  $\mathcal{A}$ . Thus by (i)  $\bar{N}$  contains only noncyclic vectors for  $\mathcal{A}$ , which says that  $N$  is closed. (iv) By (ii)  $N$  is invariant under  $\mathcal{A}$ . By hypothesis  $\mathcal{A}$  has a strictly cyclic vector so that  $N \neq X$ . These two observations essentially prove (iv) since an element  $x$  of a proper invariant subset of  $\mathcal{A}$  is necessarily an element of  $N$ . (v) If  $N$  is nonlinear, then since  $N$  is homogeneous, we know that  $N \neq N + N$ . Therefore, since  $N + N$  is invariant under  $\mathcal{A}$  (by (ii) we know that  $N + N = X$  by (iv)).

To see that there exist strictly cyclic operator algebras for which  $N$  is linear and those for which  $N$  is nonlinear let us reconsider Example 1 of [2].

**EXAMPLE.** Let  $X$  be a Banach space,  $\dim X \geq 2$  and let  $x_0 \in X$ ,  $x_0 \neq 0$ . Let each of  $x^*$  and  $y^*$  be a continuous linear functional on  $X$  such that  $x^*(x_0) = y^*(x_0) = 1$ . For each  $x$  in  $X$  define  $A_x$  by

$$A_x y = x^*(x)[y - y^*(y)x_0] + y^*(y)x$$

and let  $\mathcal{A} = \{A_x: x \in X\}$ .

It was observed in [2] that  $\mathcal{A}$  is a strictly cyclic operator algebra with strictly cyclic, separating vector  $x_0$ .

A simple argument shows that a vector  $y_0$  of  $X$  is cyclic (and hence by Theorem 5 strictly cyclic) if and only if  $y^*(y_0) \neq 0$  and  $x^*(y_0) \neq 0$ . Thus the set  $N$  of noncyclic vectors coincides with  $\ker y^* \cup \ker x^*$ . Consequently,  $N$  is linear if  $x^*$  and  $y^*$  are dependent and nonlinear otherwise.

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Received July 14, 1972 and in revised form August 25, 1972.

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Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

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