Pacific Journal of Mathematics

ON DOMINANT AND CODOMINANT DIMENSION OF QF − 3 RINGS

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Vol. 49, No. 1 May 1973

PACIFIC JOURNAL OF MATHEMATICS
Vol. 49, No. 1, 1973

ON DOMINANT AND CODOMINANT DIMENSION OF $QF-3$ RINGS

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In this paper the concept of codominant dimension is defined and studied for modules over a ring. When the ring R is artinian, a left R module M has codominant dimension at least n in case there exists a projective resolution

$$P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow M \longrightarrow 0$$

with P_i injective. It is proved that every left R-module has the above property if and only if R has dominant dimension at least n. The concept of codominant dimension is also used to study semi-perfect $QF-3$ rings.

Let R be an associative ring with an identity 1. Denote by ${}_{\mathfrak{R}}R$ (resp. $R_{\mathfrak{R}}$) the left (resp. right) R-module R. Using the terminology of [5], we have the following definitions:

(1) R is left $QF-3$, if R has a faithful projective injective left ideal.

(2) R is left $QF-3^+$ if the injective hull $E({}_{\mathfrak{R}}R)$ is projective.

(3) R is left $QF-3'$ if $E({}_{\mathfrak{R}}R)$ is torsionless, i.e., there exists a set A such that $E(R) \leqq \prod_A R$.

In general (1) $\Rightarrow$ (3). For perfect rings the three conditions are equivalent for left and right $QF-3$ rings. (See [5].)

The dominant dimension of a left (resp. right) R-module M, denoted by dom. dim $({}_{\mathfrak{R}}M)$ (resp. dom. dim $(M_{\mathfrak{R}})$) is at least n, if there exists an exact sequence

$$0 \longrightarrow M \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_n$$

of left (resp. right) R-module where each X_i is torsionless and injective for $i = 1, \cdots, n$. See [3] for details.

Note that this says when dom. dim $({}_{\mathfrak{R}}R) \geqq 1$ and R is left-artinian that $E(Re_i)$ for $i = 1, \cdots, n$ is projective where $\{e_i\}$, $i = 1, \cdots, n$ is a complete set of orthogonal idempotents, and that each X_i is projective.

We define codominant dimension as follows:

Let M be a left R-module. The codom. dim of M is at least n in case there exists an exact sequence

$$P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow M \longrightarrow 0$$

where P_i is torsionless and injective for $i = 1, \cdots, n$.

Following the notation of [3], we say that if such an exact sequence exists for $1 \leqq i \leqq n$, but no such sequence exists for $1 \leqq i \leqq n+1$, then codom. dim $({}_{\mathfrak{R}}M) = n$. If such a sequence exists for all n then codom. dim $({}_{\mathfrak{R}}M) = \infty$. If no such sequence exists codom. dim $({}_{\mathfrak{R}}M) = 0$.

An R-module U is defined to be a cogenerator if for any module M we can embed it in a product of copies of U. We have:

LEMMA. *Let U, V be left injective cogenerators then the* codom. dim $(U) =$ codom. dim (V).

The proof follows easily from properties of injective cogenerators and shall omit it.

Let U be a left injective cogenerator. If the codom. dim $(U) = n$, we say that R has l. codom. dim $({}_{\mathfrak{R}}R) = n$. In a similar manner one defines r. codom. dim $(R_{\mathfrak{R}})$. Note that if ${}_{\mathfrak{R}}R$ is artinian, products of projectives are projective and direct sums of injectives are injective. Hence l. codom. dim $({}_{\mathfrak{R}}R) = n$ is equivalent to the existence of a resolution

$$P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow U \longrightarrow 0$$

where P_i is projective and injective and $U = E(S_1) \oplus \cdots \oplus E(S_n)$ where S_i: $i = 1, \cdots, n$ is a copy of each simple left R-module.

In § 1 we characterize semi-perfect $QF-3^+$ rings in terms of their finitely generated projective, injectives.

In § 2 we show that l. dom. dim $({}_{\mathfrak{R}}R)$ and l. codom. dim $({}_{\mathfrak{R}}R)$ are the same for artinian rings. Hence, if R is artinian $QF-3$ then the l.-dom. dim (r-dom. dim) l. codom. dim (r-codom. dim) are the same.

For notation we use J to donote the Jacobson radical, and $R^{(A)}(R^A)$ denotes a direct sum (resp. direct product) of A-copies of R. Also $E(M)$ will be used to denote the injective hull of an R-module M and $P(M)$ will denote the projective cover of M when M has a projective cover. For a left R-module M, we let $\ell_{\mathfrak{R}}(M) = \{x \in R \mid x \cdot M = 0\}$, and $r_{\mathfrak{M}}(I) = \{x \in M \mid I \cdot x = 0\}$ where $I \subseteqq R$. We will use $T(M)$ to denote $M/J(M)$ where $J(M)$ is the Jacobson radical of M.

1. $QF-3$ **Rings.** Recall that if ${}_{\mathfrak{R}}R$ is noetherian $rt \cdot QF-3 \Leftrightarrow rt \cdot QF-3^+$. (See [1] and [6].)

To begin with we shall prove that under those hypotheses

$$rt \cdot QF-3^+ \Longleftrightarrow rt \cdot QF-3' .$$

PROPOSITION 1.1. *Let ${}_{\mathfrak{R}}R$ be noetherian. If $E(R_{\mathfrak{R}})$ is torsionless then $E(R_{\mathfrak{R}})$ is projective.*

Proof. Given that $0 \to E \xrightarrow{\theta} R^A$ is monic, where A is an indexing set. We show that there exists a finite number of R_α's, $\alpha \in A$ say $R_{\alpha_i}, \cdots, R_{\alpha_m}$ such that $\pi\theta|_R = \tilde{\theta}$ where π is the projection $R^A \to \bigoplus \sum_{i=1}^m R_{\alpha_i}$ is monic. Let S be the set of all finite intersections of right ideals $\{K_\alpha\}_{\alpha \in A}$ where $K_\alpha = \ker(\pi_\alpha \circ \theta|_R)$. Note that $\bigcap_{i=1}^n K_{\alpha_i}$ induces a natural embedding of

$$0 \longrightarrow R\Big/\bigcap_{i=1}^{n} K_{\alpha_i} \longrightarrow R^{(n)} .$$

Thus $R/\bigcap_{i=1}^n K_{\alpha_i}$ is torsionless. Hence by [2, Thm. I, p. 350]

$$\bigcap_{i=1}^{n} K_{\alpha_i} = r_{\mathfrak{R}} l_{\mathfrak{R}}\Big(\bigcap_{i=1}^{n} K_{\alpha_i}\Big) .$$

Now since ${}_{\mathfrak{R}}R$ noetherian, the set $\{l_{\mathfrak{R}}(\bigcap_{i=1}^n K_{\alpha_i})\}$ has a maximal element $l_{\mathfrak{R}}(\bigcap_{i=1}^m K_{\alpha_i})$ where $\bigcap_{i=1}^{n} K_{\alpha_i} \in S$. Thus $r_{\mathfrak{R}} l_{\mathfrak{R}}(\bigcap_{i=1}^m K_{\alpha_i}) = \bigcap_{i=1}^m K_{\alpha_i}$ is a minimal right ideal in S. But then $x \in \bigcap_{i=1}^m K_{\alpha_i} \Rightarrow x \in \bigcap_{\alpha \in A} K_\alpha$. Thus $\bigcap_{i=1}^m K_{\alpha_i} = 0$. This implies that $\tilde{\theta}$ is monic. But then $\pi\theta$ is monic since $\ker(\pi\theta) \cap R \neq 0$ if $\ker(\pi\theta) \neq 0$. This shows E is projective.

We next show that $QF-3^+ \Rightarrow QF-3$ for semi-perfect rings.

First we need the following lemma.

LEMMA 1.2. *Let K be finitely generated. Suppose there exists an exact sequence*

$$0 \longrightarrow K \longrightarrow E_1 \longrightarrow \cdots \longrightarrow E_n$$

where $E(K) = E_1$, $E_{i+1} = E(E_i)$ for $1 \leqq i \leqq n-1$ and each E_i is projective. Then $E_1, \cdots, E_n$ are all finitely generated.

Proof. This follows easily from the proof of [4, Lemma 1].

PROPOSITION 1.3. *Suppose R is semi-perfect. If R is left $QF-3^+$ then R is left $QF-3$.*

Proof. By Lemma 1.2 $E(R)$ is finitely generated. Since R is semi-perfect $E(R) \cong \bigoplus \sum_{i=1}^n Re_i$, where each e_i is an indecomposable idempotent.

Let $Re_1, \cdots, Re_k$ be a subset of $Re_1, \cdots, Re_n$, where the set $\{Re_1, \cdots, Re_k\}$ is a complete set of isomorphism classes of $\{Re_1, \cdots, Re_n\}$. Then $U = Re_1 \oplus \cdots \oplus Re_k$ is a minimal projective injective.

Now we come to the main theorem of this section.

THEOREM 1.4. *Let R be semi-perfect. The following are equivalent*:

(a) *R is left* $QF-3^+$.

(b) $E({}_{\mathfrak{R}}R)$ *is finitely generated and every finitely generated left injective has an injective projective cover.*

(c) *Every finitely generated left projective has a projective injective hull.*

Proof. (b) $\Rightarrow$ (a): Consider

$$P(E(R)) \longrightarrow E(R) \longrightarrow 0 .$$

Embed $R \xrightarrow{i_{\mathfrak{R}}} E(R)$ then by the projectivity of R there exists a map $\theta': R \rightarrow P(E(R))$ such that θ' is monic.

Consider the following diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & R & \xrightarrow{i_{\mathfrak{R}}} & E(R) \\ & & {\scriptstyle\theta'}\downarrow & \swarrow {\scriptstyle\theta''} & \\ & & P(E(R)) . & & \end{array}$$

Here $\theta''(r) = \theta'(r)$ for all $r \in R$. Also θ'' is monic. The injectivity of $E(R)$ forces $E(R)$ to be a direct summand of $P(E(R))$, hence projective.

(a) $\Leftrightarrow$ (c): Consider $R^{(n)}, R^{(n)} \leqq E(R)^{(n)}$. Thus $E(P) \leqq E(R)^n$, where $P \oplus P' = R^{(n)}$, as a direct summand. Hence $E(P)$ is projective. The converse is trivial.

(a) $\Rightarrow$ (b): By Lemma 1.2 $E(R)$ is finitely generated.

Consider $P(E) \xrightarrow{\theta} E \rightarrow 0$ where $P(E)$ is finitely generated injective. Let $R^{(n)} \xrightarrow{\rho} E \rightarrow 0$. Combining the above maps we have the following diagrams:

$$\begin{array}{ccccc} 0 & \longrightarrow & R^{(n)} & \xrightarrow{i_{\mathfrak{R}}^{(n)}} & E(R)^{(n)} \\ & & {\scriptstyle\rho}\downarrow & \swarrow {\scriptstyle\rho'} & \\ & & E . & & \end{array}$$

So we have ρ' epic and $\rho' \circ i_{\mathfrak{R}}^{(n)} = \rho$. Further we have

$$\begin{array}{ccccc} & & E(R)^{(n)} & & \\ & {\scriptstyle\rho''}\swarrow & \downarrow{\scriptstyle\rho'} & & \\ P(E) & \xrightarrow{\theta} & E & \longrightarrow & 0 \end{array}$$

Noting that ρ'' is epic and $P(E)$ is projective, $P(E)$ is a direct summand of $E(R)^{(n)}$. Hence injective.

A ring is perfect in case every module has a projective cover. We show that $QF-3^+$ rings can be characterized in terms of the projective cover of $E({}_{\mathfrak{R}}R)$.

THEOREM 1.5. *Let R be perfect. Then every indecomposable summand of $P(E({}_{\mathfrak{R}}R))$ is injective if and only if R is left $QF-3^{+}$.*

Proof. $\Rightarrow$ Consider the following diagram:

$$\begin{array}{ccccc} & & {}_{\mathfrak{R}}R & & \\ & \swarrow^{f} & \downarrow i & & \\ P(E({}_{\mathfrak{R}}R)) & \xrightarrow{\pi} & E({}_{\mathfrak{R}}R) & \longrightarrow & 0 . \end{array}$$

Here i is a monomorphism and π is epic. Since R is projective there exists on f such that $\pi f = i$. Clearly f is monic. Since R is perfect $P(E({}_{\mathfrak{R}}R)) \cong \sum_{\alpha \in A} Re_{\alpha}$, where e_{α} are primitive idempotents of R. Now $\operatorname{Im}(f)$ is contained in $\sum_{\alpha=1}^{n} Re_{\alpha}$, for n a positive integer, since ${}_{\mathfrak{R}}R$ is cyclic.Thus using the hypothesis, $E({}_{\mathfrak{R}}R)$ is projective and R is left $QF-3^{+}$. $\Leftarrow$ This is trivial.

2. Codominant dimension of rings. We begin with a lemma which holds the key to the main results of this section.

LEMMA 2.1. *Let R be a ring. The following conditions are equivalent.*

(1) *For every projective left R-module P, there exists an exact sequence*

$$0 \longrightarrow P \longrightarrow E_1 \longrightarrow \cdots \longrightarrow E_n$$

where E_i, $1 \leqq i \leqq n$, are injective and projective.

(2) *For every injective left R-module Q, there exists an exact sequence*

$$P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow Q \longrightarrow 0$$

where P_i, $1 \leqq i \leqq n$, are injective and projective.

Proof. (1) $\Rightarrow$ (2). For $n=1$ a modification for the proof of Theorem 1.4 will suffice. We assume the lemma is true for the nth case and prove the $n+1$ case. So consider the following exact sequences.

$$(1) \qquad 0 \longrightarrow P_{n+1} \xrightarrow{J_1} E_1 \xrightarrow{J_2} E_2 \longrightarrow \cdots \xrightarrow{J_{n+1}} E_{n+1}$$

$$(2) \qquad P_{n+1} \xrightarrow{\theta_1} P_n \xrightarrow{i_n} \cdots \longrightarrow P_1 \xrightarrow{i_1} Q \longrightarrow 0 .$$

Here Q is an arbitrary injective module and

$$P_1, \cdots, P_n, E_1, \cdots, E_{n+1}$$

are both projective and injective and P_{n+1} is projective.

Also E_k is the injective hull of Cok (J_k).

Denote by K the image of θ_1. Using the injectivity of P_n, there is a map $\theta_2\colon E_1 \to P_n$ such $\theta_2 J_1 = i_{n+1}\theta_1$ where i_{n+1} is the embedding of K into P_n. The injectivity of P_{n-1} and the exact sequence $0 \to E_1/P_{n+1} \to E_2$ induce a map $\theta_3\colon E_2 \to P_{n-1}$ which one can easily check has the property $\theta_3 J_2 = i_n \theta_2$.

In like manner we can define $\theta_k\colon E_{k-1} \to P_{n+2-k}$ such that

$$\theta_k J_{k-1} = i_{n+3-k}\theta_{k-1}\ , \quad k = 2, \cdots, n+2\ .$$

This information is summed up in the following diagram:

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & P_{n+1} & \longrightarrow & E_1 & \xrightarrow{J_2} & E_2 & \cdots & E_{n+1} & \xrightarrow{J_{n+1}} & E_{n+1} & & \\
 & & \Big\downarrow{\scriptstyle\theta_1} & {\scriptstyle h_{n+1}}\swarrow & \Big\downarrow{\scriptstyle\theta_2} & {\scriptstyle h_n}\swarrow & \Big\downarrow{\scriptstyle\theta_3} & & \Big\downarrow{\scriptstyle\theta_{n+1}} & {\scriptstyle h_1}\swarrow & \Big\downarrow{\scriptstyle\theta_{n+2}} & & \\
0 & \longrightarrow & K & \xrightarrow{i_{n+1}} & P_n & \xrightarrow{i_n} & P_{n-1} & \cdots & P_1 & \xrightarrow{i_1} & Q & \longrightarrow & 0\ . \\
 & & \Big\downarrow & & & & & & & & & & \\
 & & 0 & & & & & & & & & &
\end{array}$$

Having constructed θ_{n+2}, the projectivity of E_{n+1} induces a map $h_1\colon E_{n+1} \to P_1$ such $i_1 h_1 = \theta_{n+2}$. Now consider the map $h_1 J_{n+1} - \theta_{n+1}\colon E_n \to P_1$. We have $i_1(h_1 J_{n+1} - \theta_{n+1}) = \theta_{n+2} J_{n+1} - i_1 \theta_{n+1} = 0$. So $\operatorname{Im}(h_1 J_{n+1} - \theta_{n+1}) \leqq \ker(i_1)$.

Now consider the following diagram:

$$\begin{array}{ccccc}
 & & E_n & & \\
 & \swarrow & \Big\downarrow{\scriptstyle h_1 J_{n+1} - \theta_{n+1}} & & \\
P_2 & \longrightarrow & \operatorname{Im}(i_2) & \longrightarrow & 0\ .
\end{array}$$

We can construct h_2 using the projectivity of E_n. By a similar argument we can show that $\operatorname{Im}(h_2 J_n - \theta_n) \leqq \ker(i_2)$. By a recursive argument we can construct $h_k J_{n+2-k} - \theta_{n+2-k}$ for $k = 1, \cdots, n$ in like manner. In particular we have $h_n J_2 - \theta_2\colon E_1 \to P_n$ where $\operatorname{Im}(h_n J_2 - \theta_2) \leqq K$. We need only show equality to complete the proof. Let $k \in K$. Then there exists an $x \in P_{n+1}$ such that $\theta_1(x) = k$. Thus $(h_n J_2 - \theta_2)(J_1(-x)) = \theta_2 J_1(x) = \theta_1(x) = k$. Thus $h_n J_2 - \theta_2$ maps on to K. The proof $(2) \Rightarrow (1)$ is similar. This completes the proof.

Noting that for left artinian rings products of projectives are projective, and direct sums of injectives are injective one can easily show that dom. dim $(R) \geqq n$ implies dom. dim. $(P) \geqq n$ for all projective P.

Likewise letting $I = \bigoplus \sum E_\alpha(S_\alpha)$ be the minimal injective cogenerator of R, we find that codom. dim$(I) \geqq n$ implies codom. dim $(Q) \geqq n$ for all injectives Q. Thus we have:

THEOREM 2.2. *Let R be left artinian then the following are equivalent*:

(1) *The* inf $\{m \in Z \mid$ dom. dim $(P) = m$ *for all* P *projectives*$\} = n$.

(2) *The* inf $\{m \in Z \mid$ dom. dim $(Q) = m$ *for all* Q *injectives*$\} = n$.

(3) l. dom. dim $({}_{\mathfrak{R}}R) = n$.

(4) l. codom. dim $({}_{\mathfrak{R}}R) = n$.

If no such n *exists we say* l. dom. dim $(R) = \infty$

Proof. (3) $\Rightarrow$ (1), (4) $\Rightarrow$ (2) by our previous discussion. (1) $\Rightarrow$ (3): There exists a projective module P such dom. dim $(P) = n$.

Now $P \cong \bigoplus \sum_{\mathscr{A}} Re_\alpha$, $\{e_\alpha\}$ primitive idempotents such that for some e_β dom. dim $(Re_\beta) < n+1$ where $e_\beta \in \{e_\alpha\}$. Since $Re_\beta < R$, $n+1 >$ dom. dim $(R) \geqq n$. This yields the desired result. (2) $\Rightarrow$ (4) is similar. (1) $\Rightarrow$ (2): By Lemma 2.1 inf $\{m \in Z \mid$ codom. dim $(Q) = m\} \geqq n$. If inf of the above set is strictly greater than n, another application of the lemma forces inf $\{m \in Z \mid m =$ dom. dim (P), P projective$\} > n$ which is impossible. (2) $\Rightarrow$ (1) is similar.

Let R be left artinian and both left and right $QF-3$. Then by [4, Thm. 10] l. dom. dim $({}_{\mathfrak{R}}R)$ = r. dom. dim $(R_{\mathfrak{R}})$. Thus in view of 2.2 we have:

PROPOSITION 2.3. *Let* ${}_{\mathfrak{R}}R$ *be artinian and* $QF-3$. *Then* l. domdim $({}_{\mathfrak{R}}R)$ = r. domdin $(R_{\mathfrak{R}})$ = l. codomdin $({}_{\mathfrak{R}}R)$ = r. codomdim $(R_{\mathfrak{R}}) = n$.

Acknowledgement. The author wishes to thank the referee for his proof to Theorem 1.5 which is simpler than the author's original version.

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Received February 8, 1972 and in revised form Junuary 3, 1973.

UNIVERSITY OF WESTERN AUSTRALIA

PACIFIC JOURNAL OF MATHEMATICS

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics

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