

# Pacific Journal of Mathematics

**ROUND AND PFISTER FORMS OVER  $R(t)$**

JOHN SOLLION HSIA AND ROBERT PAUL JOHNSON

## ROUND AND PFISTER FORMS OVER $R(t)$

J. S. HSIA AND ROBERT P. JOHNSON

**An anisotropic quadratic form  $\phi$  is called round if  $\phi \cong a\phi$  whenever  $\phi$  represents  $a \neq 0$ . All round forms over  $R(t)$  are completely determined. Connections with Pfister's strongly multiplicative forms and with the reduced algebraic  $K$ -theory groups  $k_n$  of Milnor are studied.**

The concept of a round form was introduced by Witt (see [5] and [8]) to give new simple proofs of results of Pfister on the structure of the Witt ring over fields. In a previous paper [3] we determined all round forms over a global field. In this paper we completely determine all round forms over  $R(t)$ , the field of rational functions in one variable over the reals.

We now describe our main results.

Let  $\phi$  be an anisotropic form of dimension  $> 1$  over  $R(t)$ . Then  $\phi$  is round if and only if  $\phi \cong (n \times (1, f)) \oplus (1, fg)$  for some  $f, g \in R(t)$  such that  $f$  is a product of distinct linear factors and  $g$  is a product of irreducible quadratic factors. Our proof gives a method of computing  $f$  and  $g$ , which are essentially unique (see 2.5 and 2.6). We study a generalization of a round form, called a group form, over  $R(t)$  and measure how far group forms are from being round (see [3] for group forms over global fields).

In the last section we show that a form of dimension  $2^n (n \geq 2)$  is a Pfister form if and only if it is a round form of determinant one. Such a form can be written uniquely as  $2^{n-1} \times (1, f)$  for some  $f \in R[t]$  which is  $\pm$  a product of distinct monic linear factors. From this and a theorem of Elman and Lam we see that every element of  $k_n R(t)$  can be written uniquely as  $l(-1)^{n-1}l(-f)$  with  $f$  as above.

**1. Preliminaries.** We will consider only quadratic forms (often simply called "forms") over a field  $F$  of characteristic  $\neq 2$ . We write  $\phi \oplus \psi$  for the orthogonal sum and  $\phi \otimes \psi$  for the tensor product of quadratic forms [5, p. 8]. We call  $\phi$  *hyperbolic* if  $\phi \cong m \times (1, -1)$ , i.e.,  $\phi$  is a direct sum of hyperbolic planes.

Define  $\dot{D}\phi = \{a \in \dot{F} \mid \phi \text{ represents } a\}$  and  $G\phi = \{a \in \dot{F} \mid a\phi \cong \phi\}$  where  $\dot{F} = F - \{0\}$ . An anisotropic form  $\phi$  is called *round* if and only if  $\dot{D}\phi = G\phi$  (or equivalently  $\dot{D}\phi \subseteq G\phi$ ); an isotropic form is called round if and only if it is hyperbolic [5, p. 22]. A form  $\phi$  is called a *Pfister form* if  $\phi \cong (1, a_1) \otimes \cdots \otimes (1, a_n) (a_i \in \dot{F})$ .

We will frequently refer to [4] for results on quadratic forms over  $F = R(t)$ . The valuations of  $F$  which are trivial on  $R$  are of

three types: if the prime element is  $t - \alpha (\alpha \in \mathbf{R})$ , the valuation is called *real*; if the prime element is an irreducible quadratic polynomial it is called *complex*; if the prime element is  $t^{-1}$  it is called *infinite*. A *spot* is an equivalence class of valuations [7]. If  $p$  is a real or infinite spot then the completion  $F_p$  of  $F$  at  $p$  is isomorphic to  $R((\pi))$  (a *real series field*) where  $\pi$  is a prime element. If  $p$  is complex,  $F_p \cong C((\pi))$  is called a *complex series field*. See [4] for results on quadratic forms over series fields.

If  $\phi$  is a quadratic form over  $\mathbf{R}(t)$  and if  $\alpha \in \mathbf{R}$ , we define " $\phi$  at  $\alpha$ " to be the quadratic form over  $\mathbf{R}$  obtained by replacing  $t$  by  $\alpha$  in the matrix of  $\phi$ . Thus  $\phi$  at  $\alpha$  is well-defined for almost all  $\alpha \in \mathbf{R}$ . The following result is Proposition 2.1 of [4] and is due to Witt.

1.1. *A nonsingular quadratic form of dimension  $\geq 3$  over  $\mathbf{R}(t)$  is isotropic if and only if for almost all  $\alpha \in \mathbf{R}$ , the form at  $\alpha$  is isotropic over  $\mathbf{R}$ . Thus if  $\phi$  is a quadratic form of dimension  $\geq 2$  over  $\mathbf{R}(t)$  and if  $0 \neq f(t) \in \mathbf{R}(t)$ , then  $\phi$  represents  $f(t) \Leftrightarrow$  for almost all  $\alpha \in \mathbf{R}$ ,  $\phi$  at  $\alpha$  represents  $f(\alpha)$ .*

If we write  $\phi \cong (a_1, \dots, a_n)$  over a field  $F$  then  $\det \phi = a_1 \cdots a_n$  modulo  $\dot{F}^2$ . When  $F = \mathbf{R}(t)$  we assume  $\det \phi$  is written as  $\pm$  a product of distinct monic irreducible polynomials.

The following result generalizes Proposition 2.2 of [4].

1.2. *Let  $\phi, \psi$  be quadratic forms over  $\mathbf{R}(t)$ . If  $\phi \cong \psi$  at  $\alpha$  for almost all  $\alpha \in \mathbf{R}$  and if  $\det \phi, \det \psi$  have the same irreducible quadratic factors, then  $\phi \cong \psi$ .*

*Proof.* Clear for  $\dim \phi = 1$ . We assume this result is true whenever  $\dim \phi < n$  and prove it for  $\dim \phi = n > 1$ . Let  $\phi$  represent  $a \neq 0$ . Then  $\phi \oplus (-a)$  is isotropic so by 1.1,  $\psi \oplus (-a)$  is isotropic. Thus  $\psi$  represents  $a$ . Write  $\phi \cong (a) \oplus \phi_1$  and  $\psi \cong (a) \oplus \psi_1$  and apply the induction hypothesis.

1.3. *Let  $f(t) \in \mathbf{R}[t]$  and  $\alpha \in \mathbf{R}$  with  $f(\alpha) \neq 0$ . Then  $(f(t)) \cong (f(\alpha))$  (one-dimensional quadratic forms) over the completion of  $\mathbf{R}(t)$  at the spot with prime element  $t - \alpha$ .*

*Proof.* Write  $f(t) = a_0 + a_1(t - \alpha) + \cdots + a_n(t - \alpha)^n$  and apply the Local Square Theorem [7, 63: 1a], noting  $f(\alpha) = a_0$ .

2. **Round forms over  $\mathbf{R}(t)$ .** We will need the following result, which determines all round forms over a series field.

2.1. *Let  $\phi$  be an anisotropic quadratic form over a real or complex series field  $F$ .*

(a) If  $F$  is complex, then  $\phi$  is round  $\Leftrightarrow \phi$  represents 1.

(b) Let  $F$  be a real series field. Then  $F$  is pythagorean and formally real. So if  $\dim \phi$  is odd,  $\phi$  is round  $\Leftrightarrow \phi \cong (1, \dots, 1)$ . If  $\dim \phi = 2m$  is even then  $\phi$  is round  $\Leftrightarrow \phi \cong m \times (1, 1)$  or  $m \times (1, \pm \pi)$ .

*Proof.* (a) By [4, 1.2],  $\dim \phi \leq 2$  whenever  $\phi$  is anisotropic over a complex series field. Now apply [5, 2.4].

(b) It follows easily from the Local Square Theorem [7, 63:1a] that  $F$  is pythagorean. Now apply [5, 2.4] and [4, 1.6].

Now let  $F$  be a field of characteristic  $\neq 2$  and let  $\Omega$  be a set of discrete or archimedean spots on  $F$  (see [7] for terminology). We say that  $(F, \Omega)$  satisfies the *Weak Hasse-Minkowski Theorem* if whenever  $\sigma$  and  $\tau$  are quadratic forms over  $F$  with  $\sigma_p \cong \tau_p$  for all  $p \in \Omega$ , then  $\sigma \cong \tau$  ( $\sigma_p$  denotes the form  $\sigma$  viewed over the completion  $F_p$  of  $F$  at  $p$ ).

2.2. Let  $(F, \Omega)$  satisfy the *Weak Hasse-Minkowski Theorem*. Let  $\phi$  be anisotropic over  $F$ . Then  $\phi$  is round  $\Leftrightarrow$  for all  $p \in \Omega$ ,

(1)  $\phi_p$  is round

or (2)  $\phi_p$  is isotropic and  $\phi'_p$  (the anisotropic part of  $\phi_p$ ) is round and universal.

*Proof.* ( $\Rightarrow$ ): Assume  $\phi$  is round. Let  $p \in \Omega$ . We first assume  $\phi_p$  is anisotropic and show  $\phi_p$  is round. Let  $b \in \dot{D}(\phi_p)$ . Approximate  $b$  by  $a \in \dot{D}\phi$ . By the Local Square Theorem, we can obtain  $a \in b\dot{F}_p^2$ . Thus  $\phi \cong a\phi \Rightarrow \phi_p \cong b\phi_p$  so  $\phi_p$  is round.

Now assume  $\phi_p$  is isotropic and not hyperbolic. Write  $\phi_p = \phi'_p \oplus H$  with  $H$  hyperbolic. We will show  $\phi'_p \cong b\phi'_p$  for all  $b \in \dot{F}_p$  and so (2) holds. Now  $\phi_p$  represents  $b$  so we find that  $\phi_p \cong b\phi_p$  by the argument of the preceding paragraph. Thus  $\phi'_p \cong b\phi'_p$ .

( $\Leftarrow$ ): Let  $a \in \dot{D}\phi$ . Applying (1) or (2), we have  $\phi_p \cong a\phi_p$  for all  $p \in \Omega$ . By the Weak Hasse-Minkowski Theorem,  $\phi \cong a\phi$ , so  $\phi$  is round.

EXAMPLES 2.3. The Weak Hasse-Minkowski Theorem holds in the following cases:

(1) Let  $F = K(t)$  where  $K$  is an arbitrary field of characteristic  $\neq 2$  and let  $\Omega$  be the set of all spots on  $F$  that are trivial on  $K$ . Using [6, Theorem 5.3] one can show that  $(F, \Omega)$  satisfies the Weak Hasse-Minkowski Theorem.

(2) Let  $F$  be a global field and let  $\Omega$  be the set of all non-trivial spots on  $F$ . We have the following precise results in this case [3, 2.4]: let  $\phi$  be an anisotropic form over  $F$  and let  $\dim \phi > 2$ . Then  $\phi$  is round if and only if: (1)  $\dim \phi \equiv 0 \pmod{4}$ , (2) at all real

spots (if there are any)  $\phi$  is hyperbolic or positive definite, and (3)  $\det \phi = 1$ . We note that the *Strong Hasse-Minkowski Theorem* holds for  $(F, \Omega)$ , i.e., if a form  $\phi$  is isotropic for all  $p \in \Omega$  then  $\phi$  is isotropic.

(3) Cassels, Ellison, and Pfister (J. Number Theory, 3 (1971), p. 147) have recently shown that the Strong Hasse-Minkowski Theorem fails for  $F = K(t)$  where  $K = \mathbf{R}(x)$  ( $x, t$  independent indeterminants over  $\mathbf{R}$ ) though the weak theorem holds as we have mentioned in (1).

The next two results determine all round forms over  $\mathbf{R}(t)$ .

2.4. *There is no odd-dimensional round form over  $\mathbf{R}(t)$  except the form  $\phi = (1)$ .*

*Proof.* Note that  $\mathbf{R}(t)$  is non-pythogorean since  $t^2 + 1$  is not a square. Now apply [5, 2.4].

**THEOREM 2.5.** *Let  $\phi$  be an anisotropic form of dimension  $2m$  over  $\mathbf{R}(t)$ . Then the following are equivalent:*

(1)  $\phi$  is round.

(2)  $\phi \cong ((m-1) \times (1, f)) \oplus (1, fg)$  for some  $f, g \in \mathbf{R}[t]$  such that  $f$  is a product of distinct linear factors and  $f$  or  $-f$  is monic, and  $g$  is a product of monic irreducible quadratic factors (we allow  $f = 1$  or  $-1$  and allow  $g = 1$ ).

(3) For almost all  $\alpha \in \mathbf{R}$ ,  $\phi$  at  $\alpha$  is hyperbolic or positive definite.

(4)  $\phi_p$  is round for all real or infinite spots  $p$  on  $\mathbf{R}(t)$ .

*Proof.* (1)  $\Rightarrow$  (4) follows from 2.2 since there is no universal anisotropic form over a real series field. We will show (2)  $\Rightarrow$  (4)  $\Rightarrow$  (3)  $\Rightarrow$  (2). (2)  $\Rightarrow$  (4) follows from 2.1 and 1.3.

(4)  $\Rightarrow$  (3): Assume (4). Write  $\phi \cong (f_1(t), \dots, f_{2m}(t))$  with the  $f_i(t) \in \mathbf{R}[t]$ . Let  $\alpha \in \mathbf{R}$  such that  $f_i(\alpha) \neq 0$  for all  $i$ . Let  $p$  be the real spot with prime element  $t - \alpha$ . By 1.3,  $\phi_p \cong (f_1(\alpha), \dots, f_{2m}(\alpha))$ . By 2.1,  $\phi_p \cong m \times (1, 1)$  or  $m \times (1, -1)$ . So by [4, 1.6],  $\phi$  at  $\alpha$  is  $\cong m \times (1, 1)$  or  $m \times (1, -1)$ .

(3)  $\Rightarrow$  (2): Write  $\phi \cong (f_1, \dots, f_{2m})$  with the  $f_i \in \mathbf{R}[t]$ . Let  $S$  be the set of all  $a \in \mathbf{R}$  such that  $f_i(a) = 0$  for some  $i$ . Write  $S = \{a_1, \dots, a_k\}$  with  $a_1 < a_2 < \dots < a_k$ . If  $I$  is any of the intervals  $(-\infty, a_1)$ ,  $(a_1, a_2)$ ,  $\dots$ ,  $(a_k, \infty)$  then  $\phi$  at  $\alpha$  is hyperbolic for all  $\alpha \in I$  or is positive definite for all  $\alpha \in I$ . The idea now is to merge together adjacent intervals if  $\phi$  at  $\alpha$  looks the same in the adjacent intervals. If  $\phi$  at  $\alpha$  is positive definite (respectively, hyperbolic) for almost all  $\alpha \in \mathbf{R}$  then we let  $f = 1$  (respectively,  $-1$ ). Otherwise, there is an ordered subset  $\{b_1 < b_2 < \dots < b_j\}$  of  $S$  such that if  $J$  is any of the intervals  $(-\infty, b_1)$ ,  $(b_1, b_2)$ ,  $\dots$ ,  $(b_j, \infty)$  then  $\phi$  at  $\alpha$  is hyperbolic for almost all

$\alpha \in J$  or is positive definite for almost all  $\alpha \in J$ , and such that whenever  $\phi$  is hyperbolic in one of these intervals then it is positive definite in the adjacent intervals. Now let  $f = (t - b_1) \cdots (t - b_j)$  if  $\phi$  at  $\alpha$  is positive definite for almost all  $\alpha > b_j$ , and let  $f = -(t - b_1) \cdots (t - b_j)$  otherwise. Let  $g$  be the product of all the (monic) irreducible quadratic factors of  $\det \phi$ . Then by 1.2,  $\phi \cong ((m - 1) \times (1, f)) \oplus (1, fg)$ .

REMARK 2.6. (1). Part (2) of the above theorem gives us a *canonical form* for an anisotropic round form of even dimension over  $\mathbf{R}(t)$ , i.e.,  $f$  and  $g$  are uniquely determined. This fact follows easily from 1.2. The proof of (3)  $\implies$  (2) gives us a constructive method of finding  $f$  and  $g$  (provided we know the decomposition of the  $f_i$  into irreducible factors).

(2) Part (3) of the theorem provides us with the easiest way to check whether a given anisotropic form  $\phi$  of even dimension over  $\mathbf{R}(t)$  is round. If  $\phi \cong (f_1, \dots, f_{2m})$  with the  $f_i \in \mathbf{R}(t)$  and if  $\{a_1 < a_2 < \dots < a_k\}$  is the ordered set of all real roots of the  $f_i$ 's, we need only compute  $\phi$  at  $\alpha$  for one value of  $\alpha$  in each of the intervals  $(-\infty, a_1), (a_1, a_2), \dots, (a_k, \infty)$ .

As in [3], we call a quadratic form  $\phi$  over a field  $F$  a *group form* if  $\dot{D}\phi$  is a subgroup of  $\dot{F}$ . Every round form is clearly a group form. We now briefly investigate group forms over  $\mathbf{R}(t)$ .

2.7. *Let  $F$  be a field with a set  $\Omega$  of discrete or archimedean spots on  $F$ . Assume  $(F, \Omega)$  satisfies the Strong Hasse-Minkowski Theorem (local isotropy implies isotropy). Then a quadratic form  $\phi$  over  $F$  is a group form  $\implies \phi_p$  is a group form for all  $p \in \Omega$ .*

*Proof.* ( $\implies$ ): See the proof of 3.2 of [3]. ( $\impliedby$ ): Let  $a, b \in \dot{D}\phi$ . Then  $ab \in \dot{D}\phi_p$  for all  $p \in \Omega$  so  $ab \in \dot{D}\phi$ .

By [4, 2.3] and [7, 42:11],  $\mathbf{R}(t)$  satisfies the Strong Hasse-Minkowski Theorem with respect to the set of all real and complex spots. Thus by 2.7 and 1.1, we have:

2.8. *Let  $\phi$  be a quadratic form over  $\mathbf{R}(t)$ . Then  $\phi$  is a group form  $\implies \phi$  represents 1. If  $\dim \phi \geq 2$  then  $\phi$  is a group form  $\implies \phi$  at  $\alpha$  represents 1 for almost all  $\alpha \in \mathbf{R}$ .*

If  $\phi$  is an anisotropic group form over any field then  $\phi$  is round  $\iff$  the factor group  $\dot{D}\phi/G\phi = 1$ . Thus this factor group measures how far an anisotropic group form is from being round. We now investigate this factor group.

2.9. *Let  $\phi$  be a group form over  $\mathbf{R}(t)$  and assume  $\phi$  is not round.*

Then  $\dot{D}\phi/G\phi$  is infinite unless  $\phi \cong (m \times (1, -1)) \oplus (1, -g)$  where  $m \geq 1$  and  $g$  is a product of monic irreducible quadratic factors. In this latter case  $\dot{D}\phi/G\phi = 1$ .

*Proof.* (1) We first assume  $\dim \phi$  is odd and  $> 1$ . Clearly  $G\phi = \dot{F}^2$ . If  $f$  is any monic irreducible quadratic polynomial over  $\mathbf{R}$ , then  $f \in \dot{D}\phi$  by 1.1. Thus  $\dot{D}\phi/G\phi$  is infinite.

(2) Now assume  $\dim \phi$  is even and  $\phi$  is anisotropic. Then there is an interval  $I = (a, b)$  such that if  $\alpha \in I$ , then  $\phi$  at  $\alpha$  is  $\cong (m \times (1)) \oplus (n \times (-1))$  for fixed positive integers  $m, n$  with  $m \neq n$  (to see this, apply (3) of 2.5 and (2) of 2.6). Let  $a < x < y < b$  and define  $f_{xy}(t) = (t-x)(t-y) \in \mathbf{R}[t]$ . Then  $f_{xy}(\alpha) > 0$  if  $\alpha \notin I$  so  $f_{xy}(t) \in \dot{D}\phi$  by 1.1. Let  $y < y_1 < b$ , so that  $f_{xy_1}(t) \in \dot{D}\phi$  also. Let  $h(t) = f_{xy}(t) \div f_{xy_1}(t)$ . Then  $h(t) \notin G\phi$  by 1.2 since  $h(\alpha) < 0$  for  $y < \alpha < y_1$ . It is now clear that if we choose an infinite sequence of numbers  $y < y_1 < y_2 < \dots < b$  then we obtain an infinite number of distinct cosets of  $G\phi$  in  $\dot{D}\phi$ .

(3) Let  $\dim \phi$  be even and let  $\phi$  be isotropic (but not hyperbolic), and assume that  $\phi$  at  $\alpha$  is non-hyperbolic for infinitely many  $\alpha \in \mathbf{R}$ . Then there is an open interval  $I$  such that for all  $\alpha \in I$ ,  $\phi$  at  $\alpha$  is isotropic but not hyperbolic. Thus by the proof of (2) above,  $\dot{D}\phi/G\phi$  is infinite.

(4) Finally, assume  $\dim \phi$  is even and  $\phi$  is isotropic (but not hyperbolic), and assume that  $\phi$  at  $\alpha$  is hyperbolic for almost all  $\alpha \in \mathbf{R}$ . Then by 1.2,  $\phi \cong (m \times (1, -1)) \oplus (1, -g)$  where  $g$  is a product of monic irreducible quadratic factors. By 1.1,  $\dot{D}\phi = \dot{F}^1$  (where  $F = \mathbf{R}(t)$ ). Now  $G\phi = G(1, -g) = \dot{F}^1$  by 1.2 so  $\dot{D}\phi/G\phi = 1$ .

3. Pfister forms and  $k_n$  over  $\mathbf{R}(t)$ . We first consider Pfister forms over  $\mathbf{R}(t)$ .

3.1. Let  $\phi$  be a quadratic form over  $\mathbf{R}(t)$  with  $\dim \phi = 2^n (n \geq 2)$ . Then the following are equivalent:

- (1)  $\phi$  is a Pfister form.
- (2)  $\phi \cong 2^{n-1} \times (1, f)$  for some  $f \in \mathbf{R}[t]$  which is  $\pm$  a product of distinct monic linear factors (we allow  $f = \pm 1$ ).
- (3)  $\phi$  is round and  $\det \phi = 1$ .

*Proof.* (1)  $\Rightarrow$  (3) is clear. (3)  $\Rightarrow$  (2) by 2.5 (if  $\phi$  is isotropic, let  $f = -1$ ). (2)  $\Rightarrow$  (1) is clear.

In (2),  $f$  is uniquely determined by  $\phi$  (see 2.6).

We now consider, for the field  $F = \mathbf{R}(t)$ , the algebraic  $K$ -groups

$k_n F = K_n F / 2K_n F$  of Milnor [6].  $k_n$  is generated additively by the elements  $l(c_1) \cdots l(c_n) (c_i \in \dot{F})$ . We have  $l(-a_1) \cdots l(-a_n) = l(-b_1) \cdots l(-b_n) \Leftrightarrow (1, a_1) \otimes \cdots \otimes (1, a_n) \cong (1, b_1) \otimes \cdots \otimes (1, b_n)$  [2, Main theorem 3.2].

Let  $n > 1$ . By 3.1 and [2, 3.2], every element of  $k_n F$  can be written uniquely in the form  $l(-1)^{n-1} l(-f)$  for some  $f \in F$  which is  $\pm$  a product of distinct monic linear factors or is  $\pm 1$ . Thus  $k_n F$  is isomorphic to the subgroup of  $\dot{F}/\dot{F}^2$  consisting of the square classes of products of linear polynomials (note that  $l(-1)^{n-1} l(-f) + l(-1)^{n-1} l(-g) = l(-1)^{n-1} l(fg)$ ). Furthermore, there is a natural isomorphism  $s_n$  of  $k_n$  onto  $I^n/I^{n+1}$  where  $I$  is the ideal of the even-dimensional forms of the Witt ring  $W(F)$  [2, 6.1].

REMARK 3.2. By [6, 2.3], for  $n \geq 1$  and for any field  $E$  there is an isomorphism  $K_n E(t) \cong K_n E \oplus (\bigoplus K_{n-1} E[t]/(\pi))$  where the second direct sum extends over all nonzero prime ideals  $(\pi)$  of  $E[t]$ . Now let  $E = \mathbf{R}$  and let  $n \geq 2$ . The above isomorphism induces an isomorphism  $k_n \mathbf{R}(t) \cong k_n \mathbf{R} \oplus (\bigoplus k_{n-1} \mathbf{R}[t]/(\pi))$  where the second direct sum extends over all the polynomials  $\pi = t - \alpha, \alpha \in \mathbf{R}$  (note that  $k_{n-1}$  of the complex numbers is 0). Now  $k_n \mathbf{R}$  and  $k_{n-1} \mathbf{R}$  are groups of order 2 by [6, 1.6] or [2, 3.2]. Thus there is an isomorphism  $k_n \mathbf{R}(t) \cong \mathbf{Z}_2 \oplus (\bigoplus \mathbf{Z}_2)$ . This isomorphism is given explicitly as follows:  $l(-1)^{n-1} l(-f)$  (where  $f$  is  $\pm$  a product of distinct monic linear factors) maps to  $a \oplus (\bigoplus a_\alpha) (\alpha \in \mathbf{R})$  where  $a$  is 0 if and only if  $f$  is monic, and  $a_\alpha$  is 1 if and only if  $t - \alpha$  divides  $f$ .

REMARK 3.3. Let us briefly see what happens when we let our field  $F$  be a global field and let  $n \geq 3$ . Then we have:

(1) Every Pfister form of dimension  $2^n$  over  $F$  is isometric to a form  $2^{n-1} \times (1, a)$  for some  $a \in \dot{F}$ . Also  $2^{n-1} \times (1, a) \cong 2^{n-1} \times (1, b) \Leftrightarrow ab \in \dot{F}_p^2$  for all real spots  $p$  on  $F$ . These facts follow easily from the Weak Hasse-Minkowski Theorem.

(2) By (1) and by [2, Main Theorem 3.2], we see that every element of  $k_n F$  can be written as  $l(-1)^{n-1} l(-a)$  for some  $a \in \dot{F}$ , and  $l(-1)^{n-1} l(-a) = l(-1)^{n-1} l(-b) \Leftrightarrow ab \in \dot{F}_p^2$  for all  $p$  real. Thus  $k_n F \cong \bigoplus k_n F_p$  where the direct sum extends over all real spots  $p$  (note that  $k_n F_p = \mathbf{Z}_2$ ). This fact was first proved by Tate (see appendix of [6]). Elman and Lam [1] gave a simple proof (using the Strong Hasse-Minkowski Theorem) which does not depend on [2].

(3) There are round forms  $\phi$  over  $F$  of dimension  $2^n$  (with  $\det \phi = 1$ ) which are not Pfister forms [3, 2.6].

*Added in proof.* In connection with Example 2.3(3), we point out here that, without using elliptic curves theory, examples of rational function fields which do not satisfy the Strong Hasse-Min-



kowski Theorem can be found in the article: "On the Hasse Principle for Quadratic Forms", P.A.M.S., **39** (1973).

The results in §2 have been generalized recently by R. Elman in his article: "Rund forms over real algebraic function fields in one variable" (to appear). Instead of using the local-global method as we have done, Elman's approach is entirely different; he uses the algebraic theory of Pfister forms.

#### REFERENCES

1. R. Elman and T. Y. Lam, *Determination of  $k_n$  ( $n \geq 3$ ) for global fields*, Proc. Amer. Math. Soc., **31** (1972), 427-428.
2. ———, *Pfister forms and K-theory of fields*, J. Algebra, **23** (1972), 181-213.
3. J. S. Hsia and R. P. Johnson, *Round and group quadratic forms over global fields*, J. Number Theory, Vol. **5**, No. 5 (1973), 356-366.
4. J. T. Knight, *Quadratic forms over  $R(t)$* , Proc. Cambridge Philos. Soc., **62** (1966), 197-205.
5. F. Lorenz, *Quadratische Formen über Körpern*, Lecture Notes in Math., **130**, Springer-Verlag, Berlin-Heidelberg-New York, 1970.
6. J. Milnor, *Algebraic K-theory and quadratic forms*, Invent. Math., **9** (1970), 318-344.
7. O. T. O'Meara, *Introduction to Quadratic Forms*, Springer-Verlag, 1963.
8. W. Scharlau, *Quadratic Forms*, Queen's papers on pure and applied math., no. 22, Queen's Univ., Kingston, Ontario, 1969.

Received August 14, 1972. Research of the first author was supported in part by the National Science Foundation under grant NSF GP-23656.

OHIO STATE UNIVERSITY

# PACIFIC JOURNAL OF MATHEMATICS

## EDITORS

RICHARD ARENS (Managing Editor)  
University of California  
Los Angeles, California 90024

J. DUGUNDJI\*  
Department of Mathematics  
University of Southern California  
Los Angeles, California 90007

R. A. BEAUMONT  
University of Washington  
Seattle, Washington 98105

D. GILBARG AND J. MILGRAM  
Stanford University  
Stanford, California 94305

## ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

## SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA  
CALIFORNIA INSTITUTE OF TECHNOLOGY  
UNIVERSITY OF CALIFORNIA  
MONTANA STATE UNIVERSITY  
UNIVERSITY OF NEVADA  
NEW MEXICO STATE UNIVERSITY  
OREGON STATE UNIVERSITY  
UNIVERSITY OF OREGON  
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA  
STANFORD UNIVERSITY  
UNIVERSITY OF TOKYO  
UNIVERSITY OF UTAH  
WASHINGTON STATE UNIVERSITY  
UNIVERSITY OF WASHINGTON  
\* \* \*  
AMERICAN MATHEMATICAL SOCIETY  
NAVAL WEAPONS CENTER

\* C. R. DePrima California Institute of Technology, Pasadena, CA 91109, will replace J. Dugundji until August 1974.

# Pacific Journal of Mathematics

Vol. 49, No. 1

May, 1973

A. Bigard, <i>Free lattice-ordered modules</i> . . . . .	1
Richard Bolstein and Warren R. Wogen, <i>Subnormal operators in strictly cyclic operator algebras</i> . . . . .	7
Herbert Busemann and Donald E. Glassco, II, <i>Irreducible sums of simple multivectors</i> . . . . .	13
W. Wistar (William) Comfort and Victor Harold Saks, <i>Countably compact groups and finest totally bounded topologies</i> . . . . .	33
Mary Rodriguez Embry, <i>Maximal invariant subspaces of strictly cyclic operator algebras</i> . . . . .	45
Ralph S. Freese and James Bryant Nation, <i>Congruence lattices of semilattices</i> . . . . .	51
Ervin Fried and George Grätzer, <i>A nonassociative extension of the class of distributive lattices</i> . . . . .	59
John R. Giles and Donald Otto Koehler, <i>On numerical ranges of elements of locally <math>m</math>-convex algebras</i> . . . . .	79
David A. Hill, <i>On dominant and codominant dimension of <math>QF - 3</math> rings</i> . . . . .	93
John Sollion Hsia and Robert Paul Johnson, <i>Round and Pfister forms over <math>R(t)</math></i> . . . . .	101
I. Martin (Irving) Isaacs, <i>Equally partitioned groups</i> . . . . .	109
Athanassios G. Kartsatos and Edward Barry Saff, <i>Hyperpolynomial approximation of solutions of nonlinear integro-differential equations</i> . . . . .	117
Shin'ichi Kinoshita, <i>On elementary ideals of <math>\theta</math>-curves in the 3-sphere and 2-links in the 4-sphere</i> . . . . .	127
Ronald Brian Kirk, <i>Convergence of Baire measures</i> . . . . .	135
R. J. Knill, <i>The Seifert and Van Kampen theorem via regular covering spaces</i> . . . . .	149
Amos A. Kovacs, <i>Homomorphisms of matrix rings into matrix rings</i> . . . . .	161
Young K. Kwon, <i><math>HD</math>-minimal but no <math>HD</math>-minimal</i> . . . . .	171
Makoto Maejima, <i>On the renewal function when some of the mean renewal lifetimes are infinite</i> . . . . .	177
Juan José Martínez, <i>Cohomological dimension of discrete modules over profinite groups</i> . . . . .	185
W. K. Nicholson, <i>Semiperfect rings with abelian group of units</i> . . . . .	191
Louis Jackson Ratliff, Jr., <i>Three theorems on imbedded prime divisors of principal ideals</i> . . . . .	199
Billy E. Rhoades and Albert Wilansky, <i>Some commutants in <math>B(c)</math> which are almost matrices</i> . . . . .	211
John Philip Riley Jr., <i>Cross-sections of decompositions</i> . . . . .	219
Keith Duncan Stroyan, <i>A characterization of the Mackey uniformity <math>m(L^\infty, L^1)</math> for finite measures</i> . . . . .	223
Edward G. Thurber, <i>The Scholz-Brauer problem on addition chains</i> . . . . .	229
Joze Vrabec, <i>Submanifolds of acyclic 3-manifolds</i> . . . . .	243
Philip William Walker, <i>Adjoint boundary value problems for compactified singular differential operators</i> . . . . .	265
Roger P. Ware, <i>When are Witt rings group rings</i> . . . . .	279
James D. Wine, <i>Paracompactifications using filter bases</i> . . . . .	285