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John Sollion Hsia and Robert Paul Johnson

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J. S. HSIA AND ROBERT P. JOHNSON

An anisotropic quadratic form ϕ is called round if $\phi \cong a\phi$ whenever ϕ represents $a \neq 0$. All round forms over $R(t)$ are completely determined. Connections with Pfister's strongly multiplicative forms and with the reduced algebraic K-theory groups k_n of Milnor are studied.

The concept of a round form was introduced by Witt (see [5] and [8]) to give new simple proofs of results of Pfister on the structure of the Witt ring over fields. In a previous paper [3] we determined all round forms over a global field. In this paper we completely determine all round forms over $\boldsymbol{R}(t)$, the field of rational functions in one variable over the reals.

We now describe our main results.

Let ϕ be an anisotropic form of dimension > 1 over $\boldsymbol{R}(t)$. Then ϕ is round if and only if $\phi \cong (n \times (1, f)) \oplus (1, fg)$ for some $f, g \in \boldsymbol{R}(t)$ such that f is a product of distinct linear factors and g is a product of irreducible quadratic factors. Our proof gives a method of computing f and g, which are essentially unique (see 2.5 and 2.6). We study a generalization of a round form, called a group form, over $\boldsymbol{R}(t)$ and measure how far group forms are from being round (see [3] for group forms over global fields).

In the last section we show that a form of dimension $2^n (n \geqq 2)$ is a Pfister form if and only if it is a round form of determinant one. Such a form can be written uniquely as $2^{n-1} \times (1, f)$ for some $f \in \boldsymbol{R}[t]$ which is $\pm$ a product of distinct monic linear factors. From this and a theorem of Elman and Lam we see that every element of $k_n\boldsymbol{R}(t)$ can be written uniquely as $l(-1)^{n-1}l(-f)$ with f as above.

1. Preliminaries. We will consider only quadratic forms (often simply called "forms") over a field F of characteristic $\neq 2$. We write $\phi \oplus \psi$ for the orthogonal sum and $\phi \otimes \psi$ for the tensor product of quadratic forms [5, p. 8]. We call ϕ *hyperbolic* if $\phi \cong m \times (1, -1)$, i.e., ϕ is a direct sum of hyperbolic planes.

Define $\dot{D}\phi = \{a \in \dot{F} \mid \phi \text{ represents } a\}$ and $G\phi = \{a \in \dot{F} \mid a\phi \cong \phi\}$ where $\dot{F} = F - \{0\}$. An anisotropic form ϕ is called *round* if and only if $\dot{D}\phi = G\phi$ (or equivalently $\dot{D}\phi \subseteqq G\phi$); an isotropic form is called round if and only if it is hyperbolic [5, p. 22]. A form ϕ is called a *Pfister form* if $\phi \cong (1, a_1) \otimes \cdots \otimes (1, a_n) (a_i \in \dot{F})$.

We will frequently refer to [4] for results on quadratic forms over $F = \boldsymbol{R}(t)$. The valuations of F which are trivial on $\boldsymbol{R}$ are of three types: if the prime element is $t-\alpha (\alpha \in \boldsymbol{R})$, the valuation is called *real*; if the prime element is an irreducible quadratic polynomial it is called *complex*; if the prime element is t^{-1} it is called *infinite*. A *spot* is an equivalence class of valuations [7]. If p is a real or infinite spot then the completion F_p of F at p is isomorphic to $\boldsymbol{R}((\pi))$ (a *real series field*) where π is a prime element. If p is complex, $F_p \cong \boldsymbol{C}((\pi))$ is called a *complex series field*. See [4] for results on quadratic forms over series fields.

If ϕ is a quadratic form over $\boldsymbol{R}(t)$ and if $\alpha \in \boldsymbol{R}$, we define "$\phi$ at α" to be the quadratic form over $\boldsymbol{R}$ obtained by replacing t by α in the matrix of ϕ. Thus ϕ at α is well-defined for almost all $\alpha \in \boldsymbol{R}$. The following result is Proposition 2.1 of [4] and is due to Witt.

1.1. *A nonsingular quadratic form of dimension* $\geqq 3$ *over* $\boldsymbol{R}(t)$ *is isotropic if and only if for almost all* $\alpha \in \boldsymbol{R}$, *the form at* α *is isotropic over* $\boldsymbol{R}$. *Thus if* ϕ *is a quadratic form of dimension* $\geqq 2$ *over* $\boldsymbol{R}(t)$ *and if* $0 \neq f(t) \in \boldsymbol{R}(t)$, *then* ϕ *represents* $f(t) \Leftrightarrow$ *for almost all* $\alpha \in \boldsymbol{R}$, ϕ *at* α *represents* $f(\alpha)$.

If we write $\phi \cong (a_1, \cdots, a_n)$ over a field F then $\det \phi = a_1 \cdots a_n$ modulo $\dot{F}^2$. When $F = \boldsymbol{R}(t)$ we assume $\det \phi$ is written as $\pm$ a product of distinct monic irreducible polynomials.

The following result generalizes Proposition 2.2 of [4].

1.2. *Let* ϕ, ψ *be quadratic forms over* $\boldsymbol{R}(t)$. *If* $\phi \cong \psi$ *at* α *for almost all* $\alpha \in \boldsymbol{R}$ *and if* $\det \phi$, $\det \psi$ *have the same irreducible quadratic factors, then* $\phi \cong \psi$.

Proof. Clear for $\dim \phi = 1$. We assume this result is true whenever $\dim \phi < n$ and prove it for $\dim \phi = n > 1$. Let ϕ represent $a \neq 0$. Then $\phi \oplus (-a)$ is isotropic so by 1.1, $\psi \oplus (-a)$ is isotropic. Thus ψ represents a. Write $\phi \cong (a) \oplus \phi_1$ and $\psi \cong (a) \oplus \psi_1$ and apply the induction hypothesis.

1.3. *Let* $f(t) \in \boldsymbol{R}[t]$ *and* $\alpha \in \boldsymbol{R}$ *with* $f(\alpha) \neq 0$. *Then* $(f(t)) \cong (f(\alpha))$ (*one-dimensional quadratic forms*) *over the completion of* $\boldsymbol{R}(t)$ *at the spot with prime element* $t - \alpha$.

Proof. Write $f(t) = a_0 + a_1(t-\alpha) + \cdots + a_n(t-\alpha)^n$ and apply the Local Square Theorem [7, 63: 1a], noting $f(\alpha) = a_0$.

2. **Round forms over $\boldsymbol{R}(t)$.** We will need the following result, which determines all round forms over a series field.

2.1. *Let* ϕ *be an anisotropic quadratic form over a real or complex series field* F.

(a) *If F is complex, then ϕ is round* $\Leftrightarrow$ *ϕ represents* 1.

(b) *Let F be a real series field. Then F is pythagorean and formally real. So if* $\dim\phi$ *is odd, ϕ is round* $\Leftrightarrow \phi \cong (1, \cdots, 1)$. *If* $\dim\phi = 2m$ *is even then ϕ is round* $\Leftrightarrow \phi \cong m \times (1, 1)$ *or* $m \times (1, \pm\pi)$.

Proof. (a) By [4, 1.2], $\dim\phi \leqq 2$ whenever ϕ is anisotropic over a complex series field. Now apply [5, 2.4].

(b) It follows easily from the Local Square Theorem [7, 63: 1a] that F is pythagorean. Now apply [5, 2.4] and [4, 1.6].

Now let F be a field of characteristic $\neq 2$ and let Ω be a set of discrete or archimedian spots on F (see [7] for terminology). We say that (F, Ω) satisfies the *Weak Hasse-Minkowski Theorem* if whenever σ and τ are quadratic forms over F with $\sigma_p \cong \tau_p$ for all $p \in \Omega$, then $\sigma \cong \tau$ (σ_p denotes the form σ viewed over the completion F_p of F at p).

2.2. *Let (F, Ω) satisfy the Weak Hasse-Minkowski Theorem. Let ϕ be anisotropic over F. Then ϕ is round* $\Leftrightarrow$ *for all $p \in \Omega$,*

(1) *ϕ_p is round*

or (2) *ϕ_p is isotropic and ϕ'_p (the anisotropic part of ϕ_p) is round and universal.*

Proof. ($\Rightarrow$): Assume ϕ is round. Let $p \in \Omega$. We first assume ϕ_p is anisotropic and show ϕ_p is round. Let $b \in \dot{D}(\phi_p)$. Approximate b by $a \in \dot{D}\phi$. By the Local Square Theorem, we can obtain $a \in b\dot{F}_p^2$. Thus $\phi \cong a\phi \Rightarrow \phi_p \cong b\phi_p$ so ϕ_p is round.

Now assume ϕ_p is isotropic and not hyperbolic. Write $\phi_p = \phi'_p \oplus H$ with H hyperbolic. We will show $\phi'_p \cong b\phi'_p$ for all $b \in \dot{F}_p$ and so (2) holds. Now ϕ_p represents b so we find that $\phi_p \cong b\phi_p$ by the argument of the preceding paragraph. Thus $\phi'_p \cong b\phi'_p$.

($\Leftarrow$): Let $a \in \dot{D}\phi$. Applying (1) or (2), we have $\phi_p \cong a\phi_p$ for all $p \in \Omega$. By the Weak Hasse-Minkowski Theorem, $\phi \cong a\phi$, so ϕ is round.

EXAMPLES 2.3. The Weak Hasse-Minkowski Theorem holds in the following cases:

(1) Let $F = K(t)$ where K is an arbitrary field of characteristic $\neq 2$ and let Ω be the set of all spots on F that are trivial on K. Using [6, Theorem 5.3] one can show that (F, Ω) satisfies the Weak Hasse-Minkowski Theorem.

(2) Let F be a global field and let Ω be the set of all non-trivial spots on F. We have the following precise results in this case [3, 2.4]: let ϕ be an anisotropic form over F and let $\dim\phi > 2$. Then ϕ is round if and only if: (1) $\dim\phi \equiv 0 \bmod 4$, (2) at all real

spots (if there are any) ϕ is hyperbolic or positive definite, and (3) $\det \phi = 1$. We note that the *Strong Hasse-Minkowski Theorem* holds for (F, Ω), i.e., if a form ϕ is isotropic for all $p \in \Omega$ then ϕ is isotropic.

(3) Cassels, Ellison, and Pfister (J. Number Theory, 3 (1971), p. 147) have recently shown that the Strong Hasse-Minkowski Theorem fails for $F = K(t)$ where $K = \boldsymbol{R}(x)$ (x, t independent indeterminants over $\boldsymbol{R}$) though the weak theorem holds as we have mentioned in (1).

The next two results determine all round forms over $\boldsymbol{R}(t)$.

2.4. *There is no odd-dimensional round form over* $\boldsymbol{R}(t)$ *except the form* $\phi = (1)$.

Proof. Note that $\boldsymbol{R}(t)$ is non-pythogorean since $t^2 + 1$ is not a square. Now apply [5, 2.4].

THEOREM 2.5. *Let* ϕ *be an anisotropic form of dimension* $2m$ *over* $\boldsymbol{R}(t)$. *Then the following are equivalent*:

(1) ϕ *is round.*

(2) $\phi \cong ((m-1) \times (1, f)) \oplus (1, fg)$ *for some* $f, g \in \boldsymbol{R}[t]$ *such that* f *is a product of distinct linear factors and* f *or* $-f$ *is monic, and* g *is a product of monic irreducible quadratic factors* (*we allow* $f = 1$ *or* -1 *and allow* $g = 1$).

(3) *For almost all* $\alpha \in \boldsymbol{R}$, ϕ *at* α *is hyperbolic or positive definite.*

(4) ϕ_p *is round for all real or infinite spots* p *on* $\boldsymbol{R}(t)$.

Proof. (1) $\Leftrightarrow$ (4) follows from 2.2 since there is no universal anisotropic form over a real series field. We will show (2) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (2). (2) $\Rightarrow$ (4) follows from 2.1 and 1.3.

(4) $\Rightarrow$ (3): Assume (4). Write $\phi \cong (f_1(t), \cdots, f_{2m}(t))$ with the $f_i(t) \in \boldsymbol{R}[t]$. Let $\alpha \in \boldsymbol{R}$ such that $f_i(\alpha) \neq 0$ for all i. Let p be the real spot with prime element $t - \alpha$. By 1.3, $\phi_p \cong (f_1(\alpha), \cdots, f_{2m}(\alpha))$. By 2.1, $\phi_p \cong m \times (1, 1)$ or $m \times (1, -1)$. So by [4, 1.6], ϕ at α is $\cong m \times (1, 1)$ or $m \times (1, -1)$.

(3) $\Rightarrow$ (2): Write $\phi \cong (f_1, \cdots, f_{2m})$ with the $f_i \in \boldsymbol{R}[t]$. Let S be the set of all $a \in \boldsymbol{R}$ such that $f_i(a) = 0$ for some i. Write $S = \{a_1, \cdots, a_k\}$ with $a_1 < a_2 < \cdots < a_k$. If I is any of the intervals $(-\infty, a_1)$, $(a_1, a_2), \cdots, (a_k, \infty)$ then ϕ at α is hyperbolic for all $\alpha \in I$ or is positive definite for all $\alpha \in I$. The idea now is to merge together adjacent intervals if ϕ at α looks the same in the adjacent intervals. If ϕ at α is positive definite (respectively, hyperbolic) for almost all $\alpha \in \boldsymbol{R}$ then we let $f = 1$ (respectively, -1). Otherwise, there is an ordered subset $\{b_1 < b_2 < \cdots < b_j\}$ of S such that if J is any of the intervals $(-\infty, b_1), (b_1, b_2), \cdots, (b_j, \infty)$ then ϕ at α is hyperbolic for almost all

$\alpha \in J$ or is positive definite for almost all $\alpha \in J$, and such that whenever ϕ is hyperbolic in one of these intervals then it is positive definite in the adjacent intervals. Now let $f = (t - b_1) \cdots (t - b_j)$ if ϕ at α is positive definite for almost all $\alpha > b_j$, and let $f = -(t - b_1) \cdots (t - b_j)$ otherwise. Let g be the product of all the (monic) irreducible quadratic factors of $\det \phi$. Then by 1.2, $\phi \cong ((m - 1) \times (1, f)) \oplus (1, fg)$.

REMARK 2.6. (1). Part (2) of the above theorem gives us a *canonical form* for an anisotropic round form of even dimension over $\boldsymbol{R}(t)$, i.e., f and g are uniquely determined. This fact follows easily from 1.2. The proof of (3) $\Rightarrow$ (2) gives us a constructive method of finding f and g (provided we know the decomposition of the f_i into irreducible factors).

(2) Part (3) of the theorem provides us with the easiest way to check whether a given anisotropic form ϕ of even dimension over $\boldsymbol{R}(t)$ is round. If $\phi \cong (f_1, \cdots, f_{2m})$ with the $f_i \in \boldsymbol{R}(t)$ and if $\{a_1 < a_2 < \cdots < a_k\}$ is the ordered set of all real roots of the f_i's, we need only compute ϕ at α for one value of α in each of the intervals $(-\infty, a_1)$, (a_1, a_2), $\cdots$, (a_k, ∞).

As in [3], we call a quadratic form ϕ over a field F a *group form* if $\dot{D}\phi$ is a subgroup of $\dot{F}$. Every round form is clearly a group form. We now briefly investigate group forms over $\boldsymbol{R}(t)$.

2.7. *Let F be a field with a set Ω of discrete or archimedean spots on F. Assume (F, Ω) satisfies the Strong Hasse-Minkowski Theorem (local isotropy implies isotropy). Then a quadratic form ϕ over F is a group form $\Leftrightarrow \phi_p$ is a group form for all $p \in \Omega$.*

Proof. ($\Rightarrow$): See the proof of 3.2 of [3]. ($\Leftarrow$): Let $a, b \in \dot{D}\phi$. Then $ab \in \dot{D}\phi_p$ for all $p \in \Omega$ so $ab \in \dot{D}\phi$.

By [4, 2.3] and [7, 42:11], $\boldsymbol{R}(t)$ satisfies the Strong Hasse-Minkowski Theorem with respect to the set of all real and complex spots. Thus by 2.7 and 1.1, we have:

2.8. *Let ϕ be a quadratic form over $\boldsymbol{R}(t)$. Then ϕ is a group form $\Leftrightarrow \phi$ represents* 1. *If* $\dim \phi \geqq 2$ *then ϕ is a group form $\Leftrightarrow \phi$ at α represents* 1 *for almost all $\alpha \in \boldsymbol{R}$.*

If ϕ is an anisotropic group form over any field then ϕ is round $\Leftrightarrow$ the factor group $\dot{D}\phi / G\phi = 1$. Thus this factor group measures how far an anisotropic group form is from being round. We now investigate this factor group.

2.9. *Let ϕ be a group form over $\boldsymbol{R}(t)$ and assume ϕ is not round.*

Then $\dot{D}\phi/G\phi$ is infinite unless $\phi \cong (m \times (1, -1)) \oplus (1, -g)$ where $m \geqq 1$ and g is a product of monic irreducible quadratic factors. In this latter case $\dot{D}\phi/G\phi = 1$.

Proof. (1) We first assume $\dim \phi$ is odd and > 1. Clearly $G\phi = \dot{F}^2$. If f is any monic irreducible quadratic polynomial over $\boldsymbol{R}$, then $f \in \dot{D}\phi$ by 1.1. Thus $\dot{D}\phi/G\phi$ is infinite.

(2) Now assume $\dim \phi$ is even and ϕ is anisotropic. Then there is an interval $I = (a, b)$ such that if $\alpha \in I$, then ϕ at α is $\cong (m \times (1)) \oplus (n \times (-1))$ for fixed positive integers m, n with $m \neq n$ (to see this, apply (3) of 2.5 and (2) of 2.6). Let $a < x < y < b$ and define $f_{xy}(t) = (t - x)(t - y) \in \boldsymbol{R}[t]$. Then $f_{xy}(\alpha) > 0$ if $\alpha \notin I$ so $f_{xy}(t) \in \dot{D}\phi$ by 1.1. Let $y < y_1 < b$, so that $f_{xy_1}(t) \in \dot{D}\phi$ also. Let $h(t) = f_{xy}(t) \div f_{xy_1}(t)$. Then $h(t) \notin G\phi$ by 1.2 since $h(\alpha) < 0$ for $y < \alpha < y_1$. It is now clear that if we choose an infinite sequence of numbers $y < y_1 < y_2 < \cdots < b$ then we obtain an infinite number of distinct cosets of $G\phi$ in $\dot{D}\phi$.

(3) Let $\dim \phi$ be even and let ϕ be isotropic (but not hyperbolic), and assume that ϕ at α is non-hyperbolic for infinitely many $\alpha \in \boldsymbol{R}$. Then there is an open interval I such that for all $\alpha \in I$, ϕ at α is isotropic but not hyperbolic. Thus by the proof of (2) above, $\dot{D}\phi/G\phi$ is infinite.

(4) Finally, assume $\dim \phi$ is even and ϕ is isotropic (but not hyperbolic), and assume that ϕ at α is hyperbolic for almost all $\alpha \in \boldsymbol{R}$. Then by 1.2, $\phi \cong (m \times (1, -1)) \oplus (1, -g)$ where g is a product of monic irreducible quadratic factors. By 1.1, $\dot{D}\phi = \dot{F}$ (where $F = \boldsymbol{R}(t)$. Now $G\phi = G(1, -g) = \dot{F}$ by 1.2 so $\dot{D}\phi/G\phi = 1$.

3. Pfister forms and k_n over $\boldsymbol{R}(t)$. We first consider Pfister forms over $\boldsymbol{R}(t)$.

3.1. *Let ϕ be a quadratic form over $\boldsymbol{R}(t)$ with $\dim \phi = 2^n (n \geqq 2)$. Then the following are equivalent:*

(1) *ϕ is a Pfister form.*

(2) *$\phi \cong 2^{n-1} \times (1, f)$ for some $f \in \boldsymbol{R}[t]$ which is $\pm$ a product of distinct monic linear factors (we allow $f = \pm 1$).*

(3) *ϕ is round and $\det \phi = 1$.*

Proof. (1) $\Rightarrow$ (3) is clear. (3) $\Rightarrow$ (2) by 2.5 (if ϕ is isotropic, let $f = -1$). (2) $\Rightarrow$ (1) is clear.

In (2), f is uniquely determined by ϕ (see 2.6).

We now consider, for the field $F = \boldsymbol{R}(t)$, the algebraic K-groups

$k_nF = K_nF/2K_nF$ of Milnor [6]. k_n is generated additively by the elements $l(c_1)\cdots l(c_n)(c_i \in \dot{F})$. We have $l(-a_1)\cdots l(-a_n)=l(-b_1)\cdots l(-b_n)\Leftrightarrow (1, a_1)\otimes\cdots\otimes(1, a_n)\cong(1, b_1)\otimes\cdots\otimes(1, b_n)$ [2, Main theorem 3.2].

Let $n > 1$. By 3.1 and [2, 3.2], every element of k_nF can be written uniquely in the form $l(-1)^{n-1}l(-f)$ for some $f \in F$ which is $\pm$ a product of distinct monic linear factors or is ± 1. Thus k_nF is isomorphic to the subgroup of $\dot{F}/\dot{F}^2$ consisting of the square classes of products of linear polynomials (note that $l(-1)^{n-1}l(-f)+l(-1)^{n-1}l(-g)=l(-1)^{n-1}l(fg)$). Furthermore, there is a natural isomorphism s_n of k_n onto I^n/I^{n+1} where I is the ideal of the even-dimensional forms of the Witt ring $W(F)$ [2, 6.1].

REMARK 3.2. By [6, 2.3], for $n \geqq 1$ and for any field E there is an isomorphism $K_nE(t) \cong K_nE \oplus (\bigoplus K_{n-1}E[t]/(\pi))$ where the second direct sum extends over all nonzero prime ideals (π) of $E[t]$. Now let $E = \boldsymbol{R}$ and let $n \geqq 2$. The above isomorphism induces an isomorphism $k_n\boldsymbol{R}(t) \cong k_n\boldsymbol{R} \oplus (\bigoplus k_{n-1}\boldsymbol{R}[t]/(\pi))$ where the second direct sum extends over all the polynomials $\pi = t - \alpha, \alpha \in \boldsymbol{R}$ (note that k_{n-1} of the complex numbers is 0). Now $k_n\boldsymbol{R}$ and $k_{n-1}\boldsymbol{R}$ are groups of order 2 by [6, 1.6] or [2, 3.2]. Thus there is an isomorphism $k_n\boldsymbol{R}(t) \cong \boldsymbol{Z}_2 \oplus (\bigoplus_{\boldsymbol{R}} \boldsymbol{Z}_2)$. This isomorphism is given explicitly as follows: $l(-1)^{n-1}l(-f)$(where f is $\pm$ a product of distinct monic linear factors) maps to $a \oplus (\bigoplus a_\alpha)(\alpha \in \boldsymbol{R})$ where a is 0 if and only if f is monic, and a_α is 1 if and only if $t - \alpha$ divides f.

REMARK 3.3. Let us briefly see what happens when we let our field F be a global field and let $n \geqq 3$. Then we have:

(1) Every Pfister form of dimension 2^n over F is isometric to a form $2^{n-1} \times (1, a)$ for some $a \in \dot{F}$. Also $2^{n-1} \times (1, a) \cong 2^{n-1} \times (1, b) \Leftrightarrow ab \in \dot{F}_p^2$ for all real spots p on F. These facts follow easily from the Weak Hasse-Minkowski Theorem.

(2) By (1) and by [2, Main Theorem 3.2], we see that every element of k_nF can be written as $l(-1)^{n-1}l(-a)$ for some $a \in \dot{F}$, and $l(-1)^{n-1}l(-a) = l(-1)^{n-1}l(-b) \Leftrightarrow ab \in \dot{F}_p^2$ for all p real. Thus $k_nF \cong \bigoplus k_nF_p$ where the direct sum extends over all real spots p (note that $k_nF_p = \boldsymbol{Z}_2$). This fact was first proved by Tate (see appendix of [6]). Elman and Lam [1] gave a simple proof (using the Strong Hasse-Minkowski Theorem) which does not depend on [2].

(3) There are round forms ϕ over F of dimension 2^n (with $\det \phi = 1$) which are not Pfister forms [3, 2.6].

Added in proof. In connection with Example 2.3(3), we point out here that, without using elliptic curves theory, examples of rational function fields which do not satisfy the Strong Hasse-Minkowski Theorem can be found in the article: "*On the Hasse Principle for Quadratic Forms*", P.A.M.S., 39 (1973).

The results in §2 have been generalized recently by R. Elman in his article: "*Rund forms over real algebraic function fields in one variable*" (to appear). Instead of using the local-global method as we have done, Elman's approach is entirely different; he uses the algebraic theory of Pfister forms.

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