

Pacific Journal of Mathematics

**ON ELEMENTARY IDEALS OF θ -CURVES IN THE 3-SPHERE
AND 2-LINKS IN THE 4-SPHERE**

SHIN'ICHI KINOSHITA

ON ELEMENTARY IDEALS OF θ -CURVES IN THE 3-SPHERE AND 2-LINKS IN THE 4-SPHERE

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Let L be a polyhedron in an n -sphere $S^n (n \geq 3)$ that does not separate S^n . A topological invariant of the position of L in S^n can be introduced as follows: Let l be an integral $(n-2)$ -cycle on L . For each nonnegative integer d , the d th elementary ideal $E_d(l)$ is associated to l on L in S^n . If l and l' are homologous on L , then $E_d(l)$ is equal to $E_d(l')$. Now the collection of $E_d(l)$ for all possible l is a topological invariant of L in S^n .

In this paper the following two cases of $E_d(l)$ are considered:

- (1) l is a 1-cycle on a θ -curve L in S^3 , and (2) l is a 2-cycle on a 2-link L in S^4 , i.e., the union of two disjoint 2-spheres in S^4 , where each of two 2-spheres is trivially imbedded in S^4 .

The d th elementary ideal $E_d(l)$ of l on L is defined as follows (more precisely see [3]): Let G be the fundamental group $\pi(S^n - L)$ and H the multiplicative infinite cyclic group generated by t . Let ψ be a homomorphism of G into H defined by

$$g^\psi = t^{\text{link}(g,l)},$$

where $\text{link}(g, l)$ is the linking number between g and l . Using Fox's free differential calculus, we associate to ψ the d th elementary ideal E_d of the group G , evaluated in the group ring JH of H over integers. This d th elementary ideal E_d depends only on G and ψ , and hence it depends only on the position of l on L in S^n . We shall denote it by $E_d(l)$.

In this paper we shall prove the following two theorems.

THEOREM 1. *Let $f(t)$ be an integral polynomial with $f(1) = 1$. Then there exists a θ -curve L_f in S^3 , and an integral 1-cycle l on L_f such that*

$$\begin{cases} E_0(l) = E_1(l) = (0), \\ E_2(l) = (f(t)) \text{ and} \\ E_d(l) = (1), \text{ if } d > 2. \end{cases}$$

THEOREM 2. *Let $f(t)$ be an integral polynomial with $f(1) = 1$. Then there exists a 2-link L_f in S^4 , and an integral 2-cycle l on L_f such that*

- (1) *each component of L_f is a trivially imbedded 2-sphere in S^4 , and that*

(2) we have

$$\begin{cases} E_0(l) = E_1(l) = (0), \\ E_2(l) = (f(t)) \text{ and} \\ E_d(l) = (1), \text{ if } d > 2. \end{cases}$$

COROLLARY. Let $f(t)$ be an integral polynomial with $f(1) = 1$. Then there exists an oriented 2-link L_f in S^4 such that

(1) each component of L_f is a trivial 2-sphere in S^4 , and that
 (2) the d th elementary ideal of L_f , in the usual sense and in the reduced form, is as follows:

$$\begin{cases} E_0(L_f) = E_1(L_f) = (0), \\ E_2(L_f) = (f(t)) \text{ and} \\ E_d(L_f) = (1), \text{ if } d > 2. \end{cases}$$

REMARK. This kind of example was first considered in [1].

The construction of these two examples are closely related. They are also closely related to the construction of 2-spheres in S^4 in [2].

1. Let P be the family of all integral polynomials $f(t)$ which can be expressed in the following form:

$$(1) \quad \begin{aligned} & t^{-(\varepsilon_1 + \dots + \varepsilon_n)}(1 - t^{\delta_1}) + t^{-(\varepsilon_2 + \dots + \varepsilon_n)}(1 - t^{\delta_2}) \\ & + \dots + t^{-\varepsilon_n}(1 - t^{\delta_n}) + 1, \end{aligned}$$

where $\varepsilon_i = \pm 1$ and $\delta_i = \varepsilon_i$ or $\delta_i = 0$ for $i = 1, 2, \dots, n$. We assume that $1 \in P$.

LEMMA. We have $f(t) \in P$, if and only if $f(1) = 1$.

Proof. If $f(t) \in P$, then clearly we have $f(1) = 1$. Suppose that $f(1) = 1$. Then we have

$$\begin{aligned} f(t) - 1 &= (1 - t)(a_m t^m + \dots + a_0) \\ &\quad - (1 - t)(b_m t^m + \dots + b_0) \\ &= (1 - t)(a_m t^m + \dots + a_0) \\ &\quad + (1 - t^{-1})(b_m t^{m+1} + \dots + b_0 t), \end{aligned}$$

where $a_i, b_i \geq 0$ for $i = 1, 2, \dots, n$. This means that $f(t)$ with $f(1) = 1$ can be obtained from 1 by applying a finite number of operation:

$$g(t) \rightarrow g(t) + t^p(1 - t^\delta),$$

where $p \geq 0$ and $\delta = \pm 1$.

We assume $1 \in P$. Hence we should prove that if $f(t) \in P$, then $f(t) + t^p(1 - t^\delta) \in P$. Suppose that $f(t)$ has form (1). Now let

$$p = -(\varepsilon'_1 + \dots + \varepsilon'_k + \varepsilon'_{k+1} + \dots + \varepsilon'_{k+n}),$$

where $\varepsilon'_{k+i} = \varepsilon_i$ for $i = 1, 2, \dots, n$ and let

$$\delta'_1 = \delta, \delta'_2 = \dots = \delta'_k = 0 \quad \text{and} \quad \delta'_{k+i} = \delta_i$$

for $i = 1, 2, \dots, n$. Then clearly we have

$$t^{-(\varepsilon'_1 + \dots + \varepsilon'_k + \varepsilon'_{k+1} + \dots + \varepsilon'_{k+n})}(1 - t^{\delta'_1}) + \dots + t^{\varepsilon'_{k+n}}(1 - t^{\delta'_{k+n}}) = t^p(1 - t^\delta) + f(t).$$

Hence the proof is complete.

2. Let $f(t)$ be an integral polynomial with $f(1) = 1$. Suppose that $f(t)$ is expressed as (1). Now we construct a 1-dimensional polyhedron K_f in $E^3 (\subset S^3)$ as follows: The left-most side of K_f is shown in Fig. 1. Then for each i ($i = 1, \dots, n$) we add step by step one of the four figures in Fig. 2. This depends on values of ε_i and

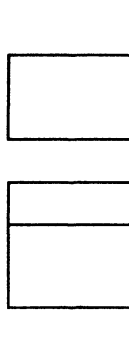


Fig. 1.

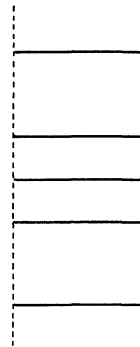


Fig. 3.

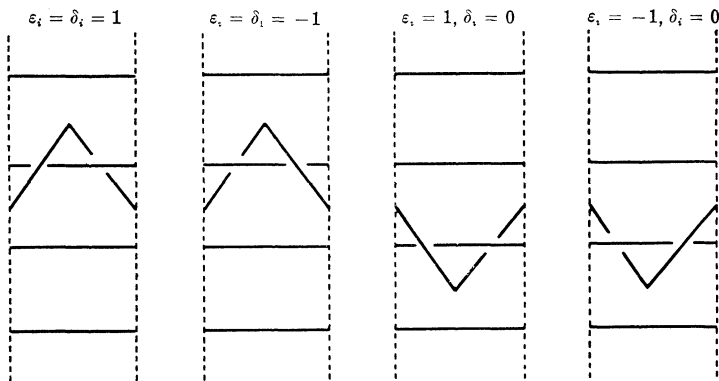


Fig. 2.

δ_i as in Fig. 2. The right-most side of K_f is shown in Fig. 3.

Now we give a presentation of the fundamental group of $E^3 - K_f$ (and that of $S^3 - K_f$, too). We use the Wirtinger presentation. If $a_0, \dots, a_n, c_0, \dots, c_m, d_0, \dots, d_m$, ($m + m' = n$) are paths in Fig. 4, and

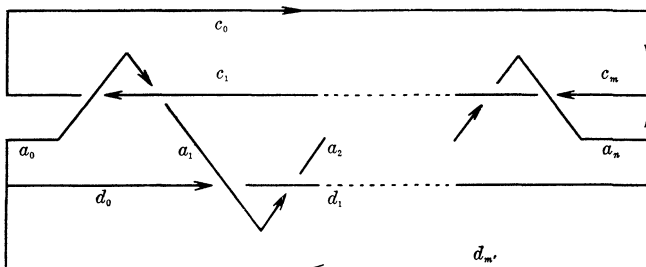


Fig. 4.

also, as usual, the paths which represent elements of the fundamental group in question, then the presentation is given as follows:

Generators:
$$\left\{ \begin{array}{l} a_0, \dots, a_n, \\ c_0, \dots, c_m, \\ d_0, \dots, d_m. (m + m' = n). \end{array} \right.$$

Relations:

(i) If $\varepsilon_i = 1, \delta_i = 1$, then

$$\left\{ \begin{array}{l} c_{j-1} = a_{i-1}c_j a_{i-1}^{-1}, \\ a_i = c_j a_{i-1} c_j^{-1}, \end{array} \right.$$

(ii) If $\varepsilon_i = -1, \delta_i = -1$, then

$$\left\{ \begin{array}{l} c_j = a_i c_{j-1} a_i^{-1}, \\ a_{i-1} = c_{j-1} a_i c_{j-1}^{-1}, \end{array} \right.$$

(iii) If $\varepsilon_i = 1, \delta_i = 0$, then

$$\left\{ \begin{array}{l} d_j = a_{i-1} d_{j-1} a_{i-1}^{-1}, \\ a_i = d_j a_{i-1} d_j^{-1}, \end{array} \right.$$

(iv) If $\varepsilon_i = -1, \delta_i = 0$, then

$$\left\{ \begin{array}{l} a_{i-1} = d_{j-1} a_i d_{j-1}^{-1}, \\ d_{j-1} = a_i d_j a_i^{-1}, \end{array} \right.$$

for each $i = 1, \dots, n$, and

$$c_0 c_m^{-1} a_n = 1.$$

3. Let k_f be a 1-cycle on K_f such that

$$\begin{cases} \text{link}(a_i, k_f) = 0, & \text{for } i = 0, 1, \dots, n, \\ \text{link}(c_i, k_f) = 1, & \text{for } i = 0, 1, \dots, m, \\ \text{link}(d_i, k_f) = 1, & \text{for } i = 0, 1, \dots, m'. \end{cases}$$

We consider the elementary ideals of k_f on K_f in S^3 . For each pair a_{i-1} and a_i the corresponding two rows in the Alexander matrix are elementary equivalent to the following:

(1) If $\varepsilon_i = \delta_i$, then

$$\begin{matrix} a_{i-1} & a_i & c_{j-1} & c_j \\ \left[\begin{array}{cccc} 1 - t^{\varepsilon_i} & 0 & -1 & 1 \\ t^{\varepsilon_i} & -1 & 0 & 0 \end{array} \right]. \end{matrix}$$

(2) If $\delta_i = 0$, then

$$\begin{matrix} a_{i-1} & a_i & d_{j-1} & d_j \\ \left[\begin{array}{cccc} 1 - t^{\varepsilon_i} & 0 & 1 & -1 \\ t^{\varepsilon_i} & -1 & 0 & 0 \end{array} \right]. \end{matrix}$$

From the last relation we have the following entries to the Alexander matrix.

$$\begin{matrix} a_n & c_0 & c_m \\ \left[\begin{array}{ccc} 1 & 1 & -1 \end{array} \right] \end{matrix}$$

Hence we have matrix (*) as an Alexander matrix of k_f on K_f in S^3 . Matrix (*) is elementary equivalent to (**). Note that we add a suitable number of rows of zeros. Matrix (**) can be reduced to (***) by elementary operations. Now it is easy to see that

$$(*) \left(\begin{array}{c|c|c} \begin{array}{cc} t^{\varepsilon_1} & -1 \\ & t^{\varepsilon_n} & -1 \end{array} & \begin{array}{c} 0 \\ 0 \end{array} & \begin{array}{c} 0 \\ 0 \end{array} \\ \hline \begin{array}{ccc} 1 & 1 & -1 \end{array} & \begin{array}{c} 0 \\ 0 \end{array} & \begin{array}{c} 0 \\ 0 \end{array} \\ \hline \begin{array}{c} 1 - t^{\varepsilon_i} \end{array} & \begin{array}{cc} -1 & 1 \\ & -1 & 1 \end{array} & \begin{array}{c} 0 \\ 0 \end{array} \\ \hline \begin{array}{c} 1 - t^{\varepsilon_i} \end{array} & \begin{array}{c} 0 \end{array} & \begin{array}{cc} 1 & -1 \\ & 1 & -1 \end{array} \end{array} \right).$$

$\varepsilon_i = \delta_i$

$\delta_i = 0$

$$(**) \left(\begin{array}{c|c|c}
 \begin{array}{cc} t^{\varepsilon_1} & -1 \\ & t^{\varepsilon_n} \end{array} & & \\
 \hline
 1 - t^{\delta_1} \dots 1 - t^{\delta_n} & 1 & \\
 \hline
 \begin{array}{cc} & -1 \\ -1 & t^{\delta_i} \end{array} & \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} & \\
 \hline
 & & \begin{matrix} 0 \\ 0 \end{matrix}
 \end{array} \right) \cdot \begin{matrix} \{\varepsilon_i = \gamma_i\} \\ \{\delta_i = 0\} \end{matrix}$$

$$(***) \left(\begin{array}{c|c|c}
 \begin{array}{cc} t^{\varepsilon_1} & -1 \\ & t^{\varepsilon_n} \end{array} & & \\
 \hline
 1 - t^{\delta_1} \dots 1 - t^{\delta_n} & 1 & \\
 \hline
 & & \begin{matrix} 0 \\ 0 \end{matrix}
 \end{array} \right) \cdot$$

$$\begin{cases} E_0(k_f) = E_1(k_f) = (0), \\ E_2(k_f) = (t^{-(\varepsilon_1 + \dots + \varepsilon_n)}(1 - t^{\delta_1}) + \dots \\ \quad + t^{-\varepsilon_n}(1 - t^{\delta_n}) + 1) = (f(t)), \\ E_d(k_f) = (1), \text{ if } d > 2. \end{cases}$$

4. *Proof of Theorem 1.* Let $f(t)$ with $f(1) = 1$ be given. First construct K_f in S^3 and k_f on K_f as in 2 and 3. The construction of the corresponding θ -curve L_f is shown in Fig. 5. The 1-cycle l_f on

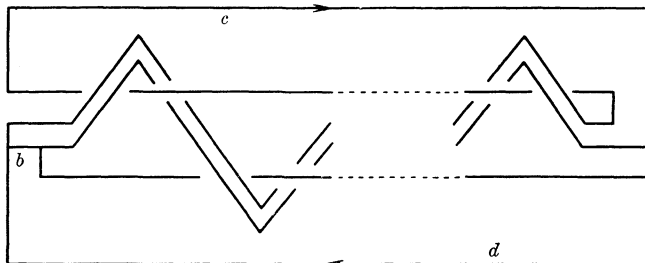


Fig. 5.

L_f has coefficient 1 on the oriented arc c and on the oriented arc d , respectively, and coefficient 0 on the arc b . It is easy to see that

$\pi(S^3 - L_f)$ is isomorphic to $\pi(S^3 - K_f)$ and $E_d(l_f) = E_d(k_f)$ for every nonnegative integer d .

REMARK. It is proved in [3] that if l is a 1-cycle on a θ -curve L in S^3 , then we have

$$\begin{cases} E_0(l) = E_1(l) = (0), & \text{and} \\ (E_d(l))^\circ = (1), & \text{if } d \geq 2, \end{cases}$$

where \circ is a trivializer (i.e., the operation to let $t = 1$ in $E_d(l(t))$).

5. *Proof of Theorem 2.* Let $f(t)$ with $f(1) = 1$ be given. First construct K_f in S^3 and k_f on K_f as in 2 and 3. Then construct the corresponding two arcs C and D in E_+^3 as in Fig. 6, where

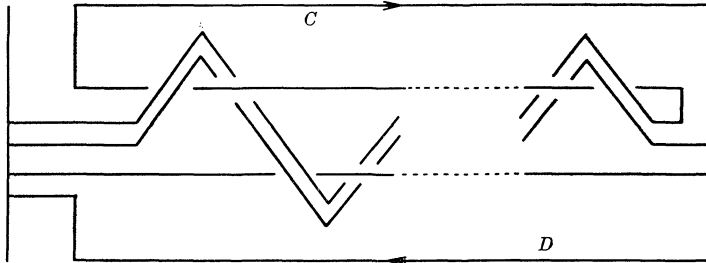


Fig. 6.

$$E_+^3 = \{(x_1, x_2, x_3) \mid x_1 \geq 0\}.$$

Then the usual construction of the spinning of these arcs around the plane

$$\{(x_1, x_2, x_3, x_4) \mid x_1 = 0, x_4 = 0\}$$

gives rise to a 2-link L_f in S^4 .

Now the arc C represents a trivial knot in E_+^3 . A part of the step to see this is shown in Fig. 7. From this it follows that the 2-sphere S_C^2 , which is the result of spinning C , is trivial in S^4 . Clearly the same is true for the 2-sphere S_D^2 , the result of spinning D .

We have

$$\pi(S^3 - K_f) \cong \pi(E_+^3 - C \cup D) \cong \pi(S^4 - L_f),$$

and to find a 2-cycle l_f on L_f that corresponds to k_f on K_f is easy. Then we have

$$E_d(k_f) = E_d(l_f)$$

for every $d \geq 0$. Hence the proof is complete.

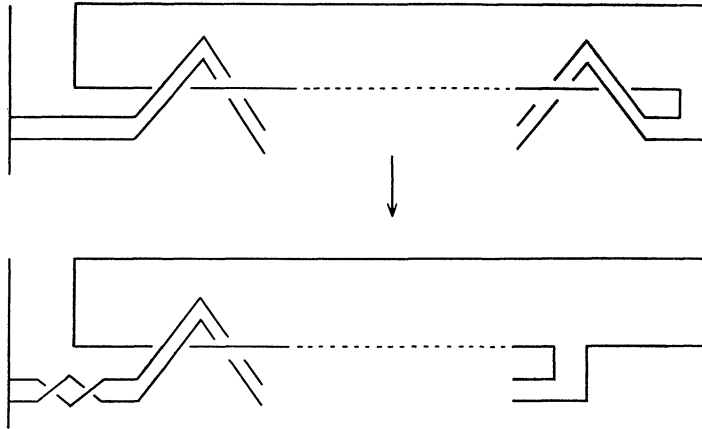


Fig. 7.

Proof of Corollary. We have $L_f = S_C^2 \cup S_D^2$ in S^4 in the example above. Then $l_f = l_c + l_d$, where l_c and l_d are fundamental cycles of S_C^2 and S_D^2 , respectively. This completes the proof.

REMARK. Let L be a 2-link in S^4 . Then it is known that for each 2-cycle l on L we always have

$$\begin{cases} E_0(l) = E_1(l) = 0, \\ (E_d(l))^\circ = (1), \text{ if } d \geq 2, \end{cases}$$

where \circ is a trivializer. (See [3] and [4].)

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Received August 3, 1972. The author of this paper is partially supported by NSF Grant GP-19964.

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