

# Pacific Journal of Mathematics

**ON ELEMENTARY IDEALS OF  $\theta$ -CURVES IN THE 3-SPHERE  
AND 2-LINKS IN THE 4-SPHERE**

SHIN'ICHI KINOSHITA

## ON ELEMENTARY IDEALS OF $\theta$ -CURVES IN THE 3-SPHERE AND 2-LINKS IN THE 4-SPHERE

SHIN'ICHI KINOSHITA

Let  $L$  be a polyhedron in an  $n$ -sphere  $S^n (n \geq 3)$  that does not separate  $S^n$ . A topological invariant of the position of  $L$  in  $S^n$  can be introduced as follows: Let  $l$  be an integral  $(n-2)$ -cycle on  $L$ . For each nonnegative integer  $d$ , the  $d$ th elementary ideal  $E_d(l)$  is associated to  $l$  on  $L$  in  $S^n$ . If  $l$  and  $l'$  are homologous on  $L$ , then  $E_d(l)$  is equal to  $E_d(l')$ . Now the collection of  $E_d(l)$  for all possible  $l$  is a topological invariant of  $L$  in  $S^n$ .

In this paper the following two cases of  $E_d(l)$  are considered: (1)  $l$  is a 1-cycle on a  $\theta$ -curve  $L$  in  $S^3$ , and (2)  $l$  is a 2-cycle on a 2-link  $L$  in  $S^4$ , i.e., the union of two disjoint 2-spheres in  $S^4$ , where each of two 2-spheres is trivially imbedded in  $S^4$ .

The  $d$ th elementary ideal  $E_d(l)$  of  $l$  on  $L$  is defined as follows (more precisely see [3]): Let  $G$  be the fundamental group  $\pi(S^n - L)$  and  $H$  the multiplicative infinite cyclic group generated by  $t$ . Let  $\psi$  be a homomorphism of  $G$  into  $H$  defined by

$$g^\psi = t^{\text{link}(g,l)},$$

where  $\text{link}(g, l)$  is the linking number between  $g$  and  $l$ . Using Fox's free differential calculus, we associate to  $\psi$  the  $d$ th elementary ideal  $E_d$  of the group  $G$ , evaluated in the group ring  $JH$  of  $H$  over integers. This  $d$ th elementary ideal  $E_d$  depends only on  $G$  and  $\psi$ , and hence it depends only on the position of  $l$  on  $L$  in  $S^n$ . We shall denote it by  $E_d(l)$ .

In this paper we shall prove the following two theorems.

**THEOREM 1.** *Let  $f(t)$  be an integral polynomial with  $f(1) = 1$ . Then there exists a  $\theta$ -curve  $L_f$  in  $S^3$ , and an integral 1-cycle  $l$  on  $L_f$  such that*

$$\begin{cases} E_0(l) = E_1(l) = (0), \\ E_2(l) = (f(t)) \text{ and} \\ E_d(l) = (1), \text{ if } d > 2. \end{cases}$$

**THEOREM 2.** *Let  $f(t)$  be an integral polynomial with  $f(1) = 1$ . Then there exists a 2-link  $L_f$  in  $S^4$ , and an integral 2-cycle  $l$  on  $L_f$  such that*

(1) *each component of  $L_f$  is a trivially imbedded 2-sphere in  $S^4$ , and that*

(2) we have

$$\begin{cases} E_0(l) = E_1(l) = (0), \\ E_2(l) = (f(t)) \text{ and} \\ E_d(l) = (1), \text{ if } d > 2. \end{cases}$$

**COROLLARY.** Let  $f(t)$  be an integral polynomial with  $f(1) = 1$ . Then there exists an oriented 2-link  $L_f$  in  $S^4$  such that

(1) each component of  $L_f$  is a trivial 2-sphere in  $S^4$ , and that

(2) the  $d$ th elementary ideal of  $L_f$ , in the usual sense and in the reduced form, is as follows:

$$\begin{cases} E_0(L_f) = E_1(L_f) = (0), \\ E_2(L_f) = (f(t)) \text{ and} \\ E_d(L_f) = (1), \text{ if } d > 2. \end{cases}$$

**REMARK.** This kind of example was first considered in [1].

The construction of these two examples are closely related. They are also closely related to the construction of 2-spheres in  $S^4$  in [2].

1. Let  $P$  be the family of all integral polynomials  $f(t)$  which can be expressed in the following form:

$$(1) \quad \begin{aligned} & t^{-(\varepsilon_1 + \dots + \varepsilon_n)}(1 - t^{\delta_1}) + t^{-(\varepsilon_2 + \dots + \varepsilon_n)}(1 - t^{\delta_2}) \\ & + \dots + t^{-\varepsilon_n}(1 - t^{\delta_n}) + 1, \end{aligned}$$

where  $\varepsilon_i = \pm 1$  and  $\delta_i = \varepsilon_i$  or  $\delta_i = 0$  for  $i = 1, 2, \dots, n$ . We assume that  $1 \in P$ .

**LEMMA.** We have  $f(t) \in P$ , if and only if  $f(1) = 1$ .

*Proof.* If  $f(t) \in P$ , then clearly we have  $f(1) = 1$ . Suppose that  $f(1) = 1$ . Then we have

$$\begin{aligned} f(t) - 1 &= (1 - t)(a_m t^m + \dots + a_0) \\ &\quad - (1 - t)(b_m t^m + \dots + b_0) \\ &= (1 - t)(a_m t^m + \dots + a_0) \\ &\quad + (1 - t^{-1})(b_m t^{m+1} + \dots + b_0 t), \end{aligned}$$

where  $a_i, b_i \geq 0$  for  $i = 1, 2, \dots, n$ . This means that  $f(t)$  with  $f(1) = 1$  can be obtained from 1 by applying a finite number of operation:

$$g(t) \rightarrow g(t) + t^p(1 - t^\delta),$$

where  $p \geq 0$  and  $\delta = \pm 1$ .

We assume  $1 \in P$ . Hence we should prove that if  $f(t) \in P$ , then  $f(t) + t^p(1 - t^\delta) \in P$ . Suppose that  $f(t)$  has form (1). Now let

$$p = -(\varepsilon'_1 + \dots + \varepsilon'_k + \varepsilon'_{k+1} + \dots + \varepsilon'_{k+n}),$$

where  $\varepsilon'_{k+i} = \varepsilon_i$  for  $i = 1, 2, \dots, n$  and let

$$\delta'_1 = \delta, \delta'_2 = \dots = \delta'_k = 0 \quad \text{and} \quad \delta'_{k+i} = \delta_i$$

for  $i = 1, 2, \dots, n$ . Then clearly we have

$$\begin{aligned} & t^{-(\varepsilon'_1 + \dots + \varepsilon'_k + \varepsilon'_{k+1} + \dots + \varepsilon'_{k+n})}(1 - t^{\delta'_1}) \\ & + \dots + t^{\varepsilon'_{k+n}}(1 - t^{\delta'_{k+n}}) = t^p(1 - t^\delta) + f(t). \end{aligned}$$

Hence the proof is complete.

2. Let  $f(t)$  be an integral polynomial with  $f(1) = 1$ . Suppose that  $f(t)$  is expressed as (1). Now we construct a 1-dimensional polyhedron  $K_f$  in  $E^3 (\subset S^3)$  as follows: The left-most side of  $K_f$  is shown in Fig. 1. Then for each  $i$  ( $i = 1, \dots, n$ ) we add step by step one of the four figures in Fig. 2. This depends on values of  $\varepsilon_i$  and

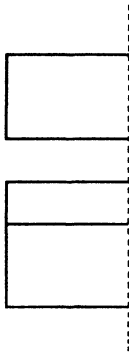


Fig. 1.

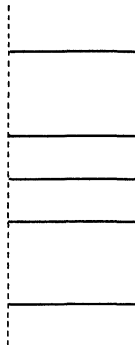


Fig. 3.

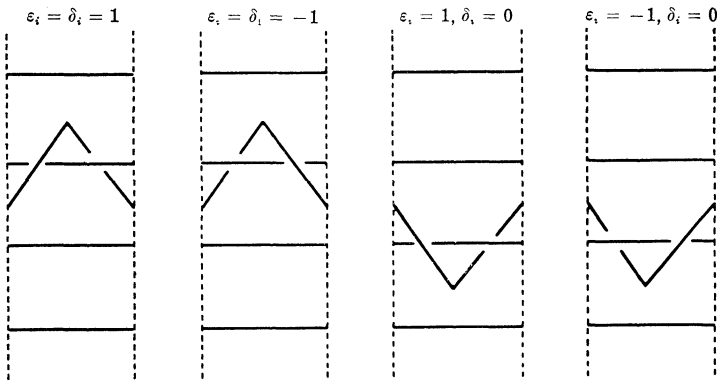


Fig. 2.

$\delta_i$  as in Fig. 2. The right-most side of  $K_f$  is shown in Fig. 3.

Now we give a presentation of the fundamental group of  $E^3 - K_f$  (and that of  $S^3 - K_f$ , too). We use the Wirtinger presentation. If  $a_0, \dots, a_n, c_0, \dots, c_m, d_0, \dots, d_m, (m + m' = n)$  are paths in Fig. 4, and

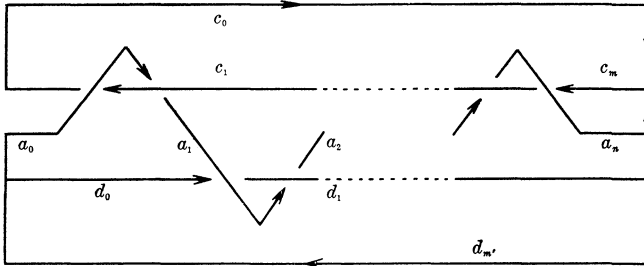


Fig. 4.

also, as usual, the paths which represent elements of the fundamental group in question, then the presentation is given as follows:

Generators: 
$$\begin{cases} a_0, \dots, a_n, \\ c_0, \dots, c_m, \\ d_0, \dots, d_{m'} (m + m' = n). \end{cases}$$

Relations:

(i) If  $\varepsilon_i = 1, \delta_i = 1$ , then

$$\begin{cases} c_{j-1} = a_{i-1}c_j a_{i-1}^{-1}, \\ a_i = c_j a_{i-1} c_j^{-1}, \end{cases}$$

(ii) If  $\varepsilon_i = -1, \delta_i = -1$ , then

$$\begin{cases} c_j = a_i c_{j-1} a_i^{-1}, \\ a_{i-1} = c_{j-1} a_i c_{j-1}^{-1}, \end{cases}$$

(iii) If  $\varepsilon_i = 1, \delta_i = 0$ , then

$$\begin{cases} d_j = a_{i-1} d_{j-1} a_{i-1}^{-1}, \\ a_i = d_j a_{i-1} d_j^{-1}, \end{cases}$$

(iv) If  $\varepsilon_i = -1, \delta_i = 0$ , then

$$\begin{cases} a_{i-1} = d_{j-1} a_i d_{j-1}^{-1}, \\ d_{j-1} = a_i d_j a_i^{-1}, \end{cases}$$

for each  $i = 1, \dots, n$ , and

$$c_0 c_m^{-1} a_n = 1.$$

3. Let  $k_f$  be a 1-cycle on  $K_f$  such that

$$\begin{cases} \text{link}(a_i, k_f) = 0, & \text{for } i = 0, 1, \dots, n, \\ \text{link}(c_i, k_f) = 1, & \text{for } i = 0, 1, \dots, m, \\ \text{link}(d_i, k_f) = 1, & \text{for } i = 0, 1, \dots, m'. \end{cases}$$

We consider the elementary ideals of  $k_f$  on  $K_f$  in  $S^3$ . For each pair  $a_{i-1}$  and  $a_i$  the corresponding two rows in the Alexander matrix are elementary equivalent to the following:

(1) If  $\varepsilon_i = \delta_i$ , then

$$\begin{bmatrix} a_{i-1} & a_i & c_{j-1} & c_j \\ 1 - t^{\varepsilon_i} & 0 & -1 & 1 \\ t^{\varepsilon_i} & -1 & 0 & 0 \end{bmatrix}.$$

(2) If  $\delta_i = 0$ , then

$$\begin{bmatrix} a_{i-1} & a_i & d_{j-1} & d_j \\ 1 - t^{\varepsilon_i} & 0 & 1 & -1 \\ t^{\varepsilon_i} & -1 & 0 & 0 \end{bmatrix}.$$

From the last relation we have the following entries to the Alexander matrix.

$$\begin{matrix} a_n & c_0 & c_m \\ [ 1 & 1 & -1 ] \end{matrix}$$

Hence we have matrix (\*) as an Alexander matrix of  $k_f$  on  $K_f$  in  $S^3$ . Matrix (\*) is elementary equivalent to (\*\*). Note that we add a suitable number of rows of zeros. Matrix (\*\*) can be reduced to (\*\*\*) by elementary operations. Now it is easy to see that

$$(*) \left[ \begin{array}{ccc|ccc} \begin{matrix} t^{\varepsilon_1} & -1 \\ & t^{\varepsilon_n} & -1 \end{matrix} & & & 0 & & 0 \\ \hline & 1 & 1 & -1 & & 0 \\ \hline & 1 - t^{\varepsilon_i} & -1 & 1 & & 0 \\ & & & -1 & 1 & \\ \hline 1 - t^{\varepsilon_i} & & & & 1 & -1 \\ & & 0 & & & 1 & -1 \end{array} \right]. \quad \begin{matrix} \varepsilon_i = \delta_i \\ \delta_i = 0 \end{matrix}$$

$$(**) \left( \begin{array}{c|cc}
 \begin{array}{cc} t^{\varepsilon_1} & -1 \\ & \vdots \\ & t^{\varepsilon_n} & -1 \end{array} & & \begin{array}{c} 0 \\ 0 \end{array} \\
 \hline
 1 - t^{\delta_1} \dots 1 - t^{\delta_n} \quad 1 & & \begin{array}{c} 0 \\ 0 \end{array} \\
 \hline
 \begin{array}{c} \vdots \\ \vdots \\ 1 - t^{\delta_i} \\ \vdots \\ \vdots \end{array} & \begin{array}{c} \vdots \\ \vdots \\ \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \\ \vdots \\ \vdots \end{array} & \begin{array}{c} 0 \\ \vdots \\ 0 \\ \vdots \end{array} \end{array} \right) \begin{array}{l} \\ \\ \left\{ \begin{array}{l} \varepsilon_i = \gamma_i \\ \delta_i = 0 \end{array} \right\} \\ \\ \end{array}$$

$$(***) \left( \begin{array}{c|cc}
 \begin{array}{cc} t^{\varepsilon_1} & -1 \\ & \vdots \\ & t^{\varepsilon_n} & -1 \end{array} & & \begin{array}{c} 0 \\ 0 \end{array} \\
 \hline
 1 - t^{\delta_1} \dots 1 - t^{\delta_n} \quad 1 & & \begin{array}{c} 0 \\ 0 \end{array} \end{array} \right)$$

$$\begin{cases} E_0(k_f) = E_1(k_f) = (0), \\ E_2(k_f) = (t^{-\varepsilon_1 + \dots + \varepsilon_n} (1 - t^{\delta_1}) + \dots \\ \quad + t^{-\varepsilon_n} (1 - t^{\delta_n}) + 1) = (f(t)), \\ E_d(k_f) = (1), \text{ if } d > 2. \end{cases}$$

4. *Proof of Theorem 1.* Let  $f(t)$  with  $f(1) = 1$  be given. First construct  $K_f$  in  $S^3$  and  $k_f$  on  $K_f$  as in 2 and 3. The construction of the corresponding  $\theta$ -curve  $L_f$  is shown in Fig. 5. The 1-cycle  $l_f$  on

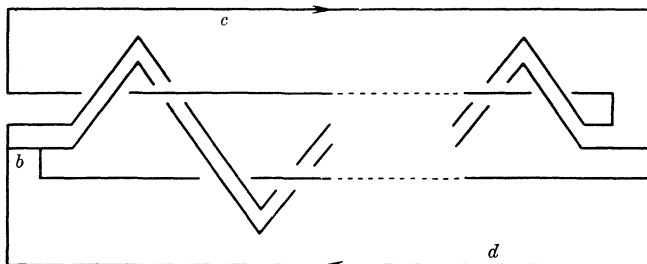


Fig. 5.

$L_f$  has coefficient 1 on the oriented arc  $c$  and on the oriented arc  $d$ , respectively, and coefficient 0 on the arc  $b$ . It is easy to see that

$\pi(S^3 - L_f)$  is isomorphic to  $\pi(S^3 - K_f)$  and  $E_d(l_f) = E_d(k_f)$  for every nonnegative integer  $d$ .

REMARK. It is proved in [3] that if  $l$  is a 1-cycle on a  $\theta$ -curve  $L$  in  $S^3$ , then we have

$$\begin{cases} E_0(l) = E_1(l) = (0), & \text{and} \\ (E_d(l))^\circ = (1), & \text{if } d \geq 2, \end{cases}$$

where  $\circ$  is a trivializer (i.e., the operation to let  $t = 1$  in  $E_d(l)(t)$ ).

5. *Proof of Theorem 2.* Let  $f(t)$  with  $f(1) = 1$  be given. First construct  $K_f$  in  $S^3$  and  $k_f$  on  $K_f$  as in 2 and 3. Then construct the corresponding two arcs  $C$  and  $D$  in  $E_+^3$  as in Fig. 6, where

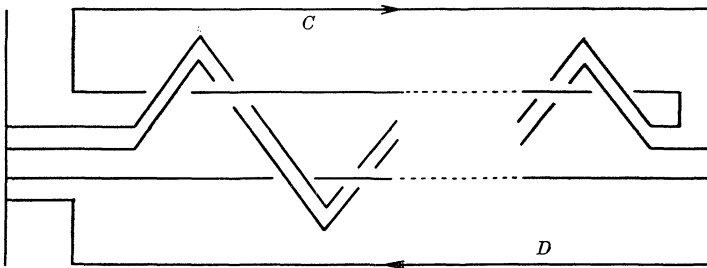


Fig. 6.

$$E_+^3 = \{(x_1, x_2, x_3) \mid x_1 \geq 0\}.$$

Then the usual construction of the spinning of these arcs around the plane

$$\{(x_1, x_2, x_3, x_4) \mid x_1 = 0, x_4 = 0\}$$

gives rise to a 2-link  $L_f$  in  $S^4$ .

Now the arc  $C$  represents a trivial knot in  $E_+^3$ . A part of the step to see this is shown in Fig. 7. From this it follows that the 2-sphere  $S_C^2$ , which is the result of spinning  $C$ , is trivial in  $S^4$ . Clearly the same is true for the 2-sphere  $S_D^2$ , the result of spinning  $D$ .

We have

$$\pi(S^3 - K_f) \cong \pi(E_+^3 - C \cup D) \cong \pi(S^4 - L_f),$$

and to find a 2-cycle  $l_f$  on  $L_f$  that corresponds to  $k_f$  on  $K_f$  is easy. Then we have

$$E_d(k_f) = E_d(l_f)$$

for every  $d \geq 0$ . Hence the proof is complete.



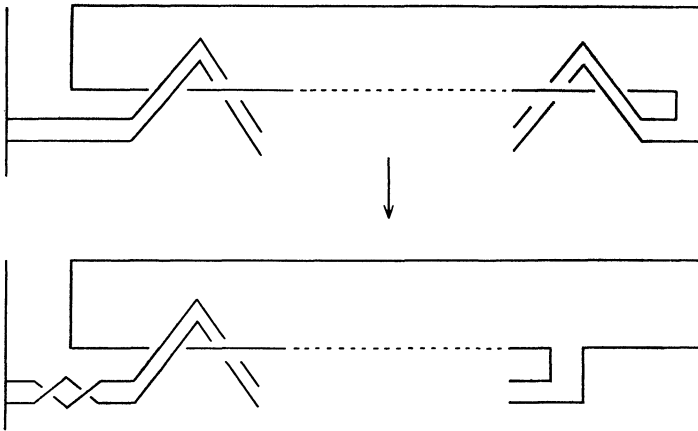


Fig. 7.

*Proof of Corollary.* We have  $L_f = S_C^2 \cup S_D^2$  in  $S^4$  in the example above. Then  $l_f = l_c + l_d$ , where  $l_c$  and  $l_d$  are fundamental cycles of  $S_C^2$  and  $S_D^2$ , respectively. This completes the proof.

REMARK. Let  $L$  be a 2-link in  $S^4$ . Then it is known that for each 2-cycle  $l$  on  $L$  we always have

$$\begin{cases} E_0(l) = E_1(l) = 0, \\ (E_d(l))^\circ = (1), \text{ if } d \geq 2, \end{cases}$$

where  $\circ$  is a trivializer. (See [3] and [4].)

#### REFERENCES

1. E. H. Van Kampen, *Zur Isotopie zweidimensionaler Flaechen im  $R^4$* , Abh. Math. Sem. Univ. Hamburg, **6** (1927), 216.
2. S. Kinoshita, *On the Alexander polynomials of 2-spheres in a 4-sphere*, Ann. of Math., **74** (1961), 518-531.
3. ———, *On elementary ideals of polyhedra in the 3-sphere*, to appear in the Pacific J. Math., **42** (1972), 89-98.
4. Y. Shinohara and D. W. Sumners, *Homology invariants of cyclic coverings with application to links*, Trans. Amer. Math. Soc., **163** (1972), 101-121.

Received August 3, 1972. The author of this paper is partially supported by NSF Grant GP-19964.

FLORIDA STATE UNIVERSITY

# PACIFIC JOURNAL OF MATHEMATICS

## EDITORS

RICHARD ARENS (Managing Editor)  
University of California  
Los Angeles, California 90024

J. DUGUNDJI\*  
Department of Mathematics  
University of Southern California  
Los Angeles, California 90007

R. A. BEAUMONT  
University of Washington  
Seattle, Washington 98105

D. GILBARG AND J. MILGRAM  
Stanford University  
Stanford, California 94305

## ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

## SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA  
CALIFORNIA INSTITUTE OF TECHNOLOGY  
UNIVERSITY OF CALIFORNIA  
MONTANA STATE UNIVERSITY  
UNIVERSITY OF NEVADA  
NEW MEXICO STATE UNIVERSITY  
OREGON STATE UNIVERSITY  
UNIVERSITY OF OREGON  
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA  
STANFORD UNIVERSITY  
UNIVERSITY OF TOKYO  
UNIVERSITY OF UTAH  
WASHINGTON STATE UNIVERSITY  
UNIVERSITY OF WASHINGTON  
\* \* \*  
AMERICAN MATHEMATICAL SOCIETY  
NAVAL WEAPONS CENTER

\* C. R. DePrima California Institute of Technology, Pasadena, CA 91109, will replace J. Dugundji until August 1974.

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

A. Bigard, <i>Free lattice-ordered modules</i> .....	1
Richard Bolstein and Warren R. Wogen, <i>Subnormal operators in strictly cyclic operator algebras</i> .....	7
Herbert Busemann and Donald E. Glassco, II, <i>Irreducible sums of simple multivectors</i> .....	13
W. Wistar (William) Comfort and Victor Harold Saks, <i>Countably compact groups and finest totally bounded topologies</i> .....	33
Mary Rodriguez Embry, <i>Maximal invariant subspaces of strictly cyclic operator algebras</i> .....	45
Ralph S. Freese and James Bryant Nation, <i>Congruence lattices of semilattices</i> .....	51
Ervin Fried and George Grätzer, <i>A nonassociative extension of the class of distributive lattices</i> .....	59
John R. Giles and Donald Otto Koehler, <i>On numerical ranges of elements of locally <math>m</math>-convex algebras</i> .....	79
David A. Hill, <i>On dominant and codominant dimension of <math>\mathbf{QF} - 3</math> rings</i> .....	93
John Sollion Hsia and Robert Paul Johnson, <i>Round and Pfister forms over <math>R(t)</math></i> .....	101
I. Martin (Irving) Isaacs, <i>Equally partitioned groups</i> .....	109
Athanasios G. Kartsatos and Edward Barry Saff, <i>Hyperpolynomial approximation of solutions of nonlinear integro-differential equations</i> .....	117
Shin'ichi Kinoshita, <i>On elementary ideals of <math>\theta</math>-curves in the 3-sphere and 2-links in the 4-sphere</i> .....	127
Ronald Brian Kirk, <i>Convergence of Baire measures</i> .....	135
R. J. Knill, <i>The Seifert and Van Kampen theorem via regular covering spaces</i> .....	149
Amos A. Kovacs, <i>Homomorphisms of matrix rings into matrix rings</i> .....	161
Young K. Kwon, <i>HD-minimal but no HD-minimal</i> .....	171
Makoto Maejima, <i>On the renewal function when some of the mean renewal lifetimes are infinite</i> .....	177
Juan José Martínez, <i>Cohomological dimension of discrete modules over profinite groups</i> .....	185
W. K. Nicholson, <i>Semiperfect rings with abelian group of units</i> .....	191
Louis Jackson Ratliff, Jr., <i>Three theorems on imbedded prime divisors of principal ideals</i> .....	199
Billy E. Rhoades and Albert Wilansky, <i>Some commutants in <math>B(c)</math> which are almost matrices</i> .....	211
John Philip Riley Jr., <i>Cross-sections of decompositions</i> .....	219
Keith Duncan Stroyan, <i>A characterization of the Mackey uniformity <math>m(L^\infty, L^1)</math> for finite measures</i> .....	223
Edward G. Thurber, <i>The Scholz-Brauer problem on addition chains</i> .....	229
Joze Vrabec, <i>Submanifolds of acyclic 3-manifolds</i> .....	243
Philip William Walker, <i>Adjoint boundary value problems for compactified singular differential operators</i> .....	265
Roger P. Ware, <i>When are Witt rings group rings</i> .....	279
James D. Wine, <i>Paracompactifications using filter bases</i> .....	285