

# Pacific Journal of Mathematics

***HD-MINIMAL BUT NO HD-MINIMAL***

YOUNG K. KWON

## $\widetilde{HD}$ -MINIMAL BUT NO $HD$ -MINIMAL

YOUNG K. KWON

Let  $U_{HD}^k$  (resp.  $U_{\widetilde{HD}}^k$ ) be the class of Riemannian  $n$ -manifolds ( $n \geq 2$ ) on which there exist  $k$  non-proportional  $HD$ -minimal (resp.  $\widetilde{HD}$ -minimal) functions. The purpose of the present paper is to construct a Riemannian  $n$ -manifold  $n \geq 3$  which carries a unique (up to constant factors)  $\widetilde{HD}$ -minimal function but no  $HD$ -minimal functions. Thus the inclusion relation

$$U_{HD}^1 \subset U_{\widetilde{HD}}^1$$

is strict for  $n \geq 3$ . By welding  $k$  copies of this Riemannian  $n$ -manifold, it is then established that the inclusion relation

$$U_{HD}^k \subset U_{\widetilde{HD}}^k$$

is strict for all  $k \geq 1$  and  $n \geq 3$ . The problem still remains open for  $n = 2$ .

1. An  $HD$ -function (harmonic and Dirichlet-finite)  $\omega$  on a Riemannian  $n$ -manifold  $M$  is called  $HD$ -minimal on  $M$  if  $\omega$  is positive on  $M$  and every  $HD$ -function  $\omega'$  with  $0 < \omega' \leq \omega$  reduces to a constant multiple of  $\omega$  on  $M$ . Let  $\{\omega_n\}$  be a sequence of positive  $HD$ -functions on  $M$ . If the sequence  $\{\omega_n\}$  decreases on  $M$ , the limit function is harmonic on  $M$  by Harnack's inequality. Such a harmonic function is called an  $\widetilde{HD}$ -function on  $M$ , and  $\widetilde{HD}$ -minimality can be defined as in the case of  $HD$ -minimal functions.

These functions were introduced by Constantinescu and Cornea [1] and systematically studied by Nakai [6]. In particular the following characterization by Nakai is important (loc. cit., cf. also Kwon-Sario [5]):

(i) A Riemannian  $n$ -manifold  $M$  carries an  $HD$ -minimal function  $\omega$  if and only if the Royden harmonic boundary  $\Delta_M$  of  $M$  contains a point  $p$ , isolated in  $\Delta_M$ . In this case  $\omega(p) > 0$  and  $\omega \equiv 0$  on  $\Delta_M - \{p\}$ .

(ii) A Riemannian  $n$ -manifold  $M$  carries an  $\widetilde{HD}$ -minimal function  $\omega$  if and only if the Royden harmonic boundary  $\Delta_M$  of  $M$  has a point  $p$  of positive harmonic measure. These are corresponded such that  $\limsup_{x \in M, x \rightarrow p} \omega(x) > 0$  and  $\limsup_{x \in M, x \rightarrow q} \omega(x) = 0$  for almost all  $q \in \Delta_M - \{p\}$  with respect to a harmonic measure on  $\Delta_M$ .

Since an isolated point of  $\Delta_M$  has a positive harmonic measure, the above characterization yields the inclusion

$$U_{HD}^k \subset U_{\widetilde{HD}}^k$$

for all  $k \geq 1$ .

For the notation and terminology we refer the reader to the monograph by Sario-Nakai [7].

2. Let  $n \geq 3$ . Denote by  $M_0$  the punctured Euclidean  $n$ -space  $R^n - 0$  with the Riemannian metric tensor

$$g_{ij}(x) = |x|^{-4}(1 + |x|^{n-2})^{4/(n-2)}\delta_{ij}, \quad 1 \leq i, j \leq n$$

where  $|x| = [\sum_{i=1}^n (x^i)^2]^{1/2}$  for  $x = (x^1, x^2, \dots, x^n) \in M_0$ .

For each pair  $(m, l)$  of positive integers  $m, l$ , set

$$H_{m,l} = \{8^k x \in M_0 \mid |x| = 1 \text{ and } x^1 \geq 0\},$$

where  $k = 2^{m-1}(2l-1) - 1$ , and  $ax = (ax^1, ax^2, \dots, ax^n)$  for  $x = (x^1, x^2, \dots, x^n) \in M_0$  and real  $a$ . Let  $M'_0$  be the slit manifold obtained from  $M_0$  by deleting all the closed hemispheres  $H_{m,l}$ . Take a sequence  $\{M'_0(l)\}_1^\infty$  of copies of  $M'_0$ . For each fixed  $m \geq 1$  and subsequently for fixed  $j \geq 0$  and  $1 \leq i \leq 2^{m-1}$ , connect  $M'_0(i + 2^m j)$ , crosswise along all the hemispheres  $H_{m,l}$  ( $l \geq 1$ ), with  $M'_0(i + 2^{m-1} + 2^m j)$ .

The resulting Riemannian  $n$ -manifold  $N$  is an infinitely sheeted covering manifold of  $M_0$ . Let  $\pi: N \rightarrow M_0$  be the natural projection.

The following result is essential to our problem (Kwon [4]):

**THEOREM 1.** *A function  $u(x)$  is harmonic on  $N$  if and only if  $[1 + |\pi(x)|^{2-n}]u(x)$  is  $\Delta_c$ -harmonic (harmonic with respect to the Euclidean structure) on  $N$ . In particular every bounded harmonic function  $u(x)$  on the submanifold*

$$G = \left\{x \in N \mid |\pi(x)| > \frac{1}{3}\right\}$$

is constant on  $\pi^{-1}(x)$  for each  $x \in M_0$  whenever it continuously vanishes on

$$\partial G = \left\{x \in N \mid |\pi(x)| = \frac{1}{3}\right\}.$$

3. For each integer  $l \geq 1$ , consider the subset of  $N$ :

$$N_l = [M'_0(l)] \cup \left[ \bigcup_{i \neq l} G_i \right]$$

where

$$G_i = \left\{x \in M'_0(i) \mid |\pi(x)| > \frac{1}{3}\right\}.$$

It is obvious that

$$G = \bigcup_{i=1}^{\infty} G_i$$

and the Riemannian  $n$ -manifold  $G$  is an infinitely sheeted covering manifold of the annulus  $\{x \in M_0 \mid 1/3 < |x| < \infty\}$ .

We are now ready to state our main result:

**THEOREM 2.** *The Riemannian  $n$ -manifold  $G$  ( $n \geq 3$ ) carries a unique (up to constant factors)  $\widetilde{HD}$ -minimal function but no  $HD$ -minimal functions. Thus the inclusion*

$$U_{HD}^1 \subset U_{\widetilde{HD}}^1$$

is strict for Riemannian manifolds of  $\dim \geq 3$ .

The proof will be given in 4 – 5.

4. For  $m \geq 1$  construct  $u_m \in HBD(N_m)$ , the class of bounded  $HD$ -functions on  $N_m$ , such that  $0 \leq u_m \leq 1$  on  $N$ ,  $u_m \equiv 0$  on  $\bigcup_{i=1}^{m-1} [M'_0(i) - G_i]$ , and  $u_m \equiv 1$  on  $\bigcup_{i=m+1}^{\infty} [M'_0(i) - G_i]$ . Clearly  $u_m \geq u_{m+1}$  on  $N$  and therefore the sequence  $\{u_m\}$  converges to an  $\widetilde{HD}$ -function  $u$  on  $G$ , uniformly on compact subsets of  $G$ . It is easy to see that  $0 \leq u < 1$  on  $G$  and  $u|_{N-G} \equiv 0$ . Since

$$u_m(x) \geq \frac{|\pi(x)|^{n-2} - 3^{2-n}}{|\pi(x)|^{n-2} + 1}$$

on  $G$  by maximum principle and Theorem 1, it follows that  $0 < u < 1$  on  $G$ . Note that  $\lim_{|\pi(x)| \rightarrow \infty} u_m(x) = 1$ .

We claim that the function  $u$  is  $\widetilde{HD}$ -minimal on  $G$ . In fact, let  $v \in \widetilde{HD}(G)$  be such that  $0 < v \leq u$  on  $G$ . In view of

$$0 \leq \limsup_{x \in G, x \rightarrow y} v(x) \leq \limsup_{x \in G, x \rightarrow y} u(x) = 0$$

for all  $y \in \partial G$ ,  $v$  can be continuously extended to  $N$  by setting  $v \equiv 0$  on  $N - G$ . By Theorem 1  $v$  attains the same value at all the points in  $N$  which lie over the same point in  $M_0$ . Thus we may assume that  $u, v$  are bounded harmonic functions on  $\pi(G) = \{\pi(x) \mid x \in G\}$  such that  $u, v \equiv 0$  on  $\pi(\partial G)$ .

Again by Theorem 1,  $(1 + |x|^{2-n})v(x)$  is  $\Delta_e$ -harmonic on  $\pi(G)$ . In view of the fact that  $\Delta_e$ -harmonicity is invariant by the Kelvin transformation, the function

$$\frac{1}{3^{n-2}|x|^{n-2}}(1 + 3^{2(n-2)}|x|^{n-2})v\left(\frac{x}{9|x|^2}\right)$$

is  $\Delta_e$ -harmonic on  $M_0$  for  $0 < |x| < 1/3$  and continuously vanishes for

$|x| = 1/3$ . Therefore, there exists a constant  $\alpha \geq 0$  such that

$$v\left(\frac{x}{9|x|^2}\right) = \frac{3^{n-2}\alpha}{1 + 3^{2(n-2)}|x|^{n-2}}$$

on  $M_0$  for  $0 < |x| < 1/3$  (cf., e.g. Helms [3, p. 81]). Thus

$$\lim_{x \rightarrow 0} v\left(\frac{x}{9|x|^2}\right) = 3^{n-2}\alpha$$

exists and  $v = 3^{n-2}\alpha u$  on  $G$ , as desired.

5. Suppose that there exists another  $\widetilde{HD}$ -minimal function  $\omega$  on  $G$ . Choose a point  $q \in \Delta_{M,G}$ , the Royden harmonic boundary of  $G$ , such that  $q$  has a positive harmonic measure and

$$\limsup_{x \in G, x \rightarrow q'} \omega(x) = 0$$

for almost all  $q' \in \Delta_{M,G} - \{q\}$  relative to a harmonic measure for  $G$ . Let  $j: G^* \rightarrow \bar{G} \subset N^*$  be the subjective continuous mapping such that  $j|_G$  is the identity mapping and  $f(x) = f(j(x))$  for all  $x \in G^*$ , the Royden compactification of  $G$ , and  $f \in M(N)$ , the Royden algebra of  $N$ . Here  $\bar{G}$  is the closure of  $G$  in  $N^*$ . Note that a Borel set  $E \subset \partial G$  has a positive harmonic measure if and only if  $j^{-1}(E)$  has a positive harmonic measure (cf. Sario-Nakai [7, p. 192]). Therefore,  $j(q) \notin \partial G$  and  $\partial G \subset j(\Delta_{M,G})$ .

For each  $m \geq 1$ ,  $u_m(q) = u_m(j(q)) = 1$  since  $j(q) \in \bar{\partial G} - \partial G$ . Thus it is not difficult to see that  $0 < \omega \leq \beta u_m$  on  $G$ , where

$$\beta = \limsup_{x \in G, x \rightarrow q} \omega(x) > 0.$$

Therefore,  $0 < \omega \leq \beta u$  on  $G$  and  $\omega$  is a constant multiple of  $u$  on  $G$  as in 4.

It remains to show that  $u$  is not  $HD$ -minimal on  $G$ . If it were,  $u$  would have a finite Dirichlet integral. But  $u$  has a continuous extension to  $G \cup \partial G$  with  $u|_{\partial G} \equiv 0$ . Then by Theorem 1  $u$  must attain the same value at all the points in  $G$  which lie over the same point in  $\pi(G)$ , a contradiction.

This completes the proof of Theorem 2.

6. Let  $G'$  be the Riemannian  $n$ -manifold obtained from  $G$  by deleting two disjoint closed subsets  $B, C$ , where

$$B = \left\{x \in M'_0(1) \mid |x| = \frac{9}{24} \text{ and } x^1 \geq 0\right\},$$

$$C = \left\{x \in M'_0(1) \mid |x| = \frac{11}{24} \text{ and } x^1 \geq 0\right\}.$$

For each  $k \geq 2$  take  $k$  copies  $G_1, G_2, \dots, G_k$  of  $G'$ , and identify, crosswise,  $B_i$  with  $C_{i+1}$  for  $1 \leq i \leq m$ . Here we set  $C_{m+1} = C_1$ . Then it is easy to see that the resulting Riemannian  $n$ -manifold  $G^{(k)}$  has exactly  $k$  non-proportional  $\widehat{HD}$ -minimal functions but no  $HD$ -minimal functions.

COROLLARY. For all  $k \geq 1$  the strict inclusion

$$U_{HD}^k < U_{\widehat{HD}}^k$$

holds for Riemannian manifolds of  $\dim \geq 3$ .

7. For the sake of completeness we shall sketch a proof of Theorem 1. In view of the simple relation

$$\Delta u = |x|^{n+2}(1 + |x|^{n-2})^{-(n+2)/(n-2)} \cdot \Delta_e[(1 + |\pi x|^{2-n})u],$$

it suffices to show the latter half.

For each integer  $k \geq 0$  let  $U_k$  be a component of the open set

$$\{x \in N \mid 2^{3k-1} < |\pi(x)| < 2^{3k+1}\},$$

and  $S_k$  a compact subset of  $U_k$  which lie over the set

$$\{x \in M_0 \mid |x| = 2^{3k}\}.$$

Since  $U_k$  is a magnification of  $U_0$  and the  $\Delta_e$ -harmonicity is invariant under a magnification, it is not difficult to see that there exists a constant  $q$ ,  $0 < q < 1$ , such that

$$|u(x)| \leq q \cdot \sup \{|u(x)| \mid x \in U_k\}$$

on  $S_k$  for any harmonic function  $u$  on  $U_k$  which changes sign on  $S_k$ . Note that  $q$  is independent of  $k$ .

Let  $u$  be a harmonic function on  $G$  such that  $|u| \leq 1$  and it continuously vanishes on  $\partial G$ . For each  $m \geq 1$ , denote by  $\pi_m$  the cover transformation of  $G$  which interchanges the sheets of  $G$ : the points in  $G \cap M'_0(i + 2^m j)$  are interchanged with points, with the same projection, in  $M'_0(i + 2^{m-1} + 2^m j)$  for  $j \geq 0$  and  $1 \leq i \leq 2^{m-1}$ . Define  $v_m$  on  $G$  by

$$v_m(x) = \frac{1}{2}[u(x) - u(\pi_m(x))].$$

Clearly  $v_m$  is harmonic on  $G$ ,  $|v_m| \leq 1$ , and  $v_m$  changes sign on  $S_k$ ,  $k = 2^{m-1}(2l - 1) - 1$ . Therefore,

$$\max \{|v_m(x)| \mid x \in S_k\} \leq q$$

for all  $l \geq 1$ . By induction on  $l$ , we derive that  $|v_m| \leq q^l$  on  $S_{k'}$ , where  $k' = 2^{m-1} - 1$ . Letting  $l \rightarrow \infty$ , we conclude that  $v_m \equiv 0$  on  $G$ , as desired.

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Received August 21, 1972 and in revised form January 17, 1973.

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