HD-MINIMAL BUT NO HD-MINIMAL

YOUNG K. KWON
\textbf{\(\widetilde{HD}\)-MINIMAL BUT NO \(HD\)-MINIMAL}

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Let \(U_{HD}^k\) (resp. \(U_{\tilde{HD}}^k\)) be the class of Riemannian \(n\)-manifolds \((n \geq 2)\) on which there exist \(k\) non-proportional \(HD\)-minimal (resp. \(\tilde{HD}\)-minimal) functions. The purpose of the present paper is to construct a Riemannian \(n\)-manifold \(n \geq 3\) which carries a unique (up to constant factors) \(\tilde{HD}\)-minimal function but no \(HD\)-minimal functions. Thus the inclusion relation

\[ U_{HD}^1 \subset U_{\tilde{HD}}^1 \]

is strict for \(n \geq 3\). By welding \(k\) copies of this Riemannian \(n\)-manifold, it is then established that the inclusion relation

\[ U_{HD}^k \subset U_{\tilde{HD}}^k \]

is strict for all \(k \geq 1\) and \(n \geq 3\). The problem still remains open for \(n = 2\).

1. An \(HD\)-function (harmonic and Dirichlet-finite) \(\omega\) on a Riemannian \(n\)-manifold \(M\) is called \(HD\)-minimal on \(M\) if \(\omega\) is positive on \(M\) and every \(HD\)-function \(\omega'\) with \(0 < \omega' \leq \omega\) reduces to a constant multiple of \(\omega\) on \(M\). Let \(\{\omega_n\}\) be a sequence of positive \(HD\)-functions on \(M\). If the sequence \(\{\omega_n\}\) decreases on \(M\), the limit function is harmonic on \(M\) by Harnack’s inequality. Such a harmonic function is called an \(\widetilde{HD}\)-function on \(M\), and \(\widetilde{HD}\)-minimality can be defined as in the case of \(HD\)-minimal functions.

These functions were introduced by Constantinescu and Cornea [1] and systematically studied by Nakai [6]. In particular the following characterization by Nakai is important (loc. cit., cf. also Kwon-Sario [5]):

(i) A Riemannian \(n\)-manifold \(M\) carries an \(HD\)-minimal function \(\omega\) if and only if the Royden harmonic boundary \(\Delta_M\) of \(M\) contains a point \(p\), isolated in \(\Delta_M\). In this case \(\omega(p) > 0\) and \(\omega = 0\) on \(\Delta_M - \{p\}\).

(ii) A Riemannian \(n\)-manifold \(M\) carries an \(\widetilde{HD}\)-minimal function \(\omega\) if and only if the Royden harmonic boundary \(\Delta_M\) of \(M\) has a point \(p\) of positive harmonic measure. These are corresponded such that \(\limsup_{x \in M, x \to p} \omega(x) > 0\) and \(\limsup_{x \in M, x \to q} \omega(x) = 0\) for almost all \(q \in \Delta_M - \{p\}\) with respect to a harmonic measure on \(\Delta_M\).

Since an isolated point of \(\Delta_M\) has a positive harmonic measure, the above characterization yields the inclusion

\[ U_{HD}^k \subset U_{\tilde{HD}}^k \]
for all $k \geq 1$.

For the notation and terminology we refer the reader to the monograph by Sario-Nakai [7].

2. Let $n \geq 3$. Denote by $M_0$ the punctured Euclidean $n$-space $\mathbb{R}^n - 0$ with the Riemannian metric tensor

$$g_{ij}(x) = |x|^{-4}(1 + |x|^{n-2})^{4/(n-2)} \delta_{ij}, \quad 1 \leq i, j \leq n$$

where $|x| = \left[\sum_{i=1}^{n} (x_i^2)^{1/2}\right]$ for $x = (x^1, x^2, \cdots, x^n) \in M_0$.

For each pair $(m, l)$ of positive integers $m, l$, set

$$H_{ml} = \{x \in M_0 \mid |x| = 1 \text{ and } x^1 \geq 0\},$$

where $k = 2^{m-1}(2l-1) - 1$, and $ax = (ax^1, ax^2, \cdots, ax^n)$ for $x = (x^1, x^2, \cdots, x^n) \in M_0$ and real $a$. Let $M'_0$ be the slit manifold obtained from $M_0$ by deleting all the closed hemispheres $H_{ml}$. Take a sequence $\{M'_0(l)\}_{l=1}^{\infty}$ of copies of $M'_0$. For each fixed $m \geq 1$ and subsequently for fixed $j \geq 0$ and $1 \leq i \leq 2^{m-1}$, connect $M'_0(i + 2^m j)$, crosswise along all the hemispheres $H_{ml}(l \geq 1)$, with $M'_0(i + 2^{m-1} + 2^m j)$.

The resulting Riemannian $n$-manifold $N$ is an infinitely sheeted covering manifold of $M_0$. Let $\pi: N \rightarrow M_0$ be the natural projection.

The following result is essential to our problem (Kwon [4]):

**Theorem 1.** A function $u(x)$ is harmonic on $N$ if and only if $[1 + |\pi(x)|^{2-n}]u(x)$ is $\Delta_e$-harmonic (harmonic with respect to the Euclidean structure) on $N$. In particular every bounded harmonic function $u(x)$ on the submanifold

$$G = \left\{x \in N \mid |\pi(x)| > \frac{1}{3}\right\}$$

is constant on $\pi^{-1}(x)$ for each $x \in M_0$ whenever it continuously vanishes on

$$\partial G = \left\{x \in N \mid |\pi(x)| = \frac{1}{3}\right\}.$$

3. For each integer $l \geq 1$, consider the subset of $N$:

$$N_l = [M'_0(l)] \cup \left[\bigcup_{i \neq l} G_i\right]$$

where

$$G_i = \left\{x \in M'_0(i) \mid |\pi(x)| > \frac{1}{3}\right\}.$$

It is obvious that
and the Riemannian $n$-manifold $G$ is an infinitely sheeted covering manifold of the annulus \( \{x \in M_0 \mid 1/3 < |x| < \infty \} \).

We are now ready to state our main result:

**Theorem 2.** The Riemannian $n$-manifold $G$ ($n \geq 3$) carries a unique (up to constant factors) $\hat{HD}$-minimal function but no $HD$-minimal functions. Thus the inclusion

\[
U_{\hat{H}D} \subset U^1_{\hat{H}D}
\]

is strict for Riemannian manifolds of $\dim \geq 3$.

The proof will be given in 4 - 5.

4. For $m \geq 1$ construct $u_m \in HBD(N_m)$, the class of bounded $HD$-functions on $N_m$, such that $0 \leq u_m \leq 1$ on $N$, $u_m \equiv 0$ on $\bigcup_{i=1}^{m-1} [M'_i(i) - G_i]$, and $u_m \equiv 1$ on $\bigcup_{i=m+1}^{\infty} [M'_i(i) - G_i]$. Clearly $u_m \geq u_{m+1}$ on $N$ and therefore the sequence \{u_m\} converges to an $\hat{HD}$-function $u$ on $G$, uniformly on compact subsets of $G$. It is easy to see that $0 \leq u < 1$ on $G$ and $u \mid N - G \equiv 0$. Since

\[
u_m(x) \geq \frac{|\pi(x)|^{n-2} - 3^{2-n}}{|\pi(x)|^{n-2} + 1}
\]

on $G$ by maximum principle and Theorem 1, it follows that $0 < u < 1$ on $G$. Note that $\lim_{|\pi(x)| \to \infty} u_m(x) = 1$.

We claim that the function $u$ is $\hat{HD}$-minimal on $G$. In fact, let $v \in \hat{HD}(G)$ be such that $0 < v \leq u$ on $G$. In view of

\[
0 \leq \lim sup_{x \in G, x \to y} v(x) \leq \lim sup_{x \in G, x \to y} u(x) = 0
\]

for all $y \in \partial G$, $v$ can be continuously extended to $N$ by setting $v \equiv 0$ on $N - G$. By Theorem 1 $v$ attains the same value at all the points in $N$ which lie over the same point in $M_0$. Thus we may assume that $u$, $v$ are bounded harmonic functions on $\pi(G) = \{\pi(x) \mid x \in G\}$ such that $u$, $v \equiv 0$ on $\pi(\partial G)$.

Again by Theorem 1, $(1 + |x|^{2-n})v(x)$ is $\Delta_\varepsilon$-harmonic on $\pi(G)$. In view of the fact that $\Delta_\varepsilon$-harmonicity is invariant by the Kelvin transformation, the function

\[
\frac{1}{3^{n-2}} |x|^{n-2}(1 + 3^{2(n-2)} |x|^{n-2})v\left(\frac{x}{9|x|^2}\right)
\]

is $\Delta_\varepsilon$-harmonic on $M_0$ for $0 < |x| < 1/3$ and continuously vanishes for
$|x| = 1/3$. Therefore, there exists a constant $a \geq 0$ such that

$$v\left(\frac{x}{9|x|^2}\right) = \frac{3^{n-2}a}{1 + 3^{2(n-3)}|x|^{n-2}}$$

on $M_0$ for $0 < |x| < 1/3$ (cf., e.g. Helms [3, p. 81]). Thus

$$\lim_{x \to 0} v\left(\frac{x}{9|x|^2}\right) = 3^{n-2}a$$

exists and $v = 3^{n-2}au$ on $G$, as desired.

5. Suppose that there exists another $\widetilde{H\!D}$-minimal function $\omega$ on $G$. Choose a point $q \in \Delta_{M,G}$, the Royden harmonic boundary of $G$, such that $q$ has a positive harmonic measure and

$$\limsup_{x \to q'} \omega(x) = 0$$

for almost all $q' \in \Delta_{M,G} - \{q\}$ relative to a harmonic measure for $G$. Let $j: G^* \to \overline{G} \subset N^*$ be the subjective continuous mapping such that $j|G$ is the identity mapping and $f(x) = f(j(x))$ for all $x \in G^*$, the Royden compactification of $G$, and $f \in M(N)$, the Royden algebra of $N$. Here $\overline{G}$ is the closure of $G$ in $N^*$. Note that a Borel set $E \subset \partial G$ has a positive harmonic measure if and only if $j^{-1}(E)$ has a positive harmonic measure (cf. Sario-Nakai [7, p. 192]). Therefore, $j(q) \in \partial G$ and $\partial G \subset j(\Delta_{M,G})$.

For each $m \geq 1$, $u_m(q) = u_m(j(q)) = 1$ since $j(q) \in \overline{G} - \partial G$. Thus it is not difficult to see that $0 < \omega \leq \beta u_m$ on $G$, where

$$\beta = \limsup_{x \to q'} \omega(x) > 0.$$ 

Therefore, $0 < \omega \leq \beta u$ on $G$ and $\omega$ is a constant multiple of $u$ on $G$ as in 4.

It remains to show that $u$ is not $H\!D$-minimal on $G$. If it were, $u$ would have a finite Dirichlet integral. But $u$ has a continuous extension to $\overline{G} \cup \partial G$ with $u|\partial G \equiv 0$. Then by Theorem 1 $u$ must attain the same value at all the points in $G$ which lie over the same point in $\pi(G)$, a contradiction.

This completes the proof of Theorem 2.

6. Let $G'$ be the Riemannian $n$-manifold obtained from $G$ by deleting two disjoint closed subsets $B$, $C$, where

$$B = \left\{ x \in M_0(1) \mid |x| = \frac{9}{24} \text{ and } x^i \geq 0 \right\},$$

$$C = \left\{ x \in M_0(1) \mid |x| = \frac{11}{24} \text{ and } x^i \geq 0 \right\}.$$
For each \( k \geq 2 \) take \( k \) copies \( G_1, G_2, \ldots, G_k \) of \( G' \), and identify, crosswise, \( B_i \) with \( C_{i+1} \) for \( 1 \leq i \leq m \). Here we set \( C_{n+1} = C_1 \). Then it is easy to see that the resulting Riemannian \( n \)-manifold \( G^{(k)} \) has exactly \( k \) non-proportional HD-minimal functions but no HD-minimal functions.

**COROLLARY.** For all \( k \geq 1 \) the strict inclusion
\[
U_{HD}^k < U_{HD}^k
\]
holds for Riemannian manifolds of \( \dim \geq 3 \).

7. For the sake of completeness we shall sketch a proof of Theorem 1. In view of the simple relation
\[
\Delta u = |x|^{n+2}(1 + |x|^{n-2})^{-2} \cdot \Delta_s[(1 + |\pi x|^{2-n})u],
\]
it suffices to show the latter half.

For each integer \( k \geq 0 \) let \( U_k \) be a component of the open set
\[
\{x \in \mathbb{N} | 2^{2k-1} < |\pi(x)| < 2^{2k+1}\},
\]
and \( S_k \) a compact subset of \( U_k \) which lie over the set
\[
\{x \in M_0 | |x| = 2^{2k}\}.
\]
Since \( U_k \) is a magnification of \( U_0 \) and the \( \Delta_s \)-harmonicity is invariant under a magnification, it is not difficult to see that there exists a constant \( q, \ 0 < q < 1 \), such that
\[
|u(x)| \leq q \cdot \sup \{|u(x)| \mid x \in U_k\}
\]
on \( S_k \) for any harmonic function \( u \) on \( U_k \) which changes sign on \( S_k \). Note that \( q \) is independent of \( k \).

Let \( u \) be a harmonic function on \( G \) such that \( |u| \leq 1 \) and it continuously vanishes on \( \partial G \). For each \( m \geq 1 \), denote by \( \pi_m \) the cover transformation of \( G \) which interchanges the sheets of \( G \): the points in \( G \cap M_0(i + 2^m j) \) are interchanged with points, with the same projection, in \( M_0(i + 2^{m-1} + 2^m j) \) for \( j \geq 0 \) and \( 1 \leq i \leq 2^{m-1} \). Define \( v_m \) on \( G \) by
\[
v_m(x) = \frac{1}{2} [u(x) - u(\pi_m(x))].
\]
Clearly \( v_m \) is harmonic on \( G \), \( |v_m| \leq 1 \), and \( v_m \) changes sign on \( S_k \), \( k = 2^{m-1}(2l - 1) - 1 \). Therefore,
\[
\max \{|v_m(x)| \mid x \in S_k\} \leq q
\]
for all \( l \geq 1 \). By induction on \( l \), we derive that \( |v_m| \leq q^l \) on \( S_{k'} \), where \( k' = 2^{m-1} - 1 \). Letting \( l \to \infty \), we conclude that \( v_m \equiv 0 \) on \( G \), as desired.

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