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ON THE RENEWAL FUNCTION WHEN SOME OF THE MEAN RENEWAL LIFETIMES ARE INFINITE

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ON THE RENEWAL FUNCTION WHEN SOME OF THE MEAN RENEWAL LIFETIMES ARE INFINITE

MAKOTO MAEJIMA

Let $\{X_i,\ i=1,2,\cdots\}$ be a sequence of independent and nonnegative random variables with the distribution function $F_i(x)$. Some of $\int_0^\infty xdF_i(x)$ may be infinite. Let $H(t)$ be the renewal function. The main object of this note is to show that in order to have the asymptotic relation $H(t)/t \sim 1/L(t)$ as $t \to \infty$, it is necessary and sufficient that $\mu(t) \sim L(t)$ as $t \to \infty$, where $L(t)$ is a function of slow growth and $\mu(t) = \lim_{n\to\infty} (1/n) \sum_{i=1}^n \mu_i(t)$, $\mu_i(t)$ being $\int_0^t [1-F_i(x)]dx$, is supposed to exist uniformly in t.

Let $H(t)$ be the renewal function for a renewal process, that is, a sequence $\{X_i,\ i=1,2,\cdots\}$ of nonnegative, independent and identically distributed random variables. Namely $H(t) = EN(t) = E[\sup\{n; S_n \leqq t\}]$, where $S_n = \sum_{i=1}^n X_i$. Smith [3] has studied the limiting behaviors of $H(t)/t$ for the case in which $EX_i = \infty$.

We now consider an extended renewal process in which X_i, $i=1,2,\cdots$ may not be identically distributed. We also in this case use the similar notations S_n and $N(t)$, and we may also define $H(t)$ in the similar manner under the condition that S_n has no finite limit point. The main object of this note is to give a generalization of a result of Smith to our extended case.

2. Some lemmas. We begin with some lemmas for an extended renewal process with the finite mean lifetimes.

Let $\{X_i, i=1,2,\cdots\}$ be a sequence of independent and nonnegative random variables with $0 < EX_i = \mu_i < \infty$ and let $F_i(x)$ be the distribution function of X_i.

LEMMA 1. *Suppose that*

$$\mu = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n \mu_i > 0 \tag{2.1}$$

exists and that

$$\lim_{A\to\infty} \int_A^\infty xdF_i(x) = 0$$

holds uniformly with respect to i. Then we have $EN^\alpha(t) < \infty$ for each $t > 0$, for $\alpha = 1, 2, \cdots$.

This lemma was first proved by Kawata [2] for $\alpha = 1$, and Hatori [1] showed it for any positive integer α.

LEMMA 2. *Suppose that $EN(t)$ and $EN^2(t)$ are finite and that* (2.1) *is true. Then we have for every t*

$$ES_{N(t)+1} = \mu(H(t) + 1) + \sum_{n=1}^{\infty} n\varepsilon_n \Pr\{N(t) + 1 = n\},$$

where ε_n is defined by

$$\varepsilon_n = \frac{1}{n}\sum_{i=1}^{n} \mu_i - \mu$$

which converges to zero as $n \to \infty$.

Proof. Letting

$$Z_n = 1, \text{ if } n \leqq N(t) + 1,$$
$$= 0, \text{ otherwise,}$$

we have

$$ES_{N(t)+1} = E\sum_{n=1}^{N(t)+1} X_n = E\sum_{n=1}^{\infty} X_n Z_n. \tag{2.2}$$

Since

$$\{Z_n = 0\} = \{N(t) + 1 < n\} = \bigcup_{k=1}^{n-1} \{N(t) + 1 = k\}$$
$$= \{X_1 > t\} \cup \bigcup_{k=2}^{n-1} \{(X_1 + \cdots + X_{k-1} \leqq t) \cap (X_1 + \cdots + X_k > t)\},$$

Z_n is independent of X_n. Thus, noticing the nonnegativeness of X_n, we see that (2.2) is

$$\sum_{n=1}^{\infty} EX_nZ_n = \sum_{n=1}^{\infty} EX_nEZ_n = \sum_{n=1}^{\infty} \mu_n \Pr\{N(t) + 1 \geqq n\},$$

which turns out to be

$$ES_{N(t)+1} = \sum_{n=1}^{\infty} (\mu + n\varepsilon_n - (n-1)\varepsilon_{n-1}) \Pr\{N(t) + 1 \geqq n\}$$
$$= \mu(H(t) + 1) + \sum_{n=1}^{\infty} (n\varepsilon_n - (n-1)\varepsilon_{n-1}) \Pr\{N(t) + 1 \geqq n\}.$$

Since

$$\sum_{n=1}^{\infty} | n\varepsilon_n \Pr\{N(t) + 1 \geqq n\} | \leqq \sup_n |\varepsilon_n| (EN^2(t) + 2) < \infty$$

by the finiteness of $EN^2(t)$, we may rewrite

$$\sum_{n=1}^{\infty}(n\varepsilon_n-(n-1)\varepsilon_{n-1})\Pr\{N(t)+1\geqq n\}=\sum_{n=1}^{\infty}n\varepsilon_n\Pr\{N(t)+1=n\}\,,$$

so that

$$ES_{N(t)+1}=\mu(H(t)+1)+\sum_{n=1}^{\infty}n\varepsilon_n\Pr\{N(t)+1=n\}\,,$$

which is the conclusion.

3. A theorem. We return to the case where X_i may have the infinite mean renewal lifetimes. Let $L(t)$ be a function of slow growth, that is, for every fixed $c>0$, $L(ct)/L(t)\to 1$ as $t\to\infty$. We shall show the following theorem which is an extension of a result due to Smith ([3], Theorem 1, (i), $\nu=1$) to the case of nonidentically distributed random variables.

THEOREM. *Let $\{X_i,\ i=1,2,\cdots\}$ be a sequence of independent and nonnegative random variables with the distribution function $F_i(x)$. Suppose that*

$$\mu(t)=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mu_i(t)>0 \tag{3.1}$$

exists uniformly in $0<t<\infty$, where

$$\mu_i(t)=\int_0^t[1-F_i(x)]dx\,.$$

Then the necessary and sufficient condition for the validity of the asymptotic relation

$$\frac{H(t)}{t}\sim\frac{1}{L(t)}\,,\quad \textit{as}\quad t\to\infty\,, \tag{3.2}$$

where $L(t)$ is a function of slow growth, is that

$$\mu(t)\sim L(t)\,,\quad \textit{as}\quad t\to\infty\,. \tag{3.3}$$

Before proving the theorem we shall show some lemmas.

We now define a new renewal process $\{X_i^*\}$ for a fixed positive number t^* by putting

$$\begin{aligned}X_i^*&=X_i\,,\quad \text{if } X_i\leqq t^*\,,\\&=t^*\,,\quad \text{otherwise}\,.\end{aligned}$$

We note that $EX_i^*=\mu_i(t^*)$ is finite. For the new variables X_i^*, we define S_n^*, $N^*(t)$ and $H^*(t)$ in obvious ways. Then we may easily verify the conditions of Lemma 1 for a fixed t^* and the following lemma is immediate.

LEMMA 3. *Suppose that* (3.1) *exists for* t^*. *Then* $E\{N^*(t)\}^\alpha < \infty$ *for* $\alpha = 1, 2, \cdots$.

The next two lemmas play essential roles in the proof of Theorem.

LEMMA 4. *Suppose that* (3.1) *exists uniformly in* t. *Then we have*

$$\liminf_{t\to\infty} \frac{H(t)\mu(t)}{t} \geqq 1 .$$

Proof. We consider X_i^* defined above. Since $EN^*(t)$ and $E\{N^*(t)\}^2$ are finite by Lemma 3, we have that for all t,

$$t < \mu(t^*)(H^*(t) + 1) + \sum_{n=1}^{\infty} n\varepsilon_n(t^*) \Pr\{N^*(t) + 1 = n\} \tag{3.4}$$

by Lemma 2 and noting $t < S_{N(t)+1}$, where $\varepsilon_n(t^*)$ is defined by

$$\frac{1}{n}\sum_{i=1}^{n} \mu_i(t^*) = \mu(t^*) + \varepsilon_n(t^*) .$$

Now (3.4) holds for $t = t^*$, in particular. Thus we have

$$t^* < \mu(t^*)(H^*(t^*) + 1) + \sum_{n=1}^{\infty} n\varepsilon_n(t^*) \Pr\{N^*(t^*) + 1 = n\} . \tag{3.5}$$

Next, we estimate of the order of $\varepsilon_n(t)$ as $t \to \infty$. Since the function $1 - F_i(x)$ decreases to zero as $x \to \infty$, so does $\mu_i(t)/t$ as $t \to \infty$. In view of the assumption that (3.1) exists uniformly in t, it follows that, for any $\varepsilon > 0$, there exists a constant N independent of t such that

$$\left|\mu(t) - \frac{1}{n}\sum_{i=1}^{n} \mu_i(t)\right| < \varepsilon , \quad \text{for} \quad n \geqq N . \tag{3.6}$$

Then we have

$$\begin{aligned}\frac{1}{t}\,|\,\mu(t)\,| &\leqq \frac{1}{t}\left|\mu(t) - \frac{1}{N}\sum_{i=1}^{N} \mu_i(t)\right| + \frac{1}{t}\cdot\frac{1}{N}\sum_{i=1}^{N} |\,\mu_i(t)\,| \\ &< \frac{\varepsilon}{t} + \varepsilon < 2\varepsilon\end{aligned}$$

for sufficiently large t, taking into account the fact that $\mu_i(t)/t \to 0$ as $t \to \infty$. Thus, we have for sufficiently large t

$$\frac{1}{t}\,|\,\varepsilon_n(t)\,| = \frac{1}{t}\left|\mu(t) - \frac{1}{n}\sum_{i=1}^{n}\mu_i(t)\right| < \frac{\varepsilon}{N^2}\,, \tag{3.7}$$

for the fixed N and for all $n \leqq N$. Therefore we have, for large t^*, from (3.6) and (3.7)

$$\begin{aligned}
&\sum_{n=1}^{\infty} n\varepsilon_n(t^*) \Pr\{N^*(t^*) + 1 = n\} \\
&< \frac{\varepsilon}{N^2}t^* \sum_{n=1}^{N} n \Pr\{N^*(t^*) + 1 = n\} + \varepsilon \sum_{n=N+1}^{\infty} n \Pr\{N^*(t^*) + 1 = n\} \qquad (3.8)\\
&< \varepsilon(t^* + H^*(t^*) + 1)\,.
\end{aligned}$$

Now we shall show that

$$\limsup_{t^*\to\infty} \frac{H^*(t^*)}{t^*} < \infty\,. \tag{3.9}$$

In order to show this, we define new truncated random variables $X_{i,A}$ for some constant A by putting

$$\begin{aligned}
X_{i,A} &= X_i\,, \quad \text{if} \quad X_i \leqq A\,,\\
&= A\,, \quad \text{otherwise}\,.
\end{aligned}$$

Clearly $EX_{i,A} = \mu_i(A)$ is finite and by the elementary renewal theorem for an extended renewal process, we have that, if $H_A(t)$ is the renewal function associated with $\{X_{i,A}\}$, then

$$\lim_{t\to\infty} \frac{H_A(t)}{t} = \frac{1}{\mu(A)}\,.$$

(For details, see Kawata [2].) (3.9) follows from the remark that $H^*(t^*) \leqq H_A(t^*)$ for $t^* \geqq A$. Since ε is arbitrary in (3.8), we have from (3.8)

$$\lim_{t^*\to\infty} \frac{1}{t^*}\left|\sum_{n=1}^{\infty} n\varepsilon_n(t^*) \Pr\{N^*(t^*) + 1 = n\}\right| = 0\,. \tag{3.10}$$

Therefore, from (3.5)

$$\liminf_{t^*\to\infty} \frac{1}{t^*}\mu(t^*)(H^*(t^*) + 1) \geqq 1\,.$$

On the other hand, we have

$$\Pr\{S_n^* \leqq t^*\} = \Pr\{S_n \leqq t^*\}\,,$$

for $n = 2, 3, \cdots$, and

$$\Pr\{S_1^* \leqq t^*\} = 1\,.$$

Thus

$$H^*(t^*) = H(t^*) + \Pr\{X_1 > t^*\}$$

and so

$$\liminf_{t\to\infty} \frac{1}{t}\, \mu(t)(H(t) + \Pr\{X_1 > t\} + 1) \geqq 1 .$$

Noticing that $\mu(t)/t \to 0$ as $t \to \infty$, we have the conclusion of the lemma.

LEMMA 5. *Under the same conditions as in Lemma 4, we have for arbitrary* $\delta > 0$

$$\limsup_{t\to\infty} \frac{H(t)\mu(\delta t)}{t} \leqq 1 + \delta .$$

Proof. Take $\delta > 0$ arbitrarily and let $\hat{X}_n$ represent new variables truncated according to the rule

$$\begin{aligned} \hat{X}_n &= X_n , \quad \text{if} \quad X_n \leqq \delta t^* , \\ &= \delta t^* , \quad \text{otherwise} . \end{aligned}$$

It is clear that $E\hat{X}_n = \mu_n(\delta t^*) < \infty$. Then, noting that $t \geqq S_{N(t)+1} - X_{N(t)+1}$, we have, by Lemma 2,

$$\begin{aligned} t &\geqq \mu(\delta t^*)(\hat{H}(t) + 1) + \sum_{n=1}^{\infty} n\varepsilon_n(\delta t^*) \Pr\{\hat{N}(t) + 1 = n\} - E\hat{X}_{\hat{N}(t)+1} \\ &\geqq \mu(\delta t^*)(\hat{H}(t) + 1) + \sum_{n=1}^{\infty} n\varepsilon_n(\delta t^*) \Pr\{\hat{N}(t) + 1 = n\} - \delta t^* , \end{aligned} \tag{3.11}$$

where $\hat{N}(t)$ and $\hat{H}(t)$ are defined in the renewal process associated with the new truncated variables $\{\hat{X}_n\}$. Since (3.11) holds for $t = t^*$, in particular, we have

$$(1 + \delta)t^* \geqq \mu(\delta t^*)(\hat{H}(t^*) + 1) + \sum_{n=1}^{\infty} n\varepsilon_n(\delta t^*) \Pr\{\hat{N}(t^*) + 1 = n\} .$$

The same arguments as in the proof of Lemma 4 yield that

$$\lim_{t^*\to\infty} \frac{1}{t^*} \left| \sum_{n=1}^{\infty} n\varepsilon_n(\delta t^*) \Pr\{\hat{N}(t^*) + 1 = n\} \right| = 0 \tag{3.12}$$

for the fixed $\delta > 0$. Noting that

$$\hat{H}(t^*) \geqq H(t^*) ,$$

we have the required result.

We now turn to the proof of the theorem.

Proof of Theorem. We first assume that

$$\frac{H(t)}{t} \sim \frac{1}{L(t)}, \quad \text{as} \quad t \to \infty .$$

By Lemma 4 we have

$$\liminf_{t\to\infty} \frac{\mu(t)}{L(t)} \geqq 1 , \tag{3.13}$$

and by Lemma 5, for any $\delta > 0$,

$$\limsup_{t\to\infty} \frac{\mu(\delta t)}{L(t)} \leqq 1 + \delta .$$

Writing δt for t, and using the fact that $L(t/\delta) \sim L(t)$ as $t \to \infty$, we have

$$\limsup_{t\to\infty} \frac{\mu(t)}{L(t)} \leqq 1 + \delta .$$

Since δ can be arbitrarily small, we, taking into account (3.13), conclude the necessity part.

Furthermore, in view of the assumption $\mu(t)$ is a function of slow growth, it follows by Lemma 5 that

$$\limsup_{t\to\infty} \frac{H(t)\mu(t)}{t} < 1 + \delta .$$

Since δ is arbitrary, Lemma 4 gives the sufficiency part.

When $\lim_{t\to\infty} \mu(t) = \infty$, we can relax slightly the condition of the uniform existence of $\mu(t)$ in the following way.

COROLLARY. *Suppose that*

$$\mu(t) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mu_i(t) > 0$$

exists for all t, *(not necessarily uniformly), and that there exists a constant* K, *independent of* t, *such that*

$$\left| \mu(t) - \frac{1}{n} \sum_{i=1}^{n} \mu_i(t) \right| = | \varepsilon_n(t) | < K$$

for $n \geqq N$, N *being some finite positive integer. If* $\lim_{t\to\infty}\mu(t) = \infty$, *then the necessary and sufficient condition for the validity of the asymptotic relation* (3.2) *is* (3.3).

Proof. In the proof of theorem, the condition relaxed has been used only in order to show (3.10) and (3.12). Thus, it suffices to show that (3.10) holds under the conditions of this corollary.

Now, we have

$$\frac{1}{t}\,|\mu(t)| \leqq \frac{1}{t}\left|\mu(t) - \frac{1}{N}\sum_{i=1}^{N}\mu_i(t)\right| + \frac{1}{t}\cdot\frac{1}{N}\sum_{i=1}^{N}|\mu_i(t)|$$
$$< \frac{K}{t} + \frac{1}{N}\sum_{i=1}^{N}\left|\frac{\mu_i(t)}{t}\right|,$$

and so $|\mu(t)|/t$ can be arbitrarily small for the sufficiently large t. Thus, $\varepsilon_n(t) = o(t)$ for all $n \leqq N$. Therefore, we have

$$\left|\sum_{n=1}^{\infty} n\varepsilon_n(t^*) \Pr\{N^*(t^*) + 1 = n\}\right|$$
$$< o(t^*)\sum_{n=1}^{N} n \Pr\{N^*(t^*) + 1 = n\} + K \sum_{n=N+1}^{\infty} n \Pr\{N^*(t^*) + 1 = n\}$$
$$< o(t^*)N^2 + K(H^*(t^*) + 1)\,.$$

Now we shall show under the condition that $\mu(t) \to \infty$, that

$$\lim_{t^*\to\infty} \frac{H^*(t^*)}{t^*} = 0\,.$$

As in the proof of the previous theorem, we have

$$\limsup_{t^*\to\infty} \frac{H^*(t^*)}{t^*} \leqq \lim_{t^*\to\infty} \frac{H_A(t^*)}{t^*} = \frac{1}{\mu(A)}\,.$$

Since A is arbitrary, this shows that

$$\lim_{t^*\to\infty} \frac{H^*(t^*)}{t^*} = 0\,,$$

and (3.10) holds.

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References

1. H. Hatori, *A note on a renewal theorem*, Kodai Math. Sem. Rep., **12** (1960), 28–37.
2. T. Kawata, *A renewal theorem*, J. Math. Soc. Japan, **8** (1956), 118–126.
3. W. L. Smith, *A note on the renewal function when the mean renewal lifetime is infinite*, J. Roy. Stat. Soc., **B23** (1961), 230–237.

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