COHOMOLOGICAL DIMENSION OF DISCRETE MODULES OVER PROFINITE GROUPS

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JUAN JOSÉ MARTÍNEZ

The main purpose of this note is to show that the finiteness of the cohomological dimension of a discrete module is closely related to the finiteness of its injective dimension. Moreover, a sufficient condition for the finiteness of the cohomological dimension is given. Both results are proved making a heavy use of the theory of cohomological triviality for finite groups.

The reader is referred to [3] for a treatment of profinite cohomology.

Throughout this note, $G$ is a profinite group. As usual, the cohomology of $G$ is denoted by $H(G, \_)$.

Recall that, if $A$ is a discrete $G$-module, the infimum of the (set of) nonnegative integers $r$ such that $H^r(S, A) = 0$, for any integer $n > r$ and any closed subgroup $S$ of $G$, is called the cohomological dimension of $A$, and is denoted by $cd(G, A)$. If $S$ is a closed subgroup of $G$, $H^r(S, A) \cong \lim \rightarrow H^r(V, A)$, where $V$ runs through all open subgroups of $G$ containing $S$ [3, Chap. I, Proposition 8, p. 1-9]. Hence, if $H^r(V, A) = 0$ for every open subgroup $V$ of $G$, then $H^r(S, A) = 0$ for every closed subgroup $S$ of $G$.

In this paper, a discrete module is called injective only when it isinjective in the corresponding category of discrete modules. If $A$ is injective, it is well-known that $cd(G, A) = 0$, because, for instance, $A$ is $V$-injective for all open subgroups $V$ of $G$. Finally, recall that the injective dimension of $A$, denoted by $id(G, A)$, is the least length of an injective resolution of $A$.

The connection between cohomologically trivial modules over finite groups [2, Chap. IX, § 3, p. 148] and discrete modules of cohomological dimension zero over profinite groups was observed, and used, by Tate in his duality theory for profinite cohomology [3, Annexe au Chapitre I, p. I-79]. Tate’s observation is quoted, for future reference, in the following.

**Lemma 1.** Let $A$ be a discrete $G$-module. Then, $cd(G, A) = 0$ if, and only if, for every open, normal subgroup $U$ of $G$, the $G/U$-module $A^U$ is cohomologically trivial.

**Proof.** See [3, Annexe au Chapitre I, Lemme 1, p. I-82]. Notice that $G/U$ is a finite group, because $G$ is compact and $U$ is open.

The Nakayama-Tate criterion for cohomological triviality takes
the following form, in the cohomology theory of profinite groups.

**Proposition 2.** Let A be a discrete G-module. If there exists a positive integer q such that $H^q(V, A) = H^{q+1}(V, A) = 0$ for all open subgroups V of G, then $cd(G, A) < q$.

**Proof.** Since A embeds in an injective, whose cohomological dimension is zero, by repeated applications of dimension-shifting it suffices to consider the case $q = 1$. Let U be an open, normal subgroup of G. If V is any subgroup of G containing U, the Hochschild-Serre spectral sequence of the $V/U$-module $A^U$ yields the exact sequence for low degrees

$$0 \rightarrow H^1(V/U, A^U) \rightarrow H^1(V, A) \rightarrow H^1(U, A)_{V/U} \rightarrow H^2(V/U, A^U) \rightarrow H^2(V, A).$$

Since U is open, so is V, and thus, $H^1(U, A) = H^1(V, A) = H^1(V, A) = 0$. Therefore, $H^1(V/U, A^U) = H^2(V/U, A^U) = 0$, and applying the Nakayama-Tate criterion [2, Chap. IX, Théorème 8, p. 152], the G/U-module $A^U$ is cohomologically trivial. By (1), the proof is complete.

The main result of this paper can be stated as follows.

**Theorem 3.** Let A be a discrete G-module, and let q be a positive integer. Then, $id(G, A) \leq q$ if, and only if, $cd(G, A) \leq q$ and $H^q(U, A)$ is a divisible abelian group for every open, normal subgroup U of G.

**Proof.** Assume the assertion true for $q - 1$, with $q > 1$. If $id(G, A) \leq q$, A has an injective resolution of length $\leq q$, say

$$0 \rightarrow A \xrightarrow{e} X_0 \xrightarrow{d_0} X_1 \rightarrow \cdots \rightarrow X_{q-1} \xrightarrow{d_{q-1}} X_q \rightarrow 0.$$  

If $B = \text{Coker } e$ and $f: X_0 \rightarrow B$ is the canonical morphism, the sequence of discrete G-modules

$$0 \rightarrow A \xrightarrow{e} X_0 \xrightarrow{f} B \rightarrow 0$$

is exact. Since $cd(G, X_0) = 0$ (injectivity of $X_0$), from the corresponding cohomology sequence it follows that

$$H^n(S, B) \cong H^{n+1}(S, A)$$

for any positive integer n and any closed subgroup S of G. Therefore, it is enough to prove that $cd(G, B) \leq q - 1$, and that $H^{q-1}(U, B)$ is divisible for all open, normal subgroups U of G. By the induction hypothesis, this follows from showing that $id(G, B) \leq q - 1$. In fact, if $e': B \rightarrow X_1$ is the morphism induced by $d_0: X_0 \rightarrow X_1$, then $\text{Ker } e' = 0$ and $\text{Im } e' = \text{Im } d_0$. Thus, the sequence
Reciprocally, if \( cd(G, A) \leq q \), let

\[
0 \rightarrow A \xrightarrow{g} Q \xrightarrow{h} C \rightarrow 0
\]

be an exact sequence of discrete \( G \)-modules, with \( Q \) injective. Then, \( cd(G, C) \leq q - 1 \), because

\[
H^q(S, C) \cong H^{q+1}(S, A)
\]

for all positive integers \( n \) and all closed subgroups \( S \) of \( G \). By the same reason, if \( H^q(U, A) \) is divisible for every open, normal subgroup \( U \) of \( G \), then so is \( H^{q-1}(U, C) \). Hence, by induction, \( C \) admits an injective resolution of length \( \leq q - 1 \), say

\[
0 \rightarrow C \xrightarrow{i} Y_0 \xrightarrow{d_0} Y_1 \rightarrow \cdots \rightarrow Y_{q-2} \xrightarrow{d_{q-2}} Y_{q-1} \rightarrow 0.
\]

Since \( \text{Ker } ih = \text{Ker } h \) and \( \text{Im } ih = \text{Im } i \), the sequence

\[
0 \rightarrow A \xrightarrow{g} Q \xrightarrow{ih} Y_0 \xrightarrow{d_0} Y_1 \rightarrow \cdots \rightarrow Y_{q-2} \xrightarrow{d_{q-2}} Y_{q-1} \rightarrow 0
\]

is exact, and so \( id(G, A) \leq q \).

It remains to prove the assertion for \( q = 1 \).

Let

\[
0 \rightarrow A \rightarrow X_0 \rightarrow X_1 \rightarrow 0
\]

be an exact sequence of discrete \( G \)-modules, where \( X_0 \) and \( X_1 \) are injectives. Since \( cd(G, X_0) = cd(G, X_1) = 0 \), passing to cohomology it follows that \( cd(G, A) \leq 1 \), and that the connecting operator \( \partial_S : X_1^S \rightarrow H^1(S, A) \) is an epimorphism for all closed subgroups \( S \) of \( G \). But, if \( D \) is any injective, discrete \( G \)-module and \( U \) is any open, normal subgroup of \( G \), it is easy to check that \( D^U \) is an injective \( G/U \)-module, whence [2, Chap. IX, Lemme 7, p. 153] implies \( D^U \) is divisible. Therefore, as the image of a divisible group, \( H^1(U, A) \) is divisible for all open, normal subgroups \( U \) of \( G \).

Reciprocally, suppose \( cd(G, A) \leq 1 \), and let

\[
0 \rightarrow A \rightarrow Y_0 \rightarrow Y_1 \rightarrow 0
\]

be an exact sequence of discrete \( G \)-modules, with \( Y_0 \) injective. Since \( cd(G, Y_0) = 0 \), taking cohomology it follows that \( cd(G, Y_1) = 0 \), and that the sequence of abelian groups

\[
Y_0^S \rightarrow Y_1^S \xrightarrow{\partial_S} H^1(S, A) \rightarrow 0
\]
is exact for all closed subgroups \( S \) of \( G \). If \( U \) is an open, normal subgroup of \( G \), \( \text{Ker}\, \partial_v \) is divisible, because so is \( Y_v^U \). Therefore, if \( \text{Im}\, \partial_v = H^i(U, A) \) is divisible, then \( \text{Dom}\, \partial_v = Y_v^U \) is also divisible, and the proof is complete applying to \( Y \), the following.

**Proposition 4.** Let \( A \) be a discrete \( G \)-module. If \( cd(G, A) = 0 \), and \( A^U \) is a divisible abelian group for every open, normal subgroup \( U \) of \( G \), then \( A \) is injective.

**Proof.** Recall that the category of discrete \( G \)-modules has injective envelopes for each of its objects. Since \( (\mathbb{Z}[G/U])_U \), where \( U \) runs through all open, normal subgroups of \( G \), is a family of generators, this result can be obtained by using a general theorem from category theory, due to Mitchell [1, Chap. III, Theorem 3.2, p. 89].

Let \( f: A \to Q \) be an injective envelope of \( A \) (in the category of discrete \( G \)-modules). If \( C = \text{Coker}\, f \) and \( g: Q \to C \) is the canonical morphism, the sequence of discrete \( G \)-modules

\[
0 \to A \xrightarrow{f} Q \xrightarrow{g} C \to 0
\]

is exact. Thus, if \( U \) is an open, normal subgroup of \( G \), the sequence of \( G/U \)-modules

\[
0 \to A^U \xrightarrow{f^U} Q^U \xrightarrow{g^U} C^U \to 0
\]

is exact, because \( cd(G, A) = 0 \). Since \( Q^U \) is an injective \( G/U \)-module and \( R \cap \text{Im}\, f^U = R \cap \text{Im}\, f \) for any sub-\( G/U \)-module \( R \) of \( Q^U \) (because, regarding \( R \) as a \( G \)-module, \( U \) operates trivially on \( R \)), \( f^U: A^U \to Q^U \) is an injective envelope of \( A^U \) (in the category of \( G/U \)-modules). On the other hand, since \( cd(G, A) = 0 \), \( A^U \) is a cohomologically trivial \( G/U \)-module, by (1). Thus, \( A^U \) is \( G/U \)-injective [2, Chap. IX, Théorème 10, p. 154], and hence, \( C^U = 0 \) [1, Chap. III, Proposition 2.5, p. 88]. Since \( C = \bigcup C^U \), \( C = 0 \), whence the result.

**Corollary 5.** Let \( A \) be a discrete \( G \)-module, and let \( r \) be a nonnegative integer. If \( cd(G, A) \leq r \), then \( id(G, A) \leq r + 1 \).

**Proof.** Take \( q = r + 1 \) in (3).

This result can be applied to profinite groups of finite dimension, as follows.

**Corollary 6.** Let \( r \) be a nonnegative integer. The following statements are true:

1. If \( p \) is a prime number and \( cd_p(G) \leq r \), then \( id(G, A) \leq r + 1 \) for all discrete \( G \)-modules \( A \) which are \( p \)-primary abelian groups.
(ii) If \( cd(G) \leq r \), then \( id(G, A) \leq r + 1 \) for all discrete \( G \)-modules \( A \) which are torsion abelian groups.

(iii) If \( scd(G) \leq r \), then \( id(G, A) \leq r + 1 \) for all discrete \( G \)-modules \( A \).

(iv) If \( cd(G) \leq r \), then \( id(G, A) \leq r + 2 \) for all discrete \( G \)-modules \( A \).

Proof. Applying [3, Chap. I, Proposition 14, p. I-20] and [3, Chap. I, Proposition 11, p. I-17], the following three equivalences are clear:

(i) \( cd_p(G) \leq r \) if, and only if, \( cd(G, A) \leq r \) for all \( p \)-primary, discrete \( G \)-modules \( A \).

(ii) \( cd(G) \leq r \) if, and only if, \( cd(G, A) \leq r \) for all torsion, discrete \( G \)-modules \( A \).

(iii) \( scd(G) \leq r \) if, and only if, \( cd(G, A) \leq r \) for all discrete \( G \)-modules \( A \).

Finally, (6, iv) is clear by [3, Chap. I, Proposition 13, p. 1-19].

REFERENCES


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