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**COHOMOLOGICAL DIMENSION OF DISCRETE MODULES
OVER PROFINITE GROUPS**

JUAN JOSÉ MARTÍNEZ

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The main purpose of this note is to show that the finiteness of the cohomological dimension of a discrete module is closely related to the finiteness of its injective dimension. Moreover, a sufficient condition for the finiteness of the cohomological dimension is given. Both results are proved making a heavy use of the theory of cohomological triviality for finite groups.

The reader is referred to [3] for a treatment of profinite cohomology.

Throughout this note, G is a profinite group. As usual, the cohomology of G is denoted by $H(G, \)$.

Recall that, if A is a discrete G -module, the infimum of the (set of) nonnegative integers r such that $H^n(S, A) = 0$, for any integer $n > r$ and any closed subgroup S of G , is called the *cohomological dimension* of A , and is denoted by $cd(G, A)$. If S is a closed subgroup of G , $H^n(S, A) \cong \varinjlim H^n(V, A)$, where V runs through all open subgroups of G containing S [3, Chap. I, Proposition 8, p. I-9]. Hence, if $H^n(V, A) = 0$ for every open subgroup V of G , then $H^n(S, A) = 0$ for every closed subgroup S of G .

In this paper, a discrete module is called *injective* only when it is injective in the corresponding category of discrete modules. If A is injective, it is well-known that $cd(G, A) = 0$, because, for instance, A is V -injective for all open subgroups V of G . Finally, recall that the *injective dimension* of A , denoted by $id(G, A)$, is the least length of an injective resolution of A .

The connection between cohomologically trivial modules over finite groups [2, Chap. IX, § 3, p. 148] and discrete modules of cohomological dimension zero over profinite groups was observed, and used, by Tate in his duality theory for profinite cohomology [3, Annexe au Chapitre I, p. I-79]. Tate's observation is quoted, for future reference, in the following.

LEMMA 1. *Let A be a discrete G -module. Then, $cd(G, A) = 0$ if, and only if, for every open, normal subgroup U of G , the G/U -module A^U is cohomologically trivial.*

Proof. See [3, Annexe au Chapitre I, Lemme 1, p. I-82]. Notice that G/U is a finite group, because G is compact and U is open.

The Nakayama-Tate criterion for cohomological triviality takes

the following form, in the cohomology theory of profinite groups.

PROPOSITION 2. *Let A be a discrete G -module. If there exists a positive integer q such that $H^q(V, A) = H^{q+1}(V, A) = 0$ for all open subgroups V of G , then $cd(G, A) < q$.*

Proof. Since A embeds in an injective, whose cohomological dimension is zero, by repeated applications of dimension-shifting it suffices to consider the case $q = 1$. Let U be an open, normal subgroup of G . If V is any subgroup of G containing U , the Hochschild-Serre spectral sequence of the V/U -module A^U yields the exact sequence for low degrees

$$\begin{aligned} 0 \longrightarrow H^1(V/U, A^U) &\longrightarrow H^1(V, A) \longrightarrow H^1(U, A)^{V/U} \\ &\longrightarrow H^2(V/U, A^U) \longrightarrow H^2(V, A) . \end{aligned}$$

Since U is open, so is V , and thus, $H^1(U, A) = H^1(V, A) = H^2(V, A) = 0$. Therefore, $H^1(V/U, A^U) = H^2(V/U, A^U) = 0$, and applying the Nakayama-Tate criterion [2, Chap. IX, Théorème 8, p. 152], the G/U -module A^U is cohomologically trivial. By (1), the proof is complete.

The main result of this paper can be stated as follows.

THEOREM 3. *Let A be a discrete G -module, and let q be a positive integer. Then, $id(G, A) \leq q$ if, and only if, $cd(G, A) \leq q$ and $H^q(U, A)$ is a divisible abelian group for every open, normal subgroup U of G .*

Proof. Assume the assertion true for $q - 1$, with $q > 1$. If $id(G, A) \leq q$, A has an injective resolution of length $\leq q$, say

$$0 \longrightarrow A \xrightarrow{e} X_0 \xrightarrow{d_0} X_1 \longrightarrow \cdots \longrightarrow X_{q-1} \xrightarrow{d_{q-1}} X_q \longrightarrow 0 .$$

If $B = \text{Coker } e$ and $f: X_0 \rightarrow B$ is the canonical morphism, the sequence of discrete G -modules

$$0 \longrightarrow A \xrightarrow{e} X_0 \xrightarrow{f} B \longrightarrow 0$$

is exact. Since $cd(G, X_0) = 0$ (injectivity of X_0), from the corresponding cohomology sequence it follows that

$$H^n(S, B) \cong H^{n+1}(S, A)$$

for any positive integer n and any closed subgroup S of G . Therefore, it is enough to prove that $cd(G, B) \leq q - 1$, and that $H^{q-1}(U, B)$ is divisible for all open, normal subgroups U of G . By the induction hypothesis, this follows from showing that $id(G, B) \leq q - 1$. In fact, if $e': B \rightarrow X_1$ is the morphism induced by $d_0: X_0 \rightarrow X_1$, then $\text{Ker } e' = 0$ and $\text{Im } e' = \text{Im } d_0$. Thus, the sequence

$$0 \longrightarrow B \xrightarrow{e'} X_1 \xrightarrow{d_1} X_2 \longrightarrow \dots \longrightarrow X_{q-1} \xrightarrow{d_{q-1}} X_q \longrightarrow 0$$

is exact.

Reciprocally, if $cd(G, A) \leq q$, let

$$0 \longrightarrow A \xrightarrow{g} Q \xrightarrow{h} C \longrightarrow 0$$

be an exact sequence of discrete G -modules, with Q injective. Then, $cd(G, C) \leq q - 1$, because

$$H^n(S, C) \cong H^{n+1}(S, A)$$

for all positive integers n and all closed subgroups S of G . By the same reason, if $H^q(U, A)$ is divisible for every open, normal subgroup U of G , then so is $H^{q-1}(U, C)$. Hence, by induction, C admits an injective resolution of length $\leq q - 1$, say

$$0 \longrightarrow C \xrightarrow{i} Y_0 \xrightarrow{d_0} Y_1 \longrightarrow \dots \longrightarrow Y_{q-2} \xrightarrow{d_{q-2}} Y_{q-1} \longrightarrow 0.$$

Since $\text{Ker } ih = \text{Ker } h$ and $\text{Im } ih = \text{Im } i$, the sequence

$$0 \longrightarrow A \xrightarrow{g} Q \xrightarrow{ih} Y_0 \xrightarrow{d_0} Y_1 \longrightarrow \dots \longrightarrow Y_{q-2} \xrightarrow{d_{q-2}} Y_{q-1} \longrightarrow 0$$

is exact, and so $id(G, A) \leq q$.

It remains to prove the assertion for $q = 1$.

Let

$$0 \longrightarrow A \longrightarrow X_0 \longrightarrow X_1 \longrightarrow 0$$

be an exact sequence of discrete G -modules, where X_0 and X_1 are injectives. Since $cd(G, X_0) = cd(G, X_1) = 0$, passing to cohomology it follows that $cd(G, A) \leq 1$, and that the connecting operator $\partial_S: X_1^S \rightarrow H^1(S, A)$ is an epimorphism for all closed subgroups S of G . But, if D is any injective, discrete G -module and U is any open, normal subgroup of G , it is easy to check that D^U is an injective G/U -module, whence [2, Chap. IX, Lemme 7, p. 153] implies D^U is divisible. Therefore, as the image of a divisible group, $H^1(U, A)$ is divisible for all open, normal subgroups U of G .

Reciprocally, suppose $cd(G, A) \leq 1$, and let

$$0 \longrightarrow A \longrightarrow Y_0 \longrightarrow Y_1 \longrightarrow 0$$

be an exact sequence of discrete G -modules, with Y_0 injective. Since $cd(G, Y_0) = 0$, taking cohomology it follows that $cd(G, Y_1) = 0$, and that the sequence of abelian groups

$$Y_0^S \longrightarrow Y_1^S \xrightarrow{\partial_S} H^1(S, A) \longrightarrow 0$$

is exact for all closed subgroups S of G . If U is an open, normal subgroup of G , $\text{Ker } \partial_U$ is divisible, because so is Y_0^U . Therefore, if $\text{Im } \partial_U = H^1(U, A)$ is divisible, then $\text{Dom } \partial_U = Y_1^U$ is also divisible, and the proof is complete applying to Y_1 the following.

PROPOSITION 4. *Let A be a discrete G -module. If $cd(G, A) = 0$, and A^U is a divisible abelian group for every open, normal subgroup U of G , then A is injective.*

Proof. Recall that the category of discrete G -modules has injective envelopes for each of its objects. Since $(\mathbf{Z}[G/U])_U$, where U runs through all open, normal subgroups of G , is a family of generators, this result can be obtained by using a general theorem from category theory, due to Mitchell [1, Chap. III, Theorem 3.2, p. 89].

Let $f: A \rightarrow Q$ be an injective envelope of A (in the category of discrete G -modules). If $C = \text{Coker } f$ and $g: Q \rightarrow C$ is the canonical morphism, the sequence of discrete G -modules

$$0 \longrightarrow A \xrightarrow{f} Q \xrightarrow{g} C \longrightarrow 0$$

is exact. Thus, if U is an open, normal subgroup of G , the sequence of G/U -modules

$$0 \longrightarrow A^U \xrightarrow{f^U} Q^U \xrightarrow{g^U} C^U \longrightarrow 0$$

is exact, because $cd(G, A) = 0$. Since Q^U is an injective G/U -module and $R \cap \text{Im } f^U = R \cap \text{Im } f$ for any sub- G/U -module R of Q^U (because, regarding R as a G -module, U operates trivially on R), $f^U: A^U \rightarrow Q^U$ is an injective envelope of A^U (in the category of G/U -modules). On the other hand, since $cd(G, A) = 0$, A^U is a cohomologically trivial G/U -module, by (1). Thus, A^U is G/U -injective [2, Chap. IX, Théorème 10, p. 154], and hence, $C^U = 0$ [1, Chap. III, Proposition 2.5, p. 88]. Since $C = \bigcup C^U$, $C = 0$, whence the result.

COROLLARY 5. *Let A be a discrete G -module, and let r be a nonnegative integer. If $cd(G, A) \leq r$, then $id(G, A) \leq r + 1$.*

Proof. Take $q = r + 1$ in (3).

This result can be applied to profinite groups of finite dimension, as follows.

COROLLARY 6. *Let r be a nonnegative integer. The following statements are true:*

(i) *If p is a prime number and $cd_p(G) \leq r$, then $id(G, A) \leq r + 1$ for all discrete G -modules A which are p -primary abelian groups.*

- (ii) If $cd(G) \leq r$, then $id(G, A) \leq r + 1$ for all discrete G -modules A which are torsion abelian groups.
- (iii) If $scd(G) \leq r$, then $id(G, A) \leq r + 1$ for all discrete G -modules A .
- (iv) If $cd(G) \leq r$, then $id(G, A) \leq r + 2$ for all discrete G -modules A .

Proof. Applying [3, Chap. I, Proposition 14, p. I-20] and [3, Chap. I, Proposition 11, p. I-17], the following three equivalences are clear:

- (i) $cd_p(G) \leq r$ if, and only if, $cd(G, A) \leq r$ for all p -primary, discrete G -modules A .
- (ii) $cd(G) \leq r$ if, and only if, $cd(G, A) \leq r$ for all torsion, discrete G -modules A .
- (iii) $scd(G) \leq r$ if, and only if, $cd(G, A) \leq r$ for all discrete G -modules A .

Finally, (6, iv) is clear by [3, Chap. I, Proposition 13, p. 1-19].

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