

Pacific Journal of Mathematics

COHOMOLOGICAL DIMENSION OF DISCRETE MODULES OVER PROFINITE GROUPS

JUAN JOSÉ MARTÍNEZ

COHOMOLOGICAL DIMENSION OF DISCRETE MODULES OVER PROFINITE GROUPS

JUAN JOSÉ MARTÍNEZ

The main purpose of this note is to show that the finiteness of the cohomological dimension of a discrete module is closely related to the finiteness of its injective dimension. Moreover, a sufficient condition for the finiteness of the cohomological dimension is given. Both results are proved making a heavy use of the theory of cohomological triviality for finite groups.

The reader is referred to [3] for a treatment of profinite cohomology.

Throughout this note, G is a profinite group. As usual, the cohomology of G is denoted by $H(G, \)$.

Recall that, if A is a discrete G -module, the infimum of the (set of) nonnegative integers r such that $H^n(S, A) = 0$, for any integer $n > r$ and any closed subgroup S of G , is called the *cohomological dimension* of A , and is denoted by $cd(G, A)$. If S is a closed subgroup of G , $H^n(S, A) \cong \lim_{\rightarrow} H^n(V, A)$, where V runs through all open subgroups of G containing S [3, Chap. I, Proposition 8, p. I-9]. Hence, if $H^n(V, A) = 0$ for every open subgroup V of G , then $H^n(S, A) = 0$ for every closed subgroup S of G .

In this paper, a discrete module is called *injective* only when it is injective in the corresponding category of discrete modules. If A is injective, it is well-known that $cd(G, A) = 0$, because, for instance, A is V -injective for all open subgroups V of G . Finally, recall that the *injective dimension* of A , denoted by $id(G, A)$, is the least length of an injective resolution of A .

The connection between cohomologically trivial modules over finite groups [2, Chap. IX, § 3, p. 148] and discrete modules of cohomological dimension zero over profinite groups was observed, and used, by Tate in his duality theory for profinite cohomology [3, Annexe au Chapitre I, p. I-79]. Tate's observation is quoted, for future reference, in the following.

LEMMA 1. *Let A be a discrete G -module. Then, $cd(G, A) = 0$ if, and only if, for every open, normal subgroup U of G , the G/U -module A^U is cohomologically trivial.*

Proof. See [3, Annexe au Chapitre I, Lemme 1, p. I-82]. Notice that G/U is a finite group, because G is compact and U is open.

The Nakayama-Tate criterion for cohomological triviality takes

the following form, in the cohomology theory of profinite groups.

PROPOSITION 2. *Let A be a discrete G -module. If there exists a positive integer q such that $H^q(V, A) = H^{q+1}(V, A) = 0$ for all open subgroups V of G , then $cd(G, A) < q$.*

Proof. Since A embeds in an injective, whose cohomological dimension is zero, by repeated applications of dimension-shifting it suffices to consider the case $q = 1$. Let U be an open, normal subgroup of G . If V is any subgroup of G containing U , the Hochschild-Serre spectral sequence of the V/U -module A^U yields the exact sequence for low degrees

$$\begin{aligned} 0 \longrightarrow H^1(V/U, A^U) &\longrightarrow H^1(V, A) \longrightarrow H^1(U, A)^{V/U} \\ &\longrightarrow H^2(V/U, A^U) \longrightarrow H^2(V, A) . \end{aligned}$$

Since U is open, so is V , and thus, $H^1(U, A) = H^1(V, A) = H^2(V, A) = 0$. Therefore, $H^1(V/U, A^U) = H^2(V/U, A^U) = 0$, and applying the Nakayama-Tate criterion [2, Chap. IX, Théorème 8, p. 152], the G/U -module A^U is cohomologically trivial. By (1), the proof is complete.

The main result of this paper can be stated as follows.

THEOREM 3. *Let A be a discrete G -module, and let q be a positive integer. Then, $id(G, A) \leq q$ if, and only if, $cd(G, A) \leq q$ and $H^q(U, A)$ is a divisible abelian group for every open, normal subgroup U of G .*

Proof. Assume the assertion true for $q - 1$, with $q > 1$. If $id(G, A) \leq q$, A has an injective resolution of length $\leq q$, say

$$0 \longrightarrow A \xrightarrow{e} X_0 \xrightarrow{d_0} X_1 \longrightarrow \cdots \longrightarrow X_{q-1} \xrightarrow{d_{q-1}} X_q \longrightarrow 0 .$$

If $B = \text{Coker } e$ and $f: X_0 \rightarrow B$ is the canonical morphism, the sequence of discrete G -modules

$$0 \longrightarrow A \xrightarrow{e} X_0 \xrightarrow{f} B \longrightarrow 0$$

is exact. Since $cd(G, X_0) = 0$ (injectivity of X_0), from the corresponding cohomology sequence it follows that

$$H^n(S, B) \cong H^{n+1}(S, A)$$

for any positive integer n and any closed subgroup S of G . Therefore, it is enough to prove that $cd(G, B) \leq q - 1$, and that $H^{q-1}(U, B)$ is divisible for all open, normal subgroups U of G . By the induction hypothesis, this follows from showing that $id(G, B) \leq q - 1$. In fact, if $e': B \rightarrow X_1$ is the morphism induced by $d_0: X_0 \rightarrow X_1$, then $\text{Ker } e' = 0$ and $\text{Im } e' = \text{Im } d_0$. Thus, the sequence

$$0 \longrightarrow B \xrightarrow{e'} X_1 \xrightarrow{d_1} X_2 \longrightarrow \dots \longrightarrow X_{q-1} \xrightarrow{d_{q-1}} X_q \longrightarrow 0$$

is exact.

Reciprocally, if $cd(G, A) \leq q$, let

$$0 \longrightarrow A \xrightarrow{g} Q \xrightarrow{h} C \longrightarrow 0$$

be an exact sequence of discrete G -modules, with Q injective. Then, $cd(G, C) \leq q - 1$, because

$$H^n(S, C) \cong H^{n+1}(S, A)$$

for all positive integers n and all closed subgroups S of G . By the same reason, if $H^q(U, A)$ is divisible for every open, normal subgroup U of G , then so is $H^{q-1}(U, C)$. Hence, by induction, C admits an injective resolution of length $\leq q - 1$, say

$$0 \longrightarrow C \xrightarrow{i} Y_0 \xrightarrow{d_0} Y_1 \longrightarrow \dots \longrightarrow Y_{q-2} \xrightarrow{d_{q-2}} Y_{q-1} \longrightarrow 0.$$

Since $\text{Ker } ih = \text{Ker } h$ and $\text{Im } ih = \text{Im } i$, the sequence

$$0 \longrightarrow A \xrightarrow{g} Q \xrightarrow{ih} Y_0 \xrightarrow{d_0} Y_1 \longrightarrow \dots \longrightarrow Y_{q-2} \xrightarrow{d_{q-2}} Y_{q-1} \longrightarrow 0$$

is exact, and so $id(G, A) \leq q$.

It remains to prove the assertion for $q = 1$.

Let

$$0 \longrightarrow A \longrightarrow X_0 \longrightarrow X_1 \longrightarrow 0$$

be an exact sequence of discrete G -modules, where X_0 and X_1 are injectives. Since $cd(G, X_0) = cd(G, X_1) = 0$, passing to cohomology it follows that $cd(G, A) \leq 1$, and that the connecting operator $\partial_s: X_1^S \rightarrow H^1(S, A)$ is an epimorphism for all closed subgroups S of G . But, if D is any injective, discrete G -module and U is any open, normal subgroup of G , it is easy to check that D^U is an injective G/U -module, whence [2, Chap. IX, Lemme 7, p. 153] implies D^U is divisible. Therefore, as the image of a divisible group, $H^1(U, A)$ is divisible for all open, normal subgroups U of G .

Reciprocally, suppose $cd(G, A) \leq 1$, and let

$$0 \longrightarrow A \longrightarrow Y_0 \longrightarrow Y_1 \longrightarrow 0$$

be an exact sequence of discrete G -modules, with Y_0 injective. Since $cd(G, Y_0) = 0$, taking cohomology it follows that $cd(G, Y_1) = 0$, and that the sequence of abelian groups

$$Y_0^S \longrightarrow Y_1^S \xrightarrow{\partial_s} H^1(S, A) \longrightarrow 0$$

is exact for all closed subgroups S of G . If U is an open, normal subgroup of G , $\text{Ker } \partial_U$ is divisible, because so is Y_0^U . Therefore, if $\text{Im } \partial_U = H^1(U, A)$ is divisible, then $\text{Dom } \partial_U = Y_1^U$ is also divisible, and the proof is complete applying to Y_1 the following.

PROPOSITION 4. *Let A be a discrete G -module. If $cd(G, A) = 0$, and A^U is a divisible abelian group for every open, normal subgroup U of G , then A is injective.*

Proof. Recall that the category of discrete G -modules has injective envelopes for each of its objects. Since $(Z[G/U])_U$, where U runs through all open, normal subgroups of G , is a family of generators, this result can be obtained by using a general theorem from category theory, due to Mitchell [1, Chap. III, Theorem 3.2, p. 89].

Let $f: A \rightarrow Q$ be an injective envelope of A (in the category of discrete G -modules). If $C = \text{Coker } f$ and $g: Q \rightarrow C$ is the canonical morphism, the sequence of discrete G -modules

$$0 \longrightarrow A \xrightarrow{f} Q \xrightarrow{g} C \longrightarrow 0$$

is exact. Thus, if U is an open, normal subgroup of G , the sequence of G/U -modules

$$0 \longrightarrow A^U \xrightarrow{f^U} Q^U \xrightarrow{g^U} C^U \longrightarrow 0$$

is exact, because $cd(G, A) = 0$. Since Q^U is an injective G/U -module and $R \cap \text{Im } f^U = R \cap \text{Im } f$ for any sub- G/U -module R of Q^U (because, regarding R as a G -module, U operates trivially on R), $f^U: A^U \rightarrow Q^U$ is an injective envelope of A^U (in the category of G/U -modules). On the other hand, since $cd(G, A) = 0$, A^U is a cohomologically trivial G/U -module, by (1). Thus, A^U is G/U -injective [2, Chap. IX, Théorème 10, p. 154], and hence, $C^U = 0$ [1, Chap. III, Proposition 2.5, p. 88]. Since $C = \bigcup C^U$, $C = 0$, whence the result.

COROLLARY 5. *Let A be a discrete G -module, and let r be a nonnegative integer. If $cd(G, A) \leq r$, then $id(G, A) \leq r + 1$.*

Proof. Take $q = r + 1$ in (3).

This result can be applied to profinite groups of finite dimension, as follows.

COROLLARY 6. *Let r be a nonnegative integer. The following statements are true:*

(i) *If p is a prime number and $cd_p(G) \leq r$, then $id(G, A) \leq r + 1$ for all discrete G -modules A which are p -primary abelian groups.*

- (ii) If $cd(G) \leq r$, then $id(G, A) \leq r + 1$ for all discrete G -modules A which are torsion abelian groups.
- (iii) If $scd(G) \leq r$, then $id(G, A) \leq r + 1$ for all discrete G -modules A .
- (iv) If $cd(G) \leq r$, then $id(G, A) \leq r + 2$ for all discrete G -modules A .

Proof. Applying [3, Chap. I, Proposition 14, p. I-20] and [3, Chap. I, Proposition 11, p. I-17], the following three equivalences are clear:

- (i) $cd_p(G) \leq r$ if, and only if, $cd(G, A) \leq r$ for all p -primary, discrete G -modules A .
- (ii) $cd(G) \leq r$ if, and only if, $cd(G, A) \leq r$ for all torsion, discrete G -modules A .
- (iii) $scd(G) \leq r$ if, and only if, $cd(G, A) \leq r$ for all discrete G -modules A .

Finally, (6, iv) is clear by [3, Chap. I, Proposition 13, p. 1-19].

REFERENCES

1. B. Mitchell, *Theory of Categories*, Pure and applied mathematics 17, Academic Press, New York, 1965.
2. J-P. Serre, *Corps Locaux*, Actualités scientifiques et industrielles 1296, Hermann, Paris, 1962.
3. ———, *Cohomologie Galoisienne*, Lecture notes in mathematics 5, Springer-Verlag, Berlin, 1965.

Received August 14, 1972 and in revised form December 18, 1972.

UNIVERSIDAD DE BUENOS AIRES

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

RICHARD ARENS (Managing Editor)
University of California
Los Angeles, California 90024

J. DUGUNDJI*
Department of Mathematics
University of Southern California
Los Angeles, California 90007

R. A. BEAUMONT
University of Washington
Seattle, Washington 98105

D. GILBARG AND J. MILGRAM
Stanford University
Stanford, California 94305

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
NAVAL WEAPONS CENTER

* C. R. DePrima California Institute of Technology, Pasadena, CA 91109, will replace J. Dugundji until August 1974.

A. Bigard, <i>Free lattice-ordered modules</i>	1
Richard Bolstein and Warren R. Wogen, <i>Subnormal operators in strictly cyclic operator algebras</i>	7
Herbert Busemann and Donald E. Glassco, II, <i>Irreducible sums of simple multivectors</i>	13
W. Wistar (William) Comfort and Victor Harold Saks, <i>Countably compact groups and finest totally bounded topologies</i>	33
Mary Rodriguez Embry, <i>Maximal invariant subspaces of strictly cyclic operator algebras</i>	45
Ralph S. Freese and James Bryant Nation, <i>Congruence lattices of semilattices</i>	51
Ervin Fried and George Grätzer, <i>A nonassociative extension of the class of distributive lattices</i>	59
John R. Giles and Donald Otto Koehler, <i>On numerical ranges of elements of locally m-convex algebras</i>	79
David A. Hill, <i>On dominant and codominant dimension of $\mathbf{QF} - 3$ rings</i>	93
John Sollion Hsia and Robert Paul Johnson, <i>Round and Pfister forms over $R(t)$</i>	101
I. Martin (Irving) Isaacs, <i>Equally partitioned groups</i>	109
Athanassios G. Kartsatos and Edward Barry Saff, <i>Hyperpolynomial approximation of solutions of nonlinear integro-differential equations</i>	117
Shin'ichi Kinoshita, <i>On elementary ideals of θ-curves in the 3-sphere and 2-links in the 4-sphere</i>	127
Ronald Brian Kirk, <i>Convergence of Baire measures</i>	135
R. J. Knill, <i>The Seifert and Van Kampen theorem via regular covering spaces</i>	149
Amos A. Kovacs, <i>Homomorphisms of matrix rings into matrix rings</i>	161
Young K. Kwon, <i>HD-minimal but no HD-minimal</i>	171
Makoto Maejima, <i>On the renewal function when some of the mean renewal lifetimes are infinite</i>	177
Juan José Martínez, <i>Cohomological dimension of discrete modules over profinite groups</i>	185
W. K. Nicholson, <i>Semiperfect rings with abelian group of units</i>	191
Louis Jackson Ratliff, Jr., <i>Three theorems on imbedded prime divisors of principal ideals</i>	199
Billy E. Rhoades and Albert Wilansky, <i>Some commutants in $B(c)$ which are almost matrices</i>	211
John Philip Riley Jr., <i>Cross-sections of decompositions</i>	219
Keith Duncan Stroyan, <i>A characterization of the Mackey uniformity $m(L^\infty, L^1)$ for finite measures</i>	223
Edward G. Thurber, <i>The Scholz-Brauer problem on addition chains</i>	229
Joze Vrabec, <i>Submanifolds of acyclic 3-manifolds</i>	243
Philip William Walker, <i>Adjoint boundary value problems for compactified singular differential operators</i>	265
Roger P. Ware, <i>When are Witt rings group rings</i>	279
James D. Wine, <i>Paracompactifications using filter bases</i>	285