

Pacific Journal of Mathematics

CROSS-SECTIONS OF DECOMPOSITIONS

JOHN PHILIP RILEY JR.

CROSS-SECTIONS OF DECOMPOSITIONS

J. P. RILEY

The following question was raised by R. H. Bing: "Is it true that if G is a monotone decomposition of E^3 into straight line intervals and one-point sets, then E^3/G is homeomorphic to E^3 ?" In his paper "Point-like decompositions of E^3 " he described a possible counter example. This example has the interesting property that it has many tame cross-sections, but if its decomposition space is homeomorphic to E^3 , its set of nondegenerate elements would have to form a wild Cantor set. This suggests that it would be interesting to study the connection between the embedding of a cross-section and the embedding of the set of nondegenerate elements in the decomposition space.

1. Introduction. Most of the terminology and notation used in this paper is standard. The reader is referred to [1], [3], [4], and [6].

If S is a 2-sphere in E^3 , then by $\text{Int } S$ we will mean the bounded component of $E^3 - S$ and by $\text{Ext } S$, the unbounded component.

Let G be an upper semi-continuous decomposition of E^3 and let H be the set of all nondegenerate elements of G . We will say that a set $R \subset E^3$ is a cross-section of G if (i) $R \cap h$ is a singleton for each $h \in H$, and (ii) the natural map P restricted to R is homeomorphism onto $\overline{P(H)}$. We note that cross-sections exist only for certain decompositions. A simple example may be constructed as follows: Let $a_n = 1/n$, for $n = 1, 2, \dots$ and let $b_n = -1/n$ for $n = 1, 2, \dots$. Let the set of nondegenerate elements of our decomposition consist of the closed interval from $(0, 1, 0)$ to $(0, -1, 0)$, the closed interval from $(a_n, 1/2, 0)$ to $(a_n, 1, 0)$ for each positive integer n , and the closed interval from $(b_n, -1/2, 0)$ to $(b_n, -1, 0)$ for each positive integer n .

II. Cross-sections of decompositions. The following question naturally arises: How are the embeddings of a cross-section R and $\overline{P(H)}$ related when E^3/G is homeomorphic to E^3 ? We will give some partial results to this question.

THEOREM 1. *Let G be an upper semi-continuous decomposition of E^3 into points and straight line intervals pointing in only a countable number of directions whose lengths are bounded away from zero such that $P(H)$ is a compact 0-dimensional set. If there exists a cross-section C of G then C is tame.*

Proof. In the special case where the elements of H point in only

one direction, we can easily show the tameness by a modification of the proof of Theorem 2 of [7].

Suppose that $H = \bigcup_{n=1}^{\infty} H_n$ where the elements of H_n are all parallel and if $h_1 \in H_i$ and $h_2 \in H_j$ where $j \neq i$ then h_1 is not parallel to h_2 . Let C_n be the set of all points $c \in C$ such that $c \in h$ for some $h \in H_n$. Let G_n be the upper semi-continuous decomposition of E^3 whose only nondegenerate elements are the elements of H_n and let P_n be the natural map. Then E^3/G_n is homeomorphic to E^3 and $P_n(H_n)$ is tame in E^3/G_n . So by the special case C_n is tame and by Corollary 2 to Theorem 3 of [7], C is tame.

The following two lemmas will be stated without proof. Their proofs are similar to that of Lemma A of [7] and use standard techniques. Lemma B is similar to Theorem 2.3 of [3].

LEMMA A. *Let G be an upper semi-continuous decomposition of E^3 such that $P(H)$ is a compact 0-dimensional set. Let $h \in H$ and suppose that there exist 2-spheres S_1 and S_2 such that $h \subset \text{Int } S_1 \cap \text{Int } S_2$ and $(S_1 \cup S_2) \cap (\cup H) = \emptyset$. Then there exists a 2-sphere S such that $h \subset \text{Int } S$, $S \cup \text{Int } S \subset S_1 \cup \text{Int } S_1$, and if $k \in H$ then $k \subset \text{Int } S$ iff $k \subset \text{Int } S_1 \cap \text{Int } S_2$.*

LEMMA B. *Let S_1, S_2, \dots, S_n be a finite collection of 2-sphere whose interiors cover $\cup H$ and which miss $\cup H$. Then there exists a finite collection of 2-spheres R_1, R_2, \dots, R_n such that $R_i = S_i$, $(R_i \cup \text{Int } R_i) \cap (R_j \cup \text{Int } R_j) = \emptyset$ if $i \neq j$, and $h \subset \text{Int } R_i$ iff $h \subset \text{Int } S_i$ and $h \cap \text{Int } S_j = \emptyset$ for $j < i$.*

THEOREM 2. *Let C be a wild Cantor set in E^3 with the property that if x and y are distinct points of C , then there exist disjoint 2-spheres S_1 and S_2 such that $(S_1 \cup S_2) \cap C = \emptyset$, $x \in \text{Int } S_1 \cap \text{Ext } S_2$ and $y \in \text{Int } S_2 \cap \text{Ext } S_1$. Then there exists a monotone decomposition G of E^3 such that C is a cross-section for G , E^3/G is homeomorphic to E^3 and $P(\bar{H})$ is tame.*

Proof. Let C be a wild Cantor set in E^3 with the required property. For each $x \in C$ we choose a 2-sphere $S_1(x)$ as follows:

Let $N_1(x)$ be a 2-sphere of radius 1/2, centered at x . Let $C_1(x) = \{t \in C \mid t \notin \text{Int } N_1(x)\}$. Then for each $y \in C_1(x)$ choose disjoint 2-spheres $S(y)$ and $R(y)$ such that $(S(y) \cup R(y)) \cap C = \emptyset$, $x \in \text{Int } S(y) \cap \text{Ext } R(y)$, and $y \in \text{Int } R(y) \cap \text{Ext } S(y)$. Now choose a set y_1, y_2, \dots, y_n of elements of $C_1(x)$ such that $\{\text{Int } R(y_1), \text{Int } R(y_2), \dots, \text{Int } R(y_n)\}$ covers $C_1(x)$. We now apply Lemma A to get a 2-sphere $S_1(x)$ such that $x \in \text{Int } S_1(x)$, $S_1(x) \cap C = \emptyset$, $C_1(x) \subset \text{Ext } S_1(x)$ and $S_1(x) \subset S(y_i) \cup \text{Int } S(y_i)$ for $i = 1, 2, \dots, n$. Therefore, there exists a finite collection of points x_1, x_2, \dots ,

$x_{m(1)}$ of C such that $C \subset \text{Int } S_1(x_1) \cup \text{Int } S_1(x_2) \cup \cdots \cup \text{Int } S_1(x_{m(1)})$. We replace $\mathcal{S}_1 = \{S_1(x_1), S_1(x_2), \cdots, S_1(x_{m(1)})\}$ by another collection of 2-spheres $\mathcal{S}_1 = \{T_{11}, T_{12}, \cdots, T_{1n(1)}\}$ satisfying the conclusions of Lemma B with respect to \mathcal{S}_1 .

We will now proceed to construct a sequence $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \cdots$ of finite covers of C . Suppose that \mathcal{S}_{k-1} has been chosen. For each point $x \in C$ we choose a 2-sphere $N_k(x)$ centered at x with radius $1/2^k$. We then proceed to choose \mathcal{S}_k by the same process as in the construction of \mathcal{S}_1 . We note that if $y_1, y_2 \in T_{kj} \cap C$ then $d(y_1, y_2) < 1/2^{k-1}$ since $T_{jk} \cap C \subset N_k(x)$ for some $x \in C$. Now for $x \in C$ we define h_x to be $\bigcap_{k=1}^{\infty} (T_{ki} \cup \text{Int } T_{ki})$ where T_{ki} is the 2-sphere in T_k whose interior contains x . Let G be the decomposition of E^3 whose only nondegenerate elements are the nondegenerate elements of $\{h_x \mid x \in C\}$. It follows easily that G is upper semi-continuous and it is clear that C is a cross-section for G . A theorem of Harrold [5] shows that E^3/G is homeomorphic to E^3 and from the criteria of [3], we see that $\overline{P(H)}$ is tame.

The Cantor set constructed in [2] is an example of a wild Cantor set satisfying the hypothesis of Theorem 2.

We can note that if C is a wild Cantor set in E^3 which does not satisfy the condition of Theorem 2, also, if C is a cross-section of a decomposition G whose decomposition space is homeomorphic to E^3 then $P(H_G)$ is a wild Cantor set which does not satisfy the condition of Theorem 2.

REFERENCES

1. S. Armentrout, *Monotone decompositions of E^3* , Topology Seminar Wisconsin, 1965, Princeton (1966), 1-25.
2. R. H. Bing, *A homeomorphism between the 3-sphere and the sum of two solid horned spheres*, Annals of Mathematics, **56** (1952), 354-362.
3. ———, *Tame Cantor sets in E^3* , Pacific J. Math., **11** (1961), 435-446.
4. ———, *Decompositions of E^3* , Topology of 3-manifolds and Related Topics, Prentice Hall (1962), 5-21.
5. O. G. Harrold, Jr., *A sufficient condition that a monotone image of the three-sphere be a topological three-sphere*, Proc. Amer. Math. Soc., **9** (1958), 846-850.
6. R. L. Moore, *Foundations of Point Set Theory*, (new ed.), Amer. Math. Soc. Colloquium Publications, **13** (1962).
7. J. P. Riley, *Decompositions of E^3 and the tameness of their sets of nondegenerate elements*, Duke Math. J., **38** No. 2, (June, 1971).

Received May 16, 1972 and in revised form September 31, 1972.

UNIVERSITY OF DELAWARE

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

RICHARD ARENS (Managing Editor)
University of California
Los Angeles, California 90024

J. DUGUNDJI*
Department of Mathematics
University of Southern California
Los Angeles, California 90007

R. A. BEAUMONT
University of Washington
Seattle, Washington 98105

D. GILBARG AND J. MILGRAM
Stanford University
Stanford, California 94305

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
NAVAL WEAPONS CENTER

* C. R. DePrima California Institute of Technology, Pasadena, CA 91109, will replace J. Dugundji until August 1974.

Pacific Journal of Mathematics

Vol. 49, No. 1

May, 1973

A. Bigard, <i>Free lattice-ordered modules</i>	1
Richard Bolstein and Warren R. Wogen, <i>Subnormal operators in strictly cyclic operator algebras</i>	7
Herbert Busemann and Donald E. Glassco, II, <i>Irreducible sums of simple multivectors</i>	13
W. Wistar (William) Comfort and Victor Harold Saks, <i>Countably compact groups and finest totally bounded topologies</i>	33
Mary Rodriguez Embry, <i>Maximal invariant subspaces of strictly cyclic operator algebras</i>	45
Ralph S. Freese and James Bryant Nation, <i>Congruence lattices of semilattices</i>	51
Ervin Fried and George Grätzer, <i>A nonassociative extension of the class of distributive lattices</i>	59
John R. Giles and Donald Otto Koehler, <i>On numerical ranges of elements of locally m-convex algebras</i>	79
David A. Hill, <i>On dominant and codominant dimension of $QF - 3$ rings</i>	93
John Sollion Hsia and Robert Paul Johnson, <i>Round and Pfister forms over $R(t)$</i>	101
I. Martin (Irving) Isaacs, <i>Equally partitioned groups</i>	109
Athanassios G. Kartsatos and Edward Barry Saff, <i>Hyperpolynomial approximation of solutions of nonlinear integro-differential equations</i>	117
Shin'ichi Kinoshita, <i>On elementary ideals of θ-curves in the 3-sphere and 2-links in the 4-sphere</i>	127
Ronald Brian Kirk, <i>Convergence of Baire measures</i>	135
R. J. Knill, <i>The Seifert and Van Kampen theorem via regular covering spaces</i>	149
Amos A. Kovacs, <i>Homomorphisms of matrix rings into matrix rings</i>	161
Young K. Kwon, <i>HD-minimal but no HD-minimal</i>	171
Makoto Maejima, <i>On the renewal function when some of the mean renewal lifetimes are infinite</i>	177
Juan José Martínez, <i>Cohomological dimension of discrete modules over profinite groups</i>	185
W. K. Nicholson, <i>Semiperfect rings with abelian group of units</i>	191
Louis Jackson Ratliff, Jr., <i>Three theorems on imbedded prime divisors of principal ideals</i>	199
Billy E. Rhoades and Albert Wilansky, <i>Some commutants in $B(c)$ which are almost matrices</i>	211
John Philip Riley Jr., <i>Cross-sections of decompositions</i>	219
Keith Duncan Stroyan, <i>A characterization of the Mackey uniformity $m(L^\infty, L^1)$ for finite measures</i>	223
Edward G. Thurber, <i>The Scholz-Brauer problem on addition chains</i>	229
Joze Vrabec, <i>Submanifolds of acyclic 3-manifolds</i>	243
Philip William Walker, <i>Adjoint boundary value problems for compactified singular differential operators</i>	265
Roger P. Ware, <i>When are Witt rings group rings</i>	279
James D. Wine, <i>Paracompactifications using filter bases</i>	285