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Let μ be a finite positive measure on a σ-algebra $\mathscr{M}$ over a set X. As usual $L^\infty(\mu)$ denotes the space of μ-essentially bounded measurable functions and $L^1(\mu)$ denotes the space of μ-integrable functions. In this article we use nonstandard analysis to give a simple description of the Mackey uniformity $m(L^\infty, L^1)$. The Mackey uniformity is the finest locally convex linear uniformity on L^∞ for which each continuous linear functional has an L^1 representation. The famous theorem of Mackey-Arens says it is given by uniform convergence on the weakly compact subsets of L^1.

Our description is simply this: Let p be a seminorm on L^∞. Then p is Mackey continuous if and only if whenever g is a finitely bounded element of the nonstandard extension $^*L^\infty$ which is infinitesimal, except possibly on a set of infinitesimal internal measure, then $p(g)$ is infinitesimal.

For the reader who is unfamiliar with nonstandard analysis we remark that ψ is L^∞-norm continuous at f if and only if whenever g is a finitely bounded element of the nonstandard extension which is infinitesimally close to f, except possibly on a set of zero internal measure, then $\psi(g)$ is infinitesimally close to $\psi(f)$. (This follows easily from Robinson's treatment of metric spaces.) We write $f \overset{n}{=} g$ if ($\|f - g\|_\infty$ is finite and) $f(x)$ is infinitesimally close to $g(x), f(x) \approx g(x)$, except possibly on a set of measure zero and say f is a norm-infinitesimal from g.

This characterization uses the idea of a linear infinitesimal relation which generalizes the nonstandard treatment of metric and uniform spaces given by Robinson [6], Luxemburg [3], and Machover and Hirschfeld [5]. The generalization first appeared in the authors dissertation in the context of bounded holomorphic functions, see Stroyan [7] and Luxemburg and Stroyan [4]. The reader is referred to the references [3, 4, 5, 6] for an introduction to standard analysis which we shall not give.

We say a measure λ is μ-continuous if for every $\varepsilon \in \boldsymbol{R}^+$ there is a $\delta \in \boldsymbol{R}^+$ so that whenever $\mu(E) < \delta$ for $E \in \mathscr{M}$, then $|\lambda(E)| < \varepsilon$.

In the nonstandard model an internal measure (or $*$-measure) λ is called *μ-S-continuous* if $\mu(E) \approx 0$, for $E \in {}^*\mathscr{M}$, implies $\lambda(E) \approx 0$. A function $f \in {}^*L^1$ is μ-S-continuous if the $*$-measure $\lambda(E) = \int_E f(x)d\mu(x)$ is μ-S-continuous. This is equivalent to saying that for every standard

$\varepsilon \in {}^{\sigma}\boldsymbol{R}^+ = \{{}^*r : r \in \boldsymbol{R}^+\}$ there exists a standard $\delta \in {}^{\sigma}\boldsymbol{R}^+$ so that if $\mu(E) < \delta$, then $|\lambda(E)| < \varepsilon$.

LEMMA 1. *If $K \subseteqq L^1(\mu)$ is weakly compact, then K is L^1-norm bounded and uniformly μ-continuous, that is, for every $\varepsilon \in \boldsymbol{R}^+$, there exists a $\delta \in \boldsymbol{R}^+$ so that $\mu(E) < \delta$ implies $\left|\int_E k(x)d\mu(x)\right| < \varepsilon$ for any $k \in K$.*

Proof. This standard result can be found, for example, in Dunford and Schwartz [1].

We wish to point out here that K is uniformly μ-continuous if and only if each member of ${}^*K$ is μ-S-continuous. To see the equivalence of these conditions observe that uniform continuity is expressed by the formal sentence

$$(\forall \varepsilon \in \boldsymbol{R}^+)(\exists \delta \in \boldsymbol{R}^+)(\forall k \in K)(\forall E \in \mathscr{M})\left[\mu(E) < \delta \Rightarrow \left|\int_E k d\mu\right| < \varepsilon\right].$$

By Leibniz principle (that 'whatever' is true or false for the standard model is true or false for the nonstandard or ideal one) we have the equivalent sentence in the nonstandard model

$$(\forall \varepsilon \in {}^*\boldsymbol{R}^+)(\exists \delta \in {}^*\boldsymbol{R}^+)(\forall k \in {}^*K)(\forall E \in {}^*\mathscr{M})\left[\mu(E) < \delta \Rightarrow \left|\int_E k d\mu\right| < \varepsilon\right].$$

If K is standard and uniformly μ-continuous and $E_0 \in {}^*\mathscr{M}$ has infinitesimal μ-measure, take $\varepsilon_0 \in {}^{\sigma}\boldsymbol{R}^+$, a standard positive tolerance, and apply the $\varepsilon - \delta$ formula in the standard model to that particular ε_0. That is, there is a standard δ_0, etc. Now shift the particular sentence

$$(\forall k \in K)(\forall E \in \mathscr{M})\left[\mu(E) < \delta_0 \Rightarrow \left|\int_E k d\mu\right| < \varepsilon_0\right]$$

to the nonstandard model (put *'s on K and $\mathscr{M}$). Since $\mu(E_0) < \delta$, the integral is less than an arbitrary standard positive ε_0, hence infinitesimal.

Conversely, if each member of ${}^*K$ is μ-S-continuous and $\varepsilon_0 \in {}^{\sigma}\boldsymbol{R}^+$ is given, then taking $\delta \approx 0$ we see that the formula

$$(\exists \delta \in {}^*\boldsymbol{R}^+)(\forall k \in {}^*K)(\forall E \in {}^*\mathscr{M})\left[\mu(E) < \delta \Rightarrow \left|\int_E k d\mu\right| < \varepsilon_0\right]$$

holds in the nonstandard model. But this formula has a standard interpretation (without the *'s) which amounts to uniform μ-continuity for that particular ε_0. Since ε_0 was an arbitrary standard tolerance we are done.

Another simple nonstandard reformulation is as follows.

LEMMA 2. *Let $\mathscr{F}$ be a family of functions from a set Y into $\boldsymbol{C}$. Let Σ be a collection of subsets of Y. The uniformity of uniform convergence on the sets of Σ is characterized by the infinitesimal relation on ${}^*\mathscr{F}$ given by "$f \overset{\Sigma}{=} g$ if and only if $f(s) \approx g(s)$ for all $s \in \bigcup [{}^*S\colon S \in \Sigma]$". More precisely, the entourages of that uniformity are exactly those subsets U of $\mathscr{F} \times \mathscr{F}$ for which ${}^*U \supseteqq \{(f, g)\colon f \overset{\Sigma}{=} g\}$ and $\{(f, g)\colon f \overset{\Sigma}{=} g\} = \bigcap [{}^*U\colon U$ is an entourage in the standard model$]$.*

Proof. The seminorms $\sup [|f(s) - g(s)|\colon s \in S]$ characterize this uniformity and Luxemburg [3] has shown that the monad of the uniformity given by $\{(f, g)\colon \lambda(f, g) \approx 0$ for all standard semimetrics λ in a gauge of a uniformity$\}$ characterizes the uniformity (in an enlargement).

If $f(s) \approx g(s)$ for $s \in \bigcup ({}^*S\colon S \in \Sigma)$ then the set $\{|f(s) - g(s)|\colon s \in {}^*S\}$ is a bounded internal set. In fact, since it contains only infinitesimals it is bounded by every positively finite number and since that is an external set it actually has infinitesimal bounds. This means that the standard semimetrics are infinitesimal and the converse is clear.

We apply Lemma 2 to the Mackey uniformity to see that in ${}^*L^\infty$ the Mackey infinitesimals are characterized by

$$\text{“}f \overset{m}{=} g \text{ if and only if } \int_X f(x)k(x)d\mu(x) \approx \int_X g(x)k(x)d\mu(x)$$
$$\text{for every weakly compact } k \in \mathrm{cpt}_w\,({}^*L^1)\text{”}\,.$$

The weakly compact points of ${}^*L^1$ are given by

$$\mathrm{cpt}_w\,({}^*L^1) = \bigcup\,[{}^*K\colon K \text{ is a weakly compact subset of } L^1]\,.$$

Now we apply Lemma 1 to see that the weakly compact points of ${}^*L^1$ are norm finitely bounded and μ-S-continuous. This observation makes it clear that the infinitesimal relation:

$$\text{“}f \overset{M}{=} g \text{ if and only if } \|f - g\|_\infty \text{ is finite and } f(x) \approx g(x)$$
$$\text{except on a set of infinitesimal internal measure”}\,,$$

is finer than the Mackey infinitesimals. This is because if k is merely a L^1-norm finite μ-S-continuous ${}^*L^1$ function and $f(x) \approx g(x)$ except on E with $\mu(E) \approx 0$, then

$$\left|\int_X (f(x) - g(x))k(x)d\mu(x)\right| \leqq \left|\int_{X\setminus E} (f - g)kd\mu\right| + \|f - g\| \left|\int_E |k d\mu\right|$$

and both terms on the right side are infinitesimal so that

$$\int_X fkd\mu \approx \int_X gkd\mu\,.$$

The relation $\overset{M}{=}$ is not the monad of a uniformity [3] but it is close enough to $\overset{m}{=}$ to recapture it.

Suppose now that $\varphi\colon L^\infty \to C$ is a standard linear functional satisfying the continuity requirement that whenever $f \overset{M}{=} g$ in ${}^*L^\infty$, then $\varphi(f) \approx \varphi(g)$. We wish to draw two immediate conclusions from this. First, φ is norm-continuous since $f \overset{n}{=} g$ implies $f \overset{M}{=} g$. Second, φ induces a μ-S-continuous standard measure on X via

$$\Phi(E) = \varphi(\chi_E)\ , \quad \text{where} \quad \chi_E(x) = \begin{cases} 0, & x \notin E \\ 1, & x \in E\ . \end{cases}$$

The next result says Φ is countably additive.

LEMMA 3. *If λ is a μ-S-continuous finite internal measure, then $\Lambda(E) = \mathrm{st}\,(\lambda({}^*E))$, for $E \in \mathscr{M}$, is countably additive.*

Proof. Λ is finitely additive by the additivity of st. Given $\varepsilon \in {}^\sigma\boldsymbol{R}^+$ there is a $\delta \in {}^\sigma\boldsymbol{R}^+$ so that if $\mu(E) < \delta$, then $|\lambda(E)| < \varepsilon$. Now take a partition E_k of X. The sum $\Sigma\mu(E_k)$ converges so given δ there is an l so that $\sum_{k=l}^{\infty} \mu(E_k) < \delta$, hence $|\lambda(\bigcup_{k \geqq l} E_k)| < \varepsilon$ and

$$\left| \Lambda(\bigcup E_k) - \sum_{k=1}^{l} \Lambda(E_k) \right| \leqq \varepsilon\ .$$

Now we can apply the Radon-Nikodym theorem to get an L^1 representation for φ. Therefore, $\overset{M}{=}$-continuous standard linear functionals are in L^1, or in other words, $\overset{M}{=}$-continuity is compatible with the dual pair $\langle L^\infty, L^1\rangle$.

Let ψ be an arbitrary (linear or not) standard functional on ${}^*L^\infty$ which satisfies $\overset{M}{=}$-continuity: $f \overset{M}{=} g$ implies $\psi(f) \approx \psi(g)$. Define a uniformity on L^∞ by the semimetrics

$$|\psi(f) - \psi(g)|$$

for ψ standard and $\overset{M}{=}$-continuous.

LEMMA 4. *If $\|f - g\|_\infty$ is finite, then*

$$f \overset{M}{=} g \text{ if and only if } \|f - g\|_1 \approx 0\ .$$

Proof. If $f \overset{M}{=} g$, then $f(x) \approx g(x)$ except on E with $\mu(E) \approx 0$, so

$$\int |f - g|\, d\mu \leqq \int_{X\setminus E} |f - g|\, d\mu + \int_E \|f - g\|_\infty\, d\mu \approx 0\ .$$

Conversely, suppose $\int |f - g|\, d\mu \approx 0$. For each $n \in {}^*N$ define the internal sequence

$$\varepsilon_n = \mu\{x: |f(x) - g(x)| > 1/n\}\,.$$

We know that for standard $n \in {}^\sigma N$, $\varepsilon_n \approx 0$ and Robinson's infinitesimal sequence lemma ([6], Theorem 3.3.20 or [4], Theorem 8. 1. 4) says ε_n is infinitesimal out to some infinite subscript, so $f \overset{M}{=} g$.

Fix a standard functional ψ, we will show that there exists a sequence ε_n so that

$$\bigcup\, [F(n, \varepsilon_n): n \in N] \subseteqq \{(f, g): |\psi(f) - \psi(g)| < 1\}$$

where

$$F(n, \varepsilon) = \{(f, g): \|f - g\|_\infty < n \quad \text{and} \quad \|f - g\|_1 < \varepsilon\}\,.$$

Take $n \in N$, since $\overset{M}{=}$ agrees with L^1-infinitesimals on the set $\{(f, g): \|f - g\|_\infty < n\}$ we know $(\exists \varepsilon \in {}^*\boldsymbol{R}^+)[F(n, \varepsilon) \subseteqq \{(f, g): |\psi(f) - \psi(g)| < 1\}]$ holds in the nonstandard model by taking $\varepsilon \approx 0$. Therefore, the same sentence holds in the standard model, so select such an ε and call it ε_n.

Sets of the form $\bigcup\, [F(n, \varepsilon_n): n \in N]$ generate a standard linear uniformity finer than that generated by the ψ's. This is the finest uniformity agreeing with the L^1-norm on L^∞-norm bounded sets.

Finally, consider the collection of standard $\overset{M}{=}$-continuous seminorms p on L^∞, that is, if $f \overset{M}{=} g$ then $p(f - g) \approx 0$. These generate the finest locally convex linear uniformity whose monad contains $\overset{M}{=}$. This is the Mackey topology since any Mackey-continuous seminorm is $\overset{M}{=}$-continuous and every $\overset{M}{=}$-continuous linear functional has an L^1 representation.

We have shown:

THEOREM. *The Mackey uniformity $m(L^\infty, L^1)$ for a finite measure is characterized by the infinitesimal relation on ${}^*L^\infty$ given by*:

"*$f \overset{M}{=} g$ if and only if $\|f - g\|_\infty$ is finite and $f(x)$ is infinitesimally close to $g(x)$, except possibility on a set of infinitesimal measure*".

Precisely, a standard seminorm $p: L^\infty \longrightarrow \boldsymbol{R}^+$ is Mackey continuous if and only if whenever $f \overset{M}{=} g$, then $p(f - g) \approx 0$.

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