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**ON FINDING THE DISTRIBUTION FUNCTION FOR AN
ORTHOGONAL POLYNOMIAL SET**

WM. R. ALLAWAY

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Let $\{a_n\}_{n=0}^\infty$ and $\{b_n\}_{n=0}^\infty$ be real sequences with $b_n > 0$ and $\{b_n\}_{n=0}^\infty$ bounded. Let $\{P_n(x)\}_{n=0}^\infty$ be a sequence of polynomials satisfying the recurrence formula

$$(1.1) \quad \begin{cases} xP_n(x) = b_{n-1}P_{n-1}(x) + a_nP_n(x) + b_nP_{n+1}(x) & (n \geq 0) \\ P_{-1}(x) = 0 & P_0(x) = 1. \end{cases}$$

Then there is a substantially unique distribution function $\psi(t)$ with respect to which the $P_n(x)$ are orthogonal. That is,

$$\int_{-\infty}^{\infty} P_n(x)P_m(x)d\psi(x) = K_n\delta_{n,m} \quad (n, m \geq 0),$$

where $K_n \neq 0$ and $\delta_{n,m}$ is the kronecker delta. This paper gives a method of constructing $\psi(x)$ for the case $\lim_{n \rightarrow \infty} b_{2n} = 0$, $\lim_{n \rightarrow \infty} b_{2n+1} = b < \infty$, the set of limit points of $\{a_n\}_{n=1}^\infty$ equals $\{-\alpha, \alpha\}$ and $\lim_{n \rightarrow \infty} \{a_{2n} + a_{2n+1}\} = 0$. The same method can be used in the case $\lim_{n \rightarrow \infty} b_n = 0$ and the set of limit points of $\{a_n\}_{n=0}^\infty$ is bounded and finite in number.

This continues the investigation started by Dickinson, Pollak, and Wannier [3] in which they studied the distribution function under the assumption $a_n = 0$ and $\sum b_n < \infty$. Goldberg [4] extended their results by considering the case $a_n = 0$ and $\lim_{n \rightarrow \infty} b_n = 0$. Finally, Maki [5] showed how to construct the distribution function when $\lim_{n \rightarrow \infty} b_n = 0$ and the set of limit points of $\{a_n\}_{n=0}^\infty$ are bounded and finite in number. In all these cases their approach was to study the continued fraction

$$(1.2) \quad K(z) = \frac{1}{|z - a_0|} - \frac{b_0^2}{|z - a_1|} - \frac{b_1^2}{|z - a_2|} \dots,$$

where $\{b_n\}_{n=0}^\infty$ and $\{a_n\}_{n=0}^\infty$ consist of the same numbers as given in (1.1).

Our approach is different from that of the above mentioned authors. If $S(\psi)$ denotes the spectrum of ψ , i.e., the set $\{\lambda \mid \psi(\lambda + \varepsilon) - \psi(\lambda - \varepsilon) > 0 \text{ for all } \varepsilon > 0\}$, then, in our case, we will show from the properties of the sequences $\{a_n\}$ and $\{b_n\}$ how to find the derived set of $S(\psi)$ and that the $S(\psi)$ consists of a denumerable set of points.

To prove our results we make use of the following theorem due to M. Krein ([1], p. 230-231).

THEOREM 1.1. *The polynomial set defined by (1.1) is associated with a determined Hamburger moment problem with solution ψ , such that $S(\psi)$ is bounded and the set of limit points of $S(\psi)$ is contained*

in $\{\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_p\}$ (α_i real) if and only if the numbers a_i and b_i ($i=0, 1, 2, \dots$) form a bounded set and $\lim_{i,k \rightarrow \infty} g_{i,k} = 0$ where $g_{i,j}$ is the element in the i th row and j th column of the matrix

$$\prod_{i=1}^p (A - \alpha_i I),$$

where

$$A = \begin{vmatrix} a_0 & b_0 & 0 & \dots \\ b_0 & a_1 & b_1 & \dots \\ 0 & b_1 & a_2 & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}.$$

2. Our main results.

THEOREM 2.1. *Let $\lim_{n \rightarrow \infty} b_{2n} = 0$ and $\lim_{n \rightarrow \infty} b_{2n+1} = b < \infty$, where $b > 0$. The set of limit points of $\{a_n\}_{n=0}^\infty$ is $\{-\alpha, \alpha\}$ and $\lim_{n \rightarrow \infty} \{a_{2n-1} + a_{2n}\} = 0$ if and only if the derived set of $S(\psi)$ equals*

$$\{-(\alpha^2 + b^2)^{1/2}, (\alpha^2 + b^2)^{1/2}\}.$$

Proof. By using the notation of Theorem 1.1, it is easy to show that the element in the i th row and j th column of the matrix $A^2 - (\alpha^2 + b^2)I$ is given by

$$g_{n,n+j} = \begin{cases} 0 & \text{if } |j| > 2, \\ b_{n-1} b_n & \text{if } j = 2, \\ b_{n-1}(a_{n-1} + a_n) & \text{if } j = 1, \\ b_{n-2}^2 + a_{n-1}^2 + b_{n-1}^2 - \alpha^2 - b^2 & \text{if } j = 0, \\ b_{n-2}(a_{n-2} + a_{n-1}) & \text{if } j = -1, \\ b_{n-2} b_{n-3} & \text{if } j = -2. \end{cases}$$

Let $\{-(\alpha^2 + b^2)^{1/2}, (\alpha^2 + b^2)^{1/2}\}$ constitute the derived set of $S(\psi)$. Because $\{b_n\}_{n=0}^\infty$ is bounded, then the Hamburger moment problem associated with (1.1) is determined (see [7], p. 59). Thus by Theorem 1.1 $\lim_{i,j \rightarrow \infty} g_{i,j} = 0$. Therefore, $\lim_{n \rightarrow \infty} (a_{2n-1} + a_{2n}) = 0$ and $\lim_{n \rightarrow \infty} (a_n^2 - \alpha^2) = 0$. But this implies that the set of limit points of $\{a_n\}_{n=0}^\infty$ is $\{-\alpha, \alpha\}$.

Conversely if the limit points of $\{a_n\}_{n=0}^\infty$ is $\{-\alpha, \alpha\}$ and

$$\lim_{n \rightarrow \infty} (a_{2n-1} + a_{2n}) = 0,$$

then $\lim_{i,j \rightarrow \infty} g_{i,j} = 0$. Thus by Theorem 1.1 this implies that the

derived set of $S(\psi)$ has $-(\alpha^2 + b^2)^{1/2}$ and $(\alpha^2 + b^2)^{1/2}$ as its only two points. This completes the proof of the theorem.

Let k be a positive integer and $\{\alpha_1, \alpha_2, \dots, \alpha_k\}$ be a set of real numbers. If $g_{i,j,k}$ is the element in the i th row and j th column of the matrix

$$\prod_{i=1}^k (A - \alpha_i I)$$

then it is easy to show by mathematical induction on k that

$$(2.1) \quad g_{n,n-i,k} = \begin{cases} h_{i,n,k} \prod_{l=1}^i b_{n-l-1} & \text{if } 1 \leq i \leq k, \\ s_{n,k} b_{n-1}^2 + q_{n,k} b_{n-2}^2 + \prod_{i=1}^k (a_{n-1} - \alpha_i) & \text{if } i = 0, \\ r_{i,n,k} \prod_{l=0}^{-i-1} b_{n+l-1} & \text{if } -k \leq i \leq -1, \\ 0 & \text{if } |i| > k, \end{cases}$$

where $\{h_{i,n,k}\}$, $\{r_{i,n,k}\}$, $\{s_{n,k}\}$, and $\{q_{n,k}\}$ are bounded sequences in n for fixed k and i .

By using Equation (2.1) and the same technique as that used in the proof of Theorem 2.1 we have

THEOREM 2.2. *Let $\lim_{n \rightarrow \infty} b_n = 0$ and $\{a_n\}_{n=0}^\infty$ be a bounded sequence. The derived set of $S(\psi)$ equals $\{\alpha_1, \alpha_2, \dots, \alpha_p\}$ if and only if the set of limit points of $\{a_n\}_{n=0}^\infty$ is $\{\alpha_1, \alpha_2, \dots, \alpha_p\}$.*

Proof. Let $L = \{\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_p\}$ be the set of limit points of $\{a_n\}_{n=0}^\infty$. From Equation (2.1) and Theorem 1.1 we have that D , the derived set of $S(\psi)$, is contained in L . Assume D is a proper subset of L . That is, $D = \{\beta_1, \beta_2, \beta_3, \dots, \beta_k\}$ where $k < p$. Thus, if $g_{i,j,k}$ is the element in the i th row and j th column of the matrix $\prod_{i=1}^k (A - \beta_i I)$, then by Theorem 1.1 and Equation (2.1)

$$\lim_{n \rightarrow \infty} \prod_{i=1}^k (a_{n-1} - \beta_i) = 0.$$

That is, L is a proper subset of D . But this is a contradiction. Thus $D = L$.

The converse may be proved in a similar manner.

Maki [6] conjectured, that in the case $\lim_{n \rightarrow \infty} b_n = 0$, the set of limit points of $S(\psi)$ equals the set of limit points of $\{a_n\}_{n=0}^\infty$. Theorem 2.2 shows that this conjecture is true for the case when $\{a_n\}_{n=0}^\infty$ is bounded and has a finite set of limit points. Chihara [2] has shown by using the theory of continued fractions that Maki's conjecture is true in general.

3. Construction of the distribution function. Because the sequence $\{b_n\}_{n=1}^\infty$ is bounded we are dealing with the determined Hamburger moment problem, so the continued fraction given in Equation (1.2) converges uniformly on every closed half plane,

$$(3.1) \quad \text{Im}(z) \geq s > 0 ,$$

to an analytic function $F(z)$ which is not a rational function. $F(z)$ has the form

$$(3.2) \quad F(z) = \int_{-\infty}^\infty (z - t)^{-1} d\psi(t) ,$$

where z satisfies (3.1). The polynomial set $\{P_n(x)\}_{n=0}^\infty$ given in (1.1) is orthogonal on $(-\infty, \infty)$ with respect to the distribution $\psi(x)$.

Let us define,

$$A(x) = \psi(x + 0) - \psi(x - 0) .$$

LEMMA 3.1. *Let T be a bounded countable set of real numbers such that the derived set of T is $B = \{\beta_1, \beta_2, \dots, \beta_n\}$. Also let*

$$\begin{aligned} H &= T \setminus B \\ &= \{h_i \mid i = 1, 2, 3, \dots\} . \end{aligned}$$

(i) $S(\psi) = H \cup B$ $A(h_j) = M_j$ ($j = 1, 2, 3, \dots$), and $A(\beta_k) = N_k$ ($k = 1, 2, 3, \dots, n$), if and only if

(ii) $M_j > 0$ ($j = 1, 2, 3, \dots$), $N_k \geq 0$ ($k = 1, 2, 3, \dots, n$),

$$\sum_{j=1}^\infty M_j + \sum_{k=1}^n N_k < \infty ,$$

and

$$F(z) = \sum_{j=1}^\infty (z - h_j)^{-1} M_j + \sum_{k=1}^n (z - \beta_k)^{-1} N_k .$$

Proof. It is easy to show that $S(\psi)$ is closed. From this and by the definition of the Lebesgue-Stieltjes Integral, (i) implies (ii). Also from the fact that $S(\psi)$ is closed and from the Stieltjes inversion formula, (ii) implies (i). This completes the proof of the lemma.

Let \mathcal{C} represent the field of complex numbers.

THEOREM 3.1. *Let $\lim_{n \rightarrow \infty} b_{2n} = 0$ and $\lim_{n \rightarrow \infty} b_{2n+1} = b < \infty$, where $b > 0$. Also let the set of limit points of $\{a_n\}_{n=0}^\infty$ be $\{-\alpha, \alpha\}$ and $\lim_{n \rightarrow \infty} \{a_{2n-1} + a_{2n}\} = 0$.*

(i) $K(z)$ as defined by Equation (1.2) is a meromorphic function in $\mathcal{C} \setminus \{-(\alpha^2 + b^2)^{1/2}, (\alpha^2 + b^2)^{1/2}\}$ and it has a representation of the form

$$(3.4) \quad K(z) = \frac{A(-(\alpha^2 + b^2)^{1/2})}{z + (\alpha^2 + b^2)^{1/2}} + \frac{A((\alpha^2 + b^2)^{1/2})}{z - (\alpha^2 + b^2)^{1/2}} + \sum_{i=0}^{\infty} \frac{A(t_i)}{z - t_i}$$

where $A(\pm(\alpha^2 + b^2)^{1/2}) \geq 0$ and $A(t_i) > 0$.

(ii) If $T = \{t_i \mid i = 1, 2, 3 \dots\}$, where t_i is as given in Equation (3.4), then $S(\psi) = T \cup \{-(\alpha^2 + b^2)^{1/2}, (\alpha^2 + b^2)^{1/2}\}$.

(iii) The limit points of $S(\psi)$ are $-(\alpha^2 + b^2)^{1/2}$ and $(\alpha^2 + b^2)^{1/2}$.

Proof. We know from Theorem 2.1 that $S(\psi)$ is countable and its derived set consists only of the points $-(\alpha^2 + b^2)^{1/2}$ and $(\alpha^2 + b^2)^{1/2}$. Thus by Lemma 3.1

$$F(z) = \frac{A(-(\alpha^2 + b^2)^{1/2})}{z + (\alpha^2 + b^2)^{1/2}} + \frac{A((\alpha^2 + b^2)^{1/2})}{z - (\alpha^2 + b^2)^{1/2}} + \sum_{i=1}^{\infty} \frac{A(t_i)}{z - t_i}$$

where $T \cup \{-(\alpha^2 + b^2)^{1/2}, (\alpha^2 + b^2)^{1/2}\} = S(\psi)$. Because ψ is monotonically non-decreasing and $-(\alpha^2 + b^2)^{1/2}, (\alpha^2 + b^2)^{1/2}$ are the only limit points of its spectrum we obtain, $A(t_i) > 0$ for $t_i \in T$ and

$$A(\pm(\alpha^2 + b^2)^{1/2}) \geq 0 .$$

But the continued fraction given in Equation 1.2 converges uniformly to $F(z)$ on any closed bounded set that doesn't contain $S(\psi)$. Thus $K(z) = F(z)$, for $z \notin S(\psi)$. This completes the proof of the theorem.

By working directly with $K(z)$ Maki ([5] Theorem (5.4)) proves that if $\lim_{n \rightarrow \infty} b_n = 0$ and the set of limit points of $\{a_n\}_{n=0}^{\infty}$ is $\{\alpha_1, \alpha_2 \dots \alpha_p\}$ with $|\alpha_i| < \infty \ i = 1, 2 \dots p$, then

(i) $K(z)$ is a meromorphic function in $\mathcal{C} \setminus \{\alpha_1, \alpha_2, \dots, \alpha_p\}$ and has a representation of the form

$$(3.5) \quad K(z) = \sum_{i=1}^p (z - \alpha_i)^{-1} A(\alpha_i) + \sum_{i=0}^{\infty} (z - t_i)^{-1} A(t_i) ,$$

where $A(\alpha_i) \geq 0$ and $A(t_i) > 0$,

(ii) if $T = \{t_i \mid i = 1, 2, 3 \dots\}$ where t_i is as given in Equation (3.5), then $S(\psi) = \{\alpha_1, \alpha_2, \dots, \alpha_p\} \cup T$, and

(iii) the derived set of $S(\psi)$ is $\{\alpha_1, \alpha_2, \dots, \alpha_p\}$.

By using Theorem 2.2 and a technique similar to the one used in our proof of Theorem 3.1 it is easy to see how to give a short proof of Maki's theorem.

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