

# Pacific Journal of Mathematics

**ON FINDING THE DISTRIBUTION FUNCTION FOR AN  
ORTHOGONAL POLYNOMIAL SET**

WM. R. ALLAWAY

## ON FINDING THE DISTRIBUTION FUNCTION FOR AN ORTHOGONAL POLYNOMIAL SET

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Let  $\{a_n\}_{n=0}^\infty$  and  $\{b_n\}_{n=0}^\infty$  be real sequences with  $b_n > 0$  and  $\{b_n\}_{n=0}^\infty$  bounded. Let  $\{P_n(x)\}_{n=0}^\infty$  be a sequence of polynomials satisfying the recurrence formula

$$(1.1) \quad \begin{cases} xP_n(x) = b_{n-1}P_{n-1}(x) + a_nP_n(x) + b_nP_{n+1}(x) & (n \geq 0) \\ P_{-1}(x) = 0 & P_0(x) = 1. \end{cases}$$

Then there is a substantially unique distribution function  $\phi(t)$  with respect to which the  $P_n(x)$  are orthogonal. That is,

$$\int_{-\infty}^{\infty} P_n(x)P_m(x)d\phi(x) = K_n\delta_{n,m} \quad (n, m \geq 0),$$

where  $K_n \neq 0$  and  $\delta_{n,m}$  is the kronecker delta. This paper gives a method of constructing  $\phi(x)$  for the case  $\lim_{n \rightarrow \infty} b_{2n} = 0$ ,  $\lim_{n \rightarrow \infty} b_{2n+1} = b < \infty$ , the set of limit points of  $\{a_n\}_{n=1}^\infty$  equals  $\{-\alpha, \alpha\}$  and  $\lim_{n \rightarrow \infty} \{a_{2n} + a_{2n+1}\} = 0$ . The same method can be used in the case  $\lim_{n \rightarrow \infty} b_n = 0$  and the set of limit points of  $\{a_n\}_{n=0}^\infty$  is bounded and finite in number.

This continues the investigation started by Dickinson, Pollak, and Wannier [3] in which they studied the distribution function under the assumption  $a_n = 0$  and  $\sum b_n < \infty$ . Goldberg [4] extended their results by considering the case  $a_n = 0$  and  $\lim_{n \rightarrow \infty} b_n = 0$ . Finally, Maki [5] showed how to construct the distribution function when  $\lim_{n \rightarrow \infty} b_n = 0$  and the set of limit points of  $\{a_n\}_{n=0}^\infty$  are bounded and finite in number. In all these cases their approach was to study the continued fraction

$$(1.2) \quad K(z) = \frac{1}{|z - a_0|} - \frac{b_0^2}{|z - a_1|} - \frac{b_1^2}{|z - a_2|} \dots,$$

where  $\{b_n\}_{n=0}^\infty$  and  $\{a_n\}_{n=0}^\infty$  consist of the same numbers as given in (1.1).

Our approach is different from that of the above mentioned authors. If  $S(\psi)$  denotes the spectrum of  $\psi$ , i.e., the set  $\{\lambda \mid \psi(\lambda + \varepsilon) - \psi(\lambda - \varepsilon) > 0 \text{ for all } \varepsilon > 0\}$ , then, in our case, we will show from the properties of the sequences  $\{a_n\}$  and  $\{b_n\}$  how to find the derived set of  $S(\psi)$  and that the  $S(\psi)$  consists of a denumerable set of points.

To prove our results we make use of the following theorem due to M. Krein ([1], p. 230-231).

**THEOREM 1.1.** *The polynomial set defined by (1.1) is associated with a determined Hamburger moment problem with solution  $\psi$ , such that  $S(\psi)$  is bounded and the set of limit points of  $S(\psi)$  is contained*

in  $\{\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_p\}$  ( $\alpha_i$  real) if and only if the numbers  $a_i$  and  $b_i$  ( $i=0, 1, 2, \dots$ ) form a bounded set and  $\lim_{i,k \rightarrow \infty} g_{i,k} = 0$  where  $g_{i,j}$  is the element in the  $i$ th row and  $j$ th column of the matrix

$$\prod_{i=1}^p (A - \alpha_i I),$$

where

$$A = \begin{pmatrix} a_0 & b_0 & 0 & \dots \\ b_0 & a_1 & b_1 & \dots \\ 0 & b_1 & a_2 & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}.$$

2. Our main results.

**THEOREM 2.1.** *Let  $\lim_{n \rightarrow \infty} b_{2n} = 0$  and  $\lim_{n \rightarrow \infty} b_{2n+1} = b < \infty$ , where  $b > 0$ . The set of limit points of  $\{a_n\}_{n=0}^\infty$  is  $\{-\alpha, \alpha\}$  and  $\lim_{n \rightarrow \infty} \{a_{2n-1} + a_{2n}\} = 0$  if and only if the derived set of  $S(\psi)$  equals*

$$\{-(\alpha^2 + b^2)^{1/2}, (\alpha^2 + b^2)^{1/2}\}.$$

*Proof.* By using the notation of Theorem 1.1, it is easy to show that the element in the  $i$ th row and  $j$ th column of the matrix  $A^2 - (\alpha^2 + b^2)I$  is given by

$$g_{n,n+j} = \begin{cases} 0 & \text{if } |j| > 2, \\ b_{n-1} b_n & \text{if } j = 2, \\ b_{n-1}(a_{n-1} + a_n) & \text{if } j = 1, \\ b_{n-2}^2 + a_{n-1}^2 + b_{n-1}^2 - \alpha^2 - b^2 & \text{if } j = 0, \\ b_{n-2}(a_{n-2} + a_{n-1}) & \text{if } j = -1, \\ b_{n-2} b_{n-3} & \text{if } j = -2. \end{cases}$$

Let  $\{-(\alpha^2 + b^2)^{1/2}, (\alpha^2 + b^2)^{1/2}\}$  constitute the derived set of  $S(\psi)$ . Because  $\{b_n\}_{n=0}^\infty$  is bounded, then the Hamburger moment problem associated with (1.1) is determined (see [7], p. 59). Thus by Theorem 1.1  $\lim_{i,j \rightarrow \infty} g_{i,j} = 0$ . Therefore,  $\lim_{n \rightarrow \infty} (a_{2n-1} + a_{2n}) = 0$  and  $\lim_{n \rightarrow \infty} (a_n^2 - \alpha^2) = 0$ . But this implies that the set of limit points of  $\{a_n\}_{n=0}^\infty$  is  $\{-\alpha, \alpha\}$ .

Conversely if the limit points of  $\{a_n\}_{n=0}^\infty$  is  $\{-\alpha, \alpha\}$  and

$$\lim_{n \rightarrow \infty} (a_{2n-1} + a_{2n}) = 0,$$

then  $\lim_{i,j \rightarrow \infty} g_{i,j} = 0$ . Thus by Theorem 1.1 this implies that the

derived set of  $S(\psi)$  has  $-(\alpha^2 + b^2)^{1/2}$  and  $(\alpha^2 + b^2)^{1/2}$  as its only two points. This completes the proof of the theorem.

Let  $k$  be a positive integer and  $\{\alpha_1, \alpha_2, \dots, \alpha_k\}$  be a set of real numbers. If  $g_{i,j,k}$  is the element in the  $i$ th row and  $j$ th column of the matrix

$$\prod_{i=1}^k (A - \alpha_i I)$$

then it is easy to show by mathematical induction on  $k$  that

$$(2.1) \quad g_{n,n-i,k} = \begin{cases} h_{i,n,k} \prod_{l=1}^i b_{n-l-1} & \text{if } 1 \leq i \leq k, \\ s_{n,k} b_{n-1}^2 + q_{n,k} b_{n-2}^2 + \prod_{i=1}^k (a_{n-1} - \alpha_i) & \text{if } i = 0, \\ r_{i,n,k} \prod_{l=0}^{-i-1} b_{n+l-1} & \text{if } -k \leq i \leq -1, \\ 0 & \text{if } |i| > k, \end{cases}$$

where  $\{h_{i,n,k}\}$ ,  $\{r_{i,n,k}\}$ ,  $\{s_{n,k}\}$ , and  $\{q_{n,k}\}$  are bounded sequences in  $n$  for fixed  $k$  and  $i$ .

By using Equation (2.1) and the same technique as that used in the proof of Theorem 2.1 we have

**THEOREM 2.2.** *Let  $\lim_{n \rightarrow \infty} b_n = 0$  and  $\{a_n\}_{n=0}^\infty$  be a bounded sequence. The derived set of  $S(\psi)$  equals  $\{\alpha_1, \alpha_2, \dots, \alpha_p\}$  if and only if the set of limit points of  $\{a_n\}_{n=0}^\infty$  is  $\{\alpha_1, \alpha_2, \dots, \alpha_p\}$ .*

*Proof.* Let  $L = \{\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_p\}$  be the set of limit points of  $\{a_n\}_{n=0}^\infty$ . From Equation (2.1) and Theorem 1.1 we have that  $D$ , the derived set of  $S(\psi)$ , is contained in  $L$ . Assume  $D$  is a proper subset of  $L$ . That is,  $D = \{\beta_1, \beta_2, \beta_3, \dots, \beta_k\}$  where  $k < p$ . Thus, if  $g_{i,j,k}$  is the element in the  $i$ th row and  $j$ th column of the matrix  $\prod_{i=1}^k (A - \beta_i I)$ , then by Theorem 1.1 and Equation (2.1)

$$\lim_{n \rightarrow \infty} \prod_{i=1}^k (a_{n-1} - \beta_i) = 0.$$

That is,  $L$  is a proper subset of  $D$ . But this is a contradiction. Thus  $D = L$ .

The converse may be proved in a similar manner.

Maki [6] conjectured, that in the case  $\lim_{n \rightarrow \infty} b_n = 0$ , the set of limit points of  $S(\psi)$  equals the set of limit points of  $\{a_n\}_{n=0}^\infty$ . Theorem 2.2 shows that this conjecture is true for the case when  $\{a_n\}_{n=0}^\infty$  is bounded and has a finite set of limit points. Chihara [2] has shown by using the theory of continued fractions that Maki's conjecture is true in general.

**3. Construction of the distribution function.** Because the sequence  $\{b_n\}_{n=1}^\infty$  is bounded we are dealing with the determined Hamburger moment problem, so the continued fraction given in Equation (1.2) converges uniformly on every closed half plane,

$$(3.1) \quad \text{Im}(z) \geq s > 0 ,$$

to an analytic function  $F(z)$  which is not a rational function.  $F(z)$  has the form

$$(3.2) \quad F(z) = \int_{-\infty}^{\infty} (z - t)^{-1} d\psi(t) ,$$

where  $z$  satisfies (3.1). The polynomial set  $\{P_n(x)\}_{n=0}^\infty$  given in (1.1) is orthogonal on  $(-\infty, \infty)$  with respect to the distribution  $\psi(x)$ .

Let us define,

$$A(x) = \psi(x + 0) - \psi(x - 0) .$$

**LEMMA 3.1.** *Let  $T$  be a bounded countable set of real numbers such that the derived set of  $T$  is  $B = \{\beta_1, \beta_2, \dots, \beta_n\}$ . Also let*

$$\begin{aligned} H &= T \setminus B \\ &= \{h_i \mid i = 1, 2, 3, \dots\} . \end{aligned}$$

(i)  $S(\psi) = H \cup B$   $A(h_j) = M_j$  ( $j = 1, 2, 3, \dots$ ), and  $A(\beta_k) = N_k$  ( $k = 1, 2, 3, \dots, n$ ), if and only if

(ii)  $M_j > 0$  ( $j = 1, 2, 3, \dots$ ),  $N_k \geq 0$  ( $k = 1, 2, 3, \dots, n$ ),

$$\sum_{j=1}^{\infty} M_j + \sum_{k=1}^n N_k < \infty ,$$

and

$$F(z) = \sum_{j=1}^{\infty} (z - h_j)^{-1} M_j + \sum_{k=1}^n (z - \beta_k)^{-1} N_k .$$

*Proof.* It is easy to show that  $S(\psi)$  is closed. From this and by the definition of the Lebesgue-Stieltjes Integral, (i) implies (ii). Also from the fact that  $S(\psi)$  is closed and from the Stieltjes inversion formula, (ii) implies (i). This completes the proof of the lemma.

Let  $\mathcal{E}$  represent the field of complex numbers.

**THEOREM 3.1.** *Let  $\lim_{n \rightarrow \infty} b_{2n} = 0$  and  $\lim_{n \rightarrow \infty} b_{2n+1} = b < \infty$ , where  $b > 0$ . Also let the set of limit points of  $\{a_n\}_{n=0}^\infty$  be  $\{-\alpha, \alpha\}$  and  $\lim_{n \rightarrow \infty} \{a_{2n-1} + a_{2n}\} = 0$ .*

(i)  $K(z)$  as defined by Equation (1.2) is a meromorphic function in  $\mathcal{E} \setminus \{-(\alpha^2 + b^2)^{1/2}, (\alpha^2 + b^2)^{1/2}\}$  and it has a representation of the form

$$(3.4) \quad K(z) = \frac{A(-(\alpha^2 + b^2)^{1/2})}{z + (\alpha^2 + b^2)^{1/2}} + \frac{A((\alpha^2 + b^2)^{1/2})}{z - (\alpha^2 + b^2)^{1/2}} + \sum_{i=0}^{\infty} \frac{A(t_i)}{z - t_i}$$

where  $A(\pm(\alpha^2 + b^2)^{1/2}) \geq 0$  and  $A(t_i) > 0$ .

(ii) If  $T = \{t_i \mid i = 1, 2, 3, \dots\}$ , where  $t_i$  is as given in Equation (3.4), then  $S(\psi) = T \cup \{-(\alpha^2 + b^2)^{1/2}, (\alpha^2 + b^2)^{1/2}\}$ .

(iii) The limit points of  $S(\psi)$  are  $-(\alpha^2 + b^2)^{1/2}$  and  $(\alpha^2 + b^2)^{1/2}$ .

*Proof.* We know from Theorem 2.1 that  $S(\psi)$  is countable and its derived set consists only of the points  $-(\alpha^2 + b^2)^{1/2}$  and  $(\alpha^2 + b^2)^{1/2}$ . Thus by Lemma 3.1

$$F(z) = \frac{A(-(\alpha^2 + b^2)^{1/2})}{z + (\alpha^2 + b^2)^{1/2}} + \frac{A((\alpha^2 + b^2)^{1/2})}{z - (\alpha^2 + b^2)^{1/2}} + \sum_{i=1}^{\infty} \frac{A(t_i)}{z - t_i}$$

where  $T \cup \{-(\alpha^2 + b^2)^{1/2}, (\alpha^2 + b^2)^{1/2}\} = S(\psi)$ . Because  $\psi$  is monotonically non-decreasing and  $-(\alpha^2 + b^2)^{1/2}, (\alpha^2 + b^2)^{1/2}$  are the only limit points of its spectrum we obtain,  $A(t_i) > 0$  for  $t_i \in T$  and

$$A(\pm(\alpha^2 + b^2)^{1/2}) \geq 0.$$

But the continued fraction given in Equation 1.2 converges uniformly to  $F(z)$  on any closed bounded set that doesn't contain  $S(\psi)$ . Thus  $K(z) = F(z)$ , for  $z \notin S(\psi)$ . This completes the proof of the theorem.

By working directly with  $K(z)$  Maki ([5] Theorem (5.4)) proves that if  $\lim_{n \rightarrow \infty} b_n = 0$  and the set of limit points of  $\{\alpha_n\}_{n=0}^{\infty}$  is  $\{\alpha_1, \alpha_2, \dots, \alpha_p\}$  with  $|\alpha_i| < \infty$   $i = 1, 2, \dots, p$ , then

(i)  $K(z)$  is a meromorphic function in  $\mathcal{E} \setminus \{\alpha_1, \alpha_2, \dots, \alpha_p\}$  and has a representation of the form

$$(3.5) \quad K(z) = \sum_{i=1}^p (z - \alpha_i)^{-1} A(\alpha_i) + \sum_{i=0}^{\infty} (z - t_i)^{-1} A(t_i),$$

where  $A(\alpha_i) \geq 0$  and  $A(t_i) > 0$ ,

(ii) if  $T = \{t_i \mid i = 1, 2, 3, \dots\}$  where  $t_i$  is as given in Equation (3.5), then  $S(\psi) = \{\alpha_1, \alpha_2, \dots, \alpha_p\} \cup T$ , and

(iii) the derived set of  $S(\psi)$  is  $\{\alpha_1, \alpha_2, \dots, \alpha_p\}$ .

By using Theorem 2.2 and a technique similar to the one used in our proof of Theorem 3.1 it is easy to see how to give a short proof of Maki's theorem.

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