

Pacific Journal of Mathematics

**SUR UN THÉORÈME DE MOONEY RELATIF AUX
FONCTIONS ANALYTIQUES BORNÉES**

ERIC AMAR

SUR UN THÉORÈME DE MOONEY RELATIF AUX FONCTIONS ANALYTIQUES BORNÉES

ERIC AMAR

We give a short proof of the following theorem of Mooney:
**If $\{\phi_n\}$ is a sequence in $L^1(-\pi, \pi)$ such that $\lim_n \int f\phi_n = l(f)$
 exists for all $f \in H^\infty$, then there is $\phi \in L^1$ such that $l(f) = \int f\phi$
 for all $f \in H^\infty$.**

NOTONS. $H^\infty(D)$ l'algèbre uniforme des fonctions analytiques bornées dans:

$$D = \{z \in \mathbb{C}; |z| < 1\}$$

$$T = \{z \in \mathbb{C}; |z| = 1\}; \quad \text{et } L^1(T, \lambda)$$

l'espace des fonctions intégrables pour la mesure de Haar λ sur T .

$$A(D) = H^\infty(D) \cap C(T).$$

Le but de cette note est de donner une autre démonstration du théorème suivant dû à Mooney [4]:

THÉORÈME. *Si $\{\phi_n\}_{n \in \mathbb{N}}$ est une suite d'éléments de $L^1(T, \lambda)$ telle que:*

$$\forall f \in H^\infty(D), l(f) = \lim_{n \rightarrow \infty} \int f\phi_n d\lambda$$

existe, alors il existe $\phi \in L^1(T, \lambda)$ tel que:

$$(*) \quad l(f) = \int f\phi d\lambda \quad \forall f \in H^\infty(D).$$

Cette démonstration est basée sur une proposition démontrée dans [3] qui permet d'appliquer à $H^\infty(D)$ la méthode que J. P. Kahane [2] a utilisé pour démontrer (*) lorsque f appartient à $A(D)$.

1. Reduction du probleme. Soit X la frontière de Shilov de $H^\infty(D)$; m la mesure représentative unique de l'origine, alors on sait [1] que $L^1(X, m)$ est isométriquement isomorphe à $L^1(T, \lambda)$.

Utilisant cet isomorphisme, les données sont alors les suivantes: Soit $\{\phi_n\}_{n \in \mathbb{N}}$ une suite de $L^1(X, m)$ telle que:

$$\forall f \in H^\infty(D); l(f) = \lim_{n \rightarrow \infty} \int_X \hat{f}\phi_n dm$$

existe; expression dans laquelle \hat{f} désigne la transformée de Gelfand de f . Il s'agit alors de trouver $\phi \in L^1(X, m)$ telle que:

$$l(f) = \int_x \hat{f} \phi \, dm \quad \forall f \in H^\infty(D).$$

Par le théorème de Banach-Steinhaus, $l(f)$ est une forme linéaire continue sur $H^\infty(D)$.

Par le théorème de Hahn-Banach, il existe une mesure μ borélienne sur X telle que:

$$l(f) = \int_x \hat{f} \, d\mu.$$

Décomposons μ par rapport à m :

$$d\mu = \eta \, dm + d\mu_s; \quad \eta \in L^1(X, m); \quad \mu_s$$

singulière par rapport à m . Posant:

$$\Phi_n = \phi_n - \eta \text{ on a: } \lim_{n \rightarrow \infty} \int_x \hat{f} \Phi_n \, dm = \int_x \hat{f} \, d\mu_s.$$

Pour démontrer le théorème il suffit alors de montrer que μ_s est orthogonale à $H^\infty(D)$.

2. Rappelons la proposition de [2]. (***) Pour tout compact K de X , de m mesure nulle, il existe un compact P tel que:

- (i) $K \subset P \subset X$
- (ii) $m(P) = 0$
- (iii) P est pic pour $H^\infty(D)$ sur X .

On en déduit le lemme suivant:

LEMME. Soit ν une mesure sur X ; ν singulière par rapport à m et telle que:

$$\nu(X) = \alpha \neq 0.$$

Alors il existe un ensemble pic P pour $H^\infty(D)$ tel que:

$$\nu(P) \neq 0 \text{ et } m(P) = 0.$$

Démonstration. $|\nu|$ est singulière donc:

$$\forall \varepsilon > 0, \exists K \text{ compact } \subset X$$

tel que:

$$m(K) = 0; \quad |\nu|(X \setminus K) < \varepsilon.$$

Par (**):

$\exists P \supset K; m(P) = 0; P$ pic pour $H^\infty(D)$ sur X .

Donc

$$|\nu|(X \setminus P) < \varepsilon \quad \text{et} \quad |\nu(P)| \geq |\nu(X)| - \varepsilon = |\alpha| - \varepsilon.$$

D'où le lemme avec $\varepsilon = |\alpha|/2$.

3. Démonstration du théorème. Supposons qu'il existe $g \in H^\infty(D)$ tel que:

$$\int \hat{g} d\mu_s \neq 0.$$

Posant: $d\nu = \hat{g} d\mu_s$, on peut appliquer le lemme à ν . Donc il existe P , pic pour $H^\infty(D)$ tel que:

$$m(P) = 0 \quad \text{et} \quad |\nu(P)| > 0.$$

Soit alors $h \in H^\infty(D)$ une fonction qui pique sur P :

$$\begin{aligned} \hat{h}(x) &= 1 \quad \text{si} \quad x \in P \\ |\hat{h}(x)| &< 1 \quad \text{si} \quad x \notin P; \end{aligned}$$

en particulier $|h(z)| < 1; z \in D$. Soit enfin:

$$\Psi_n = \hat{g} \Phi_n \in L^1(X, m).$$

On a:

$$\lim_{n \rightarrow \infty} \int \hat{f} \Psi_n dm = \lim_{n \rightarrow \infty} \int \hat{f} \hat{g} \Phi_n dm = \int \hat{f} \hat{g} d\mu_s; \quad \forall f \in H^\infty(D), \text{ car } fg \in H^\infty(D).$$

De plus:

$$(1) \quad \lim_{k \rightarrow \infty} \int_X \hat{h}^k d\nu = \nu(P) \neq 0$$

$$(2) \quad \lim_{k \rightarrow \infty} \int_X \hat{h}^k \Psi_n dm = 0 \quad \forall n \in \mathbb{N}$$

$$(3) \quad \lim_{n \rightarrow \infty} \int_X \hat{h}^k \Psi_n dm = \int_X \hat{h}^k d\nu \quad \forall k \in \mathbb{N}.$$

Posant:

$$f = \sum_0^\infty (-1)^j h^{m_j}; \quad m_j \in \mathbb{N},$$

on montre comme dans [3], que pour une suite de m_j croissant assez vite, $f \in H^\infty(D)$.

Reprenant exactement la méthode de J. P. Kahane [2], on arrive à la contradiction:

$$\int_x \hat{f} \Psi_n dm$$

n'est pas de Cauchy.

Donc il n'existe pas de $g \in H^\infty(D)$ tel que $\int \hat{g} d\mu_s \neq 0$, et μ_s est bien orthogonale à $H^\infty(D)$, ce qui achève la démonstration.

BIBLIOGRAPHIE

1. Gamelin, *Uniform Algebras*, Prentice Hall (1969).
2. J. P. Kahane, Proc. Amer. Math. Soc., **18** (1967), 827-831.
3. A. Lederer et E. Amar, C. R. Acad. Sc. Paris, p. 1449-1452 (2 juin 1971).
4. Mooney, *A theorem on bounded analytic functions*.

Received September 13, 1972.

FACULTE DES SCIENCES 91, ORSAY, FRANCE

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

RICHARD ARENS (Managing Editor)

University of California
Los Angeles, California 90024

J. DUGUNDJI*

Department of Mathematics
University of Southern California
Los Angeles, California 90007

R. A. BEAUMONT

University of Washington
Seattle, Washington 98105

D. GILBARG AND J. MILGRAM

Stanford University
Stanford, California 94305

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
NAVAL WEAPONS CENTER

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. **39**. All other communications to the editors should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$48.00 a year (6 Vols., 12 issues). Special rate: \$24.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

* C. R. DePrima California Institute of Technology, Pasadena, CA 91109, will replace J. Dugundji until August 1974.

Copyright © 1973 by
Pacific Journal of Mathematics
All Rights Reserved

Pacific Journal of Mathematics

Vol. 49, No. 2

June, 1973

Wm. R. Allaway, <i>On finding the distribution function for an orthogonal polynomial set</i>	305
Eric Amar, <i>Sur un théorème de Mooney relatif aux fonctions analytiques bornées</i>	311
Robert Morgan Brooks, <i>Analytic structure in the spectrum of a natural system</i>	315
Bahattin Cengiz, <i>On extremely regular function spaces</i>	335
Kwang-nan Chow and Moses Glasner, <i>Atoms on the Royden boundary</i>	339
Paul Frazier Duvall, Jr. and Jim Maxwell, <i>Tame Z^2-actions on E^n</i>	349
Allen Roy Freedman, <i>On the additivity theorem for n-dimensional asymptotic density</i>	357
John Griffin and Kelly Denis McKennon, <i>Multipliers and the group L_p-algebras</i>	365
Charles Lemuel Hagopian, <i>Characterizations of λ connected plane continua</i>	371
Jon Craig Helton, <i>Bounds for products of interval functions</i>	377
Ikuko Kayashima, <i>On relations between Nörlund and Riesz means</i>	391
Everett Lee Lady, <i>Slender rings and modules</i>	397
Shozo Matsuura, <i>On the Lu Qi-Keng conjecture and the Bergman representative domains</i>	407
Stephen H. McCleary, <i>The lattice-ordered group of automorphisms of an α-set</i>	417
Stephen H. McCleary, <i>o-2-transitive ordered permutation groups</i>	425
Stephen H. McCleary, <i>o-primitive ordered permutation groups. II</i>	431
Richard Rochberg, <i>Almost isometries of Banach spaces and moduli of planar domains</i>	445
R. F. Rossa, <i>Radical properties involving one-sided ideals</i>	467
Robert A. Rubin, <i>On exact localization</i>	473
S. Sribala, <i>On Σ-inverse semigroups</i>	483
H. M. (Hari Mohan) Srivastava, <i>On the Konhauser sets of biorthogonal polynomials suggested by the Laguerre polynomials</i>	489
Stuart A. Steinberg, <i>Rings of quotients of rings without nilpotent elements</i>	493
Daniel Mullane Sunday, <i>The self-equivalences of an H-space</i>	507
W. J. Thron and Richard Hawks Warren, <i>On the lattice of proximities of Čech compatible with a given closure space</i>	519
Frank Uhlig, <i>The number of vectors jointly annihilated by two real quadratic forms determines the inertia of matrices in the associated pencil</i>	537
Frank Uhlig, <i>On the maximal number of linearly independent real vectors annihilated simultaneously by two real quadratic forms</i>	543
Frank Uhlig, <i>Definite and semidefinite matrices in a real symmetric matrix pencil</i> ...	561
Arnold Lewis Villone, <i>Self-adjoint extensions of symmetric differential operators</i>	569
Cary Webb, <i>Tensor and direct products</i>	579
James Victor Whittaker, <i>On normal subgroups of differentiable homeomorphisms</i>	595
Jerome L. Paul, <i>Addendum to: "Sequences of homeomorphisms which converge to homeomorphisms"</i>	615
David E. Fields, <i>Correction to: "Dimension theory in power series rings"</i>	616
Peter Michael Curran, <i>Correction to: "Cohomology of finitely presented groups"</i>	617
Billy E. Rhoades, <i>Correction to: "Commutants of some Hausdorff matrices"</i>	617
Charles W. Trigg, <i>Corrections to: "Versum sequences in the binary system"</i>	619