

# Pacific Journal of Mathematics

**SUR UN THÉORÈME DE MOONEY RELATIF AUX  
FONCTIONS ANALYTIQUES BORNÉES**

ERIC AMAR

## SUR UN THÉORÈME DE MOONEY RELATIF AUX FONCTIONS ANALYTIQUES BORNÉES

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**We give a short proof of the following theorem of Mooney:**  
**If  $\{\phi_n\}$  is a sequence in  $L^1(-\pi, \pi)$  such that  $\lim_n \int f\phi_n = \iota(f)$**   
**exists for all  $f \in H^\infty$ , then there is  $\phi \in L^1$  such that  $\iota(f) = \int f\phi$**   
**for all  $f \in H^\infty$ .**

NOTONS.  $H^\infty(D)$  l'algèbre uniforme des fonctions analytiques bornées dans:

$$D = \{z \in \mathbb{C}; |z| < 1\}$$

$$T = \{z \in \mathbb{C}; |z| = 1\}; \quad \text{et } L^1(T, \lambda)$$

l'espace des fonctions intégrables pour la mesure de Haar  $\lambda$  sur  $T$ .

$$A(D) = H^\infty(D) \cap C(T).$$

Le but de cette note est de donner une autre démonstration du théorème suivant dû à Mooney [4]:

**THÉORÈME.** *Si  $\{\phi_n\}_{n \in \mathbb{N}}$  est une suite d'éléments de  $L^1(T, \lambda)$  telle que:*

$$\forall f \in H^\infty(D), \iota(f) = \lim_{n \rightarrow \infty} \int f\phi_n d\lambda$$

*existe, alors il existe  $\phi \in L^1(T, \lambda)$  tel que:*

$$(*) \quad \iota(f) = \int f\phi d\lambda \quad \forall f \in H^\infty(D).$$

Cette démonstration est basée sur une proposition démontrée dans [3] qui permet d'appliquer à  $H^\infty(D)$  la méthode que J. P. Kahane [2] a utilisé pour démontrer (\*) lorsque  $f$  appartient à  $A(D)$ .

**1. Réduction du problème.** Soit  $X$  la frontière de Shilov de  $H^\infty(D)$ ;  $m$  la mesure représentative unique de l'origine, alors on sait [1] que  $L^1(X, m)$  est isométriquement isomorphe à  $L^1(T, \lambda)$ .

Utilisant cet isomorphisme, les données sont alors les suivantes: Soit  $\{\phi_n\}_{n \in \mathbb{N}}$  une suite de  $L^1(X, m)$  telle que:

$$\forall f \in H^\infty(D); \iota(f) = \lim_{n \rightarrow \infty} \int_X \hat{f}\phi_n dm$$

existe; expression dans laquelle  $\hat{f}$  désigne la transformée de Gelfand de  $f$ . Il s'agit alors de trouver  $\phi \in L^1(X, m)$  telle que:

$$l(f) = \int_X \hat{f} \phi \, dm \quad \forall f \in H^\infty(D).$$

Par le théorème de Banach-Steinhaus,  $l(f)$  est une forme linéaire continue sur  $H^\infty(D)$ .

Par le théorème de Hahn-Banach, il existe une mesure  $\mu$  borélienne sur  $X$  telle que:

$$l(f) = \int_X \hat{f} \, d\mu.$$

Décomposons  $\mu$  par rapport à  $m$ :

$$d\mu = \eta \, dm + d\mu_s; \quad \eta \in L^1(X, m); \quad \mu_s$$

singulière par rapport à  $m$ . Posant:

$$\Phi_n = \phi_n - \eta \text{ on a: } \lim_{n \rightarrow \infty} \int_X \hat{f} \Phi_n \, dm = \int_X \hat{f} \, d\mu_s.$$

Pour démontrer le théorème il suffit alors de montrer que  $\mu_s$  est orthogonale à  $H^\infty(D)$ .

2. Rappelons la proposition de [2]. (\*\*). Pour tout compact  $K$  de  $X$ , de  $m$  mesure nulle, il existe un compact  $P$  tel que:

- (i)  $K \subset P \subset X$
- (ii)  $m(P) = 0$
- (iii)  $P$  est pic pour  $H^\infty(D)$  sur  $X$ .

On en déduit le lemme suivant:

LEMME. Soit  $\nu$  une mesure sur  $X$ ;  $\nu$  singulière par rapport à  $m$  et telle que:

$$\nu(X) = \alpha \neq 0.$$

Alors il existe un ensemble pic  $P$  pour  $H^\infty(D)$  tel que:

$$\nu(P) \neq 0 \text{ et } m(P) = 0.$$

Démonstration.  $|\nu|$  est singulière donc:

$$\forall \varepsilon > 0, \exists K \text{ compact } \subset X$$

tel que:

$$m(K) = 0; \quad |\nu|(X \setminus K) < \varepsilon.$$

Par (\*\*):

$\exists P \supset K; m(P) = 0; P$  pic pour  $H^\infty(D)$  sur  $X$ .

Donc

$$|\nu|(X \setminus P) < \varepsilon \quad \text{et} \quad |\nu(P)| \geq |\nu(X)| - \varepsilon = |\alpha| - \varepsilon.$$

D'où le lemme avec  $\varepsilon = |\alpha|/2$ .

**3. Démonstration du théorème.** Supposons qu'il existe  $g \in H^\infty(D)$  tel que:

$$\int \hat{g} d\mu_s \neq 0.$$

Posant:  $d\nu = \hat{g} d\mu_s$ , on peut appliquer le lemme à  $\nu$ . Donc il existe  $P$ , pic pour  $H^\infty(D)$  tel que:

$$m(P) = 0 \quad \text{et} \quad |\nu(P)| > 0.$$

Soit alors  $h \in H^\infty(D)$  une fonction qui pique sur  $P$ :

$$\begin{aligned} \hat{h}(x) &= 1 \quad \text{si} \quad x \in P \\ |\hat{h}(x)| &< 1 \quad \text{si} \quad x \notin P; \end{aligned}$$

en particulier  $|h(z)| < 1; z \in D$ . Soit enfin:

$$\Psi_n = \hat{g}\Phi_n \in L^1(X, m).$$

On a:

$$\lim_{n \rightarrow \infty} \int \hat{f} \Psi_n dm = \lim_{n \rightarrow \infty} \int \hat{f} \hat{g} \Phi_n dm = \int \hat{f} \hat{g} d\mu_s; \quad \forall f \in H^\infty(D), \text{ car } fg \in H^\infty(D).$$

De plus:

$$(1) \quad \lim_{k \rightarrow \infty} \int_X \hat{h}^k d\nu = \nu(P) \neq 0$$

$$(2) \quad \lim_{k \rightarrow \infty} \int_X \hat{h}^k \Psi_n dm = 0 \quad \forall n \in \mathbb{N}$$

$$(3) \quad \lim_{n \rightarrow \infty} \int_X \hat{h}^k \Psi_n dm = \int_X \hat{h}^k d\nu \quad \forall k \in \mathbb{N}.$$

Posant:

$$f = \sum_0^\infty (-1)^j h^{m_j}; \quad m_j \in \mathbb{N},$$

on montre comme dans [3], que pour une suite de  $m_j$  croissant assez vite,  $f \in H^\infty(D)$ .

Reprenant exactement la méthode de J. P. Kahane [2], on arrive à la contradiction:

$$\int_x \hat{f} \Psi_n dm$$

n'est pas de Cauchy.

Donc il n'existe pas de  $g \in H^\infty(D)$  tel que  $\int \hat{g} d\mu_s \neq 0$ , et  $\mu_s$  est bien orthogonale à  $H^\infty(D)$ , ce qui achève la démonstration.

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Wm. R. Allaway, <i>On finding the distribution function for an orthogonal polynomial set</i> .....	305
Eric Amar, <i>Sur un théorème de Mooney relatif aux fonctions analytiques bornées</i> .....	311
Robert Morgan Brooks, <i>Analytic structure in the spectrum of a natural system</i> .....	315
Bahattin Cengiz, <i>On extremely regular function spaces</i> .....	335
Kwang-nan Chow and Moses Glasner, <i>Atoms on the Royden boundary</i> .....	339
Paul Frazier Duvall, Jr. and Jim Maxwell, <i>Tame <math>Z^2</math>-actions on <math>E^n</math></i> .....	349
Allen Roy Freedman, <i>On the additivity theorem for <math>n</math>-dimensional asymptotic density</i> .....	357
John Griffin and Kelly Denis McKennon, <i>Multipliers and the group <math>L_p</math>-algebras</i> .....	365
Charles Lemuel Hagopian, <i>Characterizations of <math>\lambda</math> connected plane continua</i> .....	371
Jon Craig Helton, <i>Bounds for products of interval functions</i> .....	377
Ikuko Kayashima, <i>On relations between Nörlund and Riesz means</i> .....	391
Everett Lee Lady, <i>Slender rings and modules</i> .....	397
Shozo Matsuura, <i>On the Lu Qi-Keng conjecture and the Bergman representative domains</i> .....	407
Stephen H. McCleary, <i>The lattice-ordered group of automorphisms of an <math>\alpha</math>-set</i> .....	417
Stephen H. McCleary, <i><math>o - 2</math>-transitive ordered permutation groups</i> .....	425
Stephen H. McCleary, <i><math>o</math>-primitive ordered permutation groups. II</i> .....	431
Richard Rochberg, <i>Almost isometries of Banach spaces and moduli of planar domains</i> .....	445
R. F. Rossa, <i>Radical properties involving one-sided ideals</i> .....	467
Robert A. Rubin, <i>On exact localization</i> .....	473
S. Sribala, <i>On <math>\Sigma</math>-inverse semigroups</i> .....	483
H. M. (Hari Mohan) Srivastava, <i>On the Konhauser sets of biorthogonal polynomials suggested by the Laguerre polynomials</i> .....	489
Stuart A. Steinberg, <i>Rings of quotients of rings without nilpotent elements</i> .....	493
Daniel Mullane Sunday, <i>The self-equivalences of an <math>H</math>-space</i> .....	507
W. J. Thron and Richard Hawks Warren, <i>On the lattice of proximities of Čech compatible with a given closure space</i> .....	519
Frank Uhlig, <i>The number of vectors jointly annihilated by two real quadratic forms determines the inertia of matrices in the associated pencil</i> .....	537
Frank Uhlig, <i>On the maximal number of linearly independent real vectors annihilated simultaneously by two real quadratic forms</i> .....	543
Frank Uhlig, <i>Definite and semidefinite matrices in a real symmetric matrix pencil</i> .....	561
Arnold Lewis Villone, <i>Self-adjoint extensions of symmetric differential operators</i> .....	569
Cary Webb, <i>Tensor and direct products</i> .....	579
James Victor Whittaker, <i>On normal subgroups of differentiable homeomorphisms</i> .....	595
Jerome L. Paul, <i>Addendum to: "Sequences of homeomorphisms which converge to homeomorphisms"</i> .....	615
David E. Fields, <i>Correction to: "Dimension theory in power series rings"</i> .....	616
Peter Michael Curran, <i>Correction to: "Cohomology of finitely presented groups"</i> .....	617
Billy E. Rhoades, <i>Correction to: "Commutants of some Hausdorff matrices"</i> .....	617
Charles W. Trigg, <i>Corrections to: "Versum sequences in the binary system"</i> .....	619