

Pacific Journal of Mathematics

TAME Z^2 -ACTIONS ON E^n

PAUL FRAZIER DUVALL, JR. AND JIM MAXWELL

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P. F. DUVAL, JR. AND J. W. MAXWELL

Let $\mathcal{H}(E^n)$ denote the group of homeomorphisms of euclidean n -space, and G a subgroup isomorphic to $Z \oplus Z$. G is said to be a Z^2 -action on E^n and two such actions are said to be equivalent if they are conjugate in $\mathcal{H}(E^n)$. In §2, the notion of a tame Z^2 -action is introduced and for $n \geq 5$ tame Z^2 -actions are shown to be classified by $\pi_1(SO_{n-2}) \cong Z_2$. In §3, tameness is shown to be inherited by a subaction of a tame Z^2 -action and an example of a nontame Z^2 -action with tame subactions is given.

1. Introduction. Let U be an n -dimensional manifold and let $\mathcal{H}(U)$ denote the group of homeomorphisms of U onto itself with the compact open topology. If G is a subgroup of $\mathcal{H}(U)$, we say that G acts on U and refer to G as an action. If K is a topological group which is isomorphic to G , we refer to G as a K -action. Two actions are (topologically) equivalent if they are conjugate in $\mathcal{H}(U)$. We say that G satisfies Sperner's condition if for each compact set $X \subset U$ the set $\{g \in G \mid g(X) \cap X \neq \emptyset\}$ is finite. Unless otherwise stated, all actions on E^n will be assumed to be orientation preserving, i.e., we require that each member of an action be orientation preserving.

If Z^i denotes the free abelian group on i generators, we have, for $i \leq n$, the standard Z^i -action on E^n , generated by the maps h_j , $j = 1, \dots, i$, where $h_j(x_1, \dots, x_n) = (x_1, \dots, x_j + 1, \dots, x_n)$. It is a classical result that a Z -action on E^2 is equivalent to the standard action if and only if it satisfies Sperner's condition, and Duvall and Husch [3] showed that for $n \neq 4$, a Z^n -action on E^n is equivalent to the standard action if and only if it satisfies Sperner's condition. In general, however, Sperner's condition is not sufficient to insure that a Z^k -action is equivalent to the standard action. Examples of non-standard actions which satisfy Sperner's condition may be found in [9], [10], [6], and [3]. Husch [6], and Husch and Row [7] have shown that the standard Z -action for $n > 4$ and the standard Z and Z^2 -actions for $n = 3$ are characterized by Sperner's condition together with an additional homotopy condition.

In this note, we define the notion of a tame action (inspired by [6]) and show that the tame Z^2 -actions on E^n , $n \geq 5$, are classified by $\pi_1(SO_{n-2}) \cong Z_2$. We also give examples of some nonstandard Z^2 -actions. We will often use the fact that if G satisfies Sperner's condition and has no elements of finite order, then the orbit space U/G is a (T_2) manifold and the natural projection $U \rightarrow U/G$ is a regular covering

map [8]. We use the symbol \cong to denote homeomorphism or isomorphism, depending on the context. For equivalent formulations of Sperner's condition and any notation not specifically explained here, the reader is referred to [3].

2. Tame actions. Recall that a sequence $\{G_i, \alpha_i\}_{i=1}^\infty$

$$G_1 \xleftarrow{\alpha_1} G_2 \xleftarrow{\alpha_2} \dots$$

of groups and homomorphisms is *stable* if for some subsequence

$$G_{i_1} \xleftarrow{\beta_1} G_{i_2} \xleftarrow{\beta_2} G_{i_3} \xleftarrow{\dots}$$

we have

$$\beta_n |_{\text{image } \beta_{n+1}} : \text{image } \beta_{n+1} \longrightarrow \text{image } \beta_n$$

is an isomorphism for each n , where $\beta_n = \alpha_{i_n} \alpha_{i_n+1} \dots \alpha_{i_{n+1}-1}$. We omit the proof of the following proposition. One implication is proved in [7]; the other may be verified by a routine diagram chase.

PROPOSITION 1. *In the commutative diagram*

$$\begin{array}{ccccccc}
 & & 1 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 \dots & \longrightarrow & A_{i+1} & \xrightarrow{\alpha_i} & A_i & \longrightarrow & A_{i-1} & \longrightarrow & \dots & \longrightarrow & A_1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 \dots & \longrightarrow & B_{i+1} & \xrightarrow{\beta_i} & B_i & \longrightarrow & B_{i-1} & \longrightarrow & \dots & \longrightarrow & B_1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 \dots & \longrightarrow & C_{i+1} & \xrightarrow{\gamma_i} & C_i & \xrightarrow{\gamma_{i-1}} & C_{i-1} & \longrightarrow & \dots & \longrightarrow & C_1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & & 1 & &
 \end{array}$$

of groups and homomorphisms, suppose that the columns are exact and that the γ_i are isomorphisms. Then, $\{A_i, \alpha_i\}_{i=1}^\infty$ is stable if and only if $\{B_i, \beta_i\}_{i=1}^\infty$ is stable. If either sequence is stable, the induced sequence $1 \rightarrow \varprojlim A_i \rightarrow \varprojlim B_i \rightarrow \varprojlim C_i \rightarrow 1$ is exact.

Let G be a Z^2 -action on E^n . For each $X \subset E^n$, let GX denote the set $\{g(X) \mid g \in G\}$. We say that G is *tame* provided that:

- (1) G satisfies Sperner's condition and
- (2) For each compact set $C \subset E^n$, there is a compact set $D \subset E^n$

containing C such that the inclusion induced map $l_*: \pi_j(E^n - GD) \rightarrow \pi_j(E^n - GC)$ is the zero map for $j = 0, 1$.

We assume now that G is a tame Z^2 -action and that $n \geq 5$. Let O_G be the orbit space E^n/G . Then O_G is a manifold and the projection $p: E^n \rightarrow O_G$ is a covering map. Since O_G is an Eilenberg-McLane $K(Z^2, 1)$ space, it follows [15] that O_G has the homotopy type of the torus $T^2 = S^1 \times S^1$. Since $H_n(O_G) \cong H_{n-1}(O_G) \cong 0$, it follows that O_G is noncompact and has one end [14].

Let $\{D_i\}_{i=1}^\infty$ be a collection of compact subsets of O_G such that $\bigcup D_i = O_G$, $D_{i+1} \supset D_i$ for each i , and $O_G - D_i$ is connected for each i . Using (2) above, we can find a nested sequence $\{C_i\}_{i=1}^\infty$ of compact subsets of E^n such that $GC_i \supset p^{-1}D_i$ and $l_*: \pi_j(E^n - GC_{i+1}) \rightarrow \pi_j(E^n - GC_i)$ is zero for each i and $j = 0, 1$. By choosing a subsequence of D_i 's if necessary, we can assume without loss of generality that $GC_i = p^{-1}D_i$.

PROPOSITION 2. $\pi_1(O_G, O_G - D_i) = 0$ for each i , and $E^n - GC_i$ is connected.

Proof. Let x be a base point for $O_G - D_i$, and assume without loss of generality that $x \in O_G - D_{i+1}$. Let \hat{x} be a point in $E^n - GC_{i+1}$ such that $p(\hat{x}) = x$. Let $\alpha: (I, \{0, 1\}, \{0\}) \rightarrow (O_G, O_G - D_i, x)$ be a map, and assume (without loss of generality) that $\alpha(1) \in O_G - D_{i+1}$. Let $\hat{\alpha}$ be the lift of α based at \hat{x} . We have $\hat{x}, \hat{\alpha}(1) \in E^n - GC_{i+1}$ so by (2) there is a path β in $E^n - GC_i$ joining \hat{x} and $\hat{\alpha}(1)$. Since $\hat{\alpha}$ and β are homotopic with endpoints fixed in E^n , α and $p\beta$ are homotopic in O_G , so that $[\alpha] = 0$ in $\pi_1(O_G, O_G - D_i)$. The second conclusion follows from the first by a covering space argument.

PROPOSITION 3. If ε is the end of O_G , π_1 is stable at ε and the natural projection $\pi_1(\varepsilon) \rightarrow \pi_1(O_G)$ is an isomorphism.

Proof. For each i , let $x_i \in E^n - GC_i$ be a base point, $y_i = p(x_i)$, and let $\alpha_i, p\alpha_i$ be connecting paths between x_i, x_{i+1} and y_i, y_{i+1} . We have the following diagram from the exact homotopy sequences of a fibration

$$\begin{array}{ccccccc}
 & & & & & & 1 \\
 & & & & & & \uparrow \\
 & & & & & & 1 \\
 1 & \longrightarrow & \pi_1(E^n - GC_i, x_i) & \longrightarrow & \pi_1(O_G - D_i, y_i) & \longrightarrow & \pi_0(p^{-1}(y_i)) & \longrightarrow & 1 \\
 & & & & \searrow & & \uparrow & & \nearrow \\
 & & & & \cong & & \cong & & \\
 & & & & & & \pi_1(O_G, y_i) & & \\
 & & & & & & \uparrow & & \\
 & & & & & & 1 & &
 \end{array}$$

which gives rise to the commutative diagram

$$\begin{array}{ccccccc}
 & \vdots & & \vdots & & \vdots & \\
 & \uparrow & & \uparrow & & \uparrow & \\
 1 \longrightarrow & \pi_1(E^n - GC_i, x_i) & \longrightarrow & \pi_1(O_G - D_i, y_i) & \longrightarrow & \pi_1(O_G, y_i) & \longrightarrow 1 \\
 & \uparrow & & \uparrow & & \uparrow \gamma_i & \\
 1 \longrightarrow & \pi_1(E^n - GC_{i+1}, x_{i+1}) & \longrightarrow & \pi_1(O_G - D_{i+1}, y_{i+1}) & \longrightarrow & \pi_1(O_G, y_{i+1}) & \longrightarrow 1 \\
 & \uparrow & & \uparrow & & \uparrow & \\
 & \vdots & & \vdots & & \vdots &
 \end{array}$$

where the columns are change of base point maps and each γ_i is an isomorphism. We apply Proposition 1 to see that π_1 is stable at ε and that

$$\begin{array}{c}
 1 \longrightarrow \varprojlim \{\pi_1(E^n - GC_i, x_i)\} \longrightarrow \varprojlim \{\pi_1(O_G - D_i, y_i)\} \\
 \longrightarrow \varprojlim \{\pi_1(O_G, y_i)\} \longrightarrow 1
 \end{array}$$

is exact. This translates into $1 \rightarrow 1 \rightarrow \pi_1(\varepsilon) \rightarrow \pi_1(O_G) \rightarrow 1$, and the proof is complete.

Now by [11], we may assume that O_G has a (unique) *PL* structure. There is a map $f: T^2 \rightarrow O_G$ which is a homotopy equivalence. We may assume f to be an embedding by general position. Let $\tau_G = f(T^2)$. By a theorem of Hudson [5], τ_G is unique up to concordance, hence, up to ambient isotopy [4].

THEOREM 4. *If G is a tame Z^2 -action on E^n , $n \geq 5$, then O_G is *PL*-homeomorphic to the interior of a regular neighborhood of τ_G .*

Proof. Let N be a regular neighborhood of τ_G in O_G . From the exact sequence of the triad $(O_G, O_G - \tau_G, \text{int } N)$ [1], we have

$$\begin{array}{l}
 \dots \longrightarrow \pi_i(O_G - \tau_G, \text{int } N - \tau_G) \longrightarrow \pi_i(O_G, \text{int } N) \\
 \longrightarrow \pi_i(O_G; O_G - \tau_G, \text{int } N) \longrightarrow \pi_{i-1}(O_G - \tau_G, \text{int } N - \tau_G) \longrightarrow \dots \\
 \longrightarrow \pi_2(O_G, O_G - \tau_G, \text{int } N) \longrightarrow \pi_1(O_G - \tau_G, \text{int } N - \tau_G) \\
 \longrightarrow \pi_1(O_G, \text{int } N) .
 \end{array}$$

Since $\pi_i(O_G, \text{int } N) \cong 0$ for all i , we can apply [4, Lemma 12.4] to get $\pi_i(O_G, O_G - \tau_G, \text{int } N) \cong 0$ for all i , and thus $\pi_i(O_G - \tau_G, \text{int } N - \tau_G) \cong 0$ for all i . It follows that the inclusion $\text{int } N - \tau_G \rightarrow O_G - \tau_G$ is a homotopy equivalence and hence the inclusion $\text{bdy } N \rightarrow O_G - \text{int } N$ is a homotopy equivalence. General position gives an inclusion induced isomorphism $\pi_1(O_G - \text{int } N) \cong \pi_1(O_G)$ so the projection $\pi_1(\varepsilon) \rightarrow \pi_1(O_G - \text{int } N)$ is an isomorphism by Proposition 3. Applying Siebenmann's Open

Collar Theorem [14], we have that $O_G - \text{int } N$ is PL homeomorphic to $\text{bdy } N \times [0, 1)$. Thus, $O_G \cong \text{int } N$.

COROLLARY 5. *There are exactly two topological types of tame Z^2 -actions on E^n , $n \geq 5$.*

Proof. Two tame Z^2 -actions are equivalent if and only if their orbit spaces are homeomorphic [3]. By a theorem of T. Price [13, p. 336] and uniqueness of PL structures [11], orientable regular neighborhoods of T^2 in codimension three or greater are SO_{n-2} bundles and are classified up to homeomorphism by $\pi_1(SO_{n-2}) \cong Z_2$, $n \geq 5$.

3. Subactions.

PROPOSITION 6. *Suppose G is a Z^2 -action on E^n , $n \geq 2$ and $H \subset G$ is a subgroup of index two. Then G is a tame action if and only if H is a tame action.*

Proof. Suppose first that G is a tame action. Since G satisfies Sperner's condition, clearly H does. Now let $\{C_j\}$ be a sequence of compact sets in E^n such that $C_j \subset C_{j+1}$ for each j and the inclusion $E^n - GC_{j+1}$ into $E^n - GC_j$ is zero on π_0 and π_1 . Since H is a subgroup of index two, one can find a homeomorphism k in G such that $GC_j = HD_j$ where $D_j = C_j \cup k(C_j)$. Hence, H is tame.

Now suppose H is tame and suppose G fails to satisfy Sperner's condition. Then for some compact set $C \subset E^n$, the set

$$M = \{g \in G \mid g(C) \cap C \neq \emptyset\}$$

is infinite. We can find a basis $\{h, k\}$ for G such that $\{h, k^2\}$ is a basis for H . Since H satisfies Sperner's condition, M must contain an infinite number of elements of the form $h^i k^{2i+1}$. Let $D = k(C) \cup C$. Then if $h^i k^{2i+1} \in M$, $h^i k^{2i}(D) = h^i k^{2i+1}(C) \cup h^i k^{2i}(C)$ so that $h^i k^{2i}(D) \cap D \neq \emptyset$. But then H does not satisfy Sperner's condition, a contradiction. Thus, G satisfies Sperner's condition.

Now let $\{C_j\}$ be a sequence of compact sets in E^n such that $C_j \subset C_{j+1}$ for each j and $\bigcup C_j = E^n$. As above there is a homeomorphism k in G such that $GC_j = HD_j$ where $D_j = C_j \cup k(C_j)$ (the homeomorphism k in the above basis will suffice). Taking a subsequence of the C_j 's we find that inclusion $E^n - HD_{j+1} = E^n - GC_{j+1}$ into $E^n - HD_j = E^n - GC_j$ is zero on π_0 and π_1 . Thus, G is a tame action.

PROPOSITION 7. *Suppose that X is the total space of an orientable $O(q)$ bundle over T^2 , $q \geq 3$ and $p: \tilde{X} \rightarrow X$ is a double cover of X . Then \tilde{X} is the total space of the trivial $O(q)$ bundle over T^2 .*

Proof. Orientable $O(q)$ bundles over T^2 are classified by $\pi_1(SO_q)$ which is Z_2 for $q \geq 3$. Applying the classification described by Price in [13] yields the proposition.

THEOREM 8. *Suppose that H is a tame Z^2 -action on E^n , $n \geq 5$. Then H is topologically equivalent to the standard Z^2 -action if and only if there exists a Z^2 -action G on E^n such that H is a subgroup of G and $G/H \cong Z_2$.*

Proof. If H is equivalent to the standard action, the required G clearly exists.

Now suppose H is tame and G is given such that $G/H \cong Z^2$. By Proposition 6, one has that G is tame. Thus, O_G is homeomorphic to the interior of a orientable regular neighborhood of T^2 and, therefore, is the total space of an orientable $O(q)$ bundle over T^2 . But the natural covering projection of O_H onto O_G is a double cover of O_G . The theorem follows from Proposition 7.

COROLLARY 9. *Suppose G is a tame Z^2 -action on E^n , $n \geq 5$. Then every homeomorphism in G except the identity is topologically equivalent to a translation of E^n .*

Proof. Suppose f is a member of G . Then one can find a basis $\{h, k\}$ for G such that there is a positive integer n for which $h^n = f$. Clearly if h is topologically equivalent to a translation then so is f . Let H be the subgroup of G generated by $\{h, k^2\}$. Then H is a subgroup of index two and by Proposition 6 is tame. Applying Theorem 8, one gets H topologically equivalent to the standard Z^2 -action and, hence, h is topologically equivalent to a translation.

REMARK. Proposition 6 is true whenever H is a subgroup of finite index. This together with Corollary 9 says that every subaction of a tame Z^2 -action is tame.

EXAMPLE. For $n \geq 5$, let W^{n-2} be a contractible open manifold and let $Q^n = S^1 \times S^1 \times W^{n-2}$. Then the universal cover of Q^n is $E^2 \times W^{n-2} \cong E^n$ [12], so that if G is the corresponding group of covering transformations, G is a Z^2 -action satisfying Sperner's condition. Let ε_Q and ε_W denote the ends of Q^n and W^{n-2} respectively. It follows by applying Proposition 1 that if $\pi_1(\varepsilon_Q)$ is stable so is $\pi_1(\varepsilon_W)$ and there is a short exact sequence $1 \rightarrow \pi_1(\varepsilon_W) \rightarrow \pi_1(\varepsilon_Q) \rightarrow Z^2 \rightarrow 1$. If W^{n-2} is the Whitehead example [17] for $n = 5$ or the interior of a contractible manifold with nonsimply connected boundary for $n > 5$ [2], the above shows that Q^n is not the interior of a regular neighborhood of T^2 , so

that G is not a tame action. However, if h_1, h_2 are the generators of G corresponding to the standard cover of $S^1 \times S^1$, the orbit space of the subgroup generated by h_i is homeomorphic to $S^1 \times E^{n-1}$, so that h_1 and h_2 are both topologically equivalent to translations.

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Received August 3, 1972. Research partially supported by the College of Arts and Sciences Office of Research and Graduate Studies, Oklahoma State University.

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The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$48.00 a year (6 Vols., 12 issues). Special rate: \$24.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

* C. R. DePrima California Institute of Technology, Pasadena, CA 91109, will replace J. Dugundji until August 1974.

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