MULTIPLIERS AND THE GROUP $L_p$-ALGEBRAS

JOHN GRIFFIN AND KELLY DENIS MCKENNON
MULTIPLIERS AND THE GROUP $L_p$-ALGEBRAS

JOHN GRIFFIN AND KELLY McKENNON

Let $G$ be a locally compact group, $p$ a number in $[1, \infty[$, and $L_p$ the usual $L_p$-space with respect to left Haar measure on $G$. The space $L_p'$ consists of those functions $f$ in $L_p'$ such that $g*f$ is well-defined and in $L_p$ for each $g$ in $L_p$. Since each function in $L_p'$ may be identified with a linear operator on $L_p$ which, as it turns out, is bounded; the operator norm may be super-imposed on $L_p'$ and, under this norm $\| \|_p$, $L_p'$ is a normed algebra. The family of right multipliers (i.e., bounded linear operators which commute with left multiplication operators) on any normed algebra $A$ will be written as $m_r(A)$ and the family of left multipliers as $m_l(A)$. The family of all bounded linear operators on $L_p$ which commute with left translations will be written as $\mathfrak{M}_p$.

It was shown in a previous issue of this journal that the Banach algebra $\mathfrak{M}_p$ is linearly isomorphic to the normed algebra $m_r(L_p')$, whenever the group $G$ is either Abelian or compact. This fact is shown, in the present paper, to hold for general locally compact $G$. The norm $\| \|_p$ is defective in that, unless $p = 1$, $(L_p, \| \|_p)$ is never complete.

An attempt will be made in the sequel to supply this deficiency by the introduction of a second norm $\| \|'_p$ on $L_p'$ under which $L_p'$ is always a Banach algebra. It will be seen that, for $p = 2$ (and of course for $p = 1$), the Banach algebra $m_r(L_p', \| \|'_p)$ is linearly isometric to $\mathfrak{M}_p$.

Let $G$ be a fixed, but arbitrary, locally compact topological group with left Haar measure $\lambda$. Write $C_0$ for the family of continuous, complex-valued functions on $G$ with compact support.

Let $p$ be a fixed, but arbitrary, number in $[1, \infty[$ and write $\| \|$ for the norm on the Banach space $L_p = L_p(G, \lambda)$. The group $L_p$-algebra $L_p'$ is the set

$$\{ f \in L_p : g*f \in L_p \text{ for all } g \in L_p \}.$$  

A function $f \in L_p$ is said to be $p$-tempered and, as shown in [3], the number

$$\| f \|'_p = \sup \{ \| g*f \|_p : g \in C_0, \| g \| \leq 1 \}$$

is finite. Conversely, if $\| f \|'_p$ is finite for some $f \in L_p$, then—as proved in [3]—$f$ is $p$-tempered and there exists precisely one operator $W_f$ in $\mathfrak{M}_p$ such that

$$\| W_f \| = \| f \|'_p \text{ and } W_f(g) = g*f$$

The number $\| f \|'_p$ is finite if and only if $f = w*h$ for some $w \in L_1$ and $h \in L_p$. The number $\| f \|'_p$ is independent of the choice of $h \in L_p$ corresponding to $f = w*h$.
for all $g \in L_p$.

Let $\Delta$ be the modular function for $G$ and let

$$L_{1,p'} = \{ f^{\Delta:i} : f \in L_1 \} \quad (p' = p/(p - 1))$$

which is linearly isometric to $L_1$ when it bears the norm $|| \cdot ||_{1,p'}$ defined by

$$(2) \quad || h ||_{1,p'} = \int_G |h| \Delta^{-1/p'} d\lambda$$

for each $h \in L_{1,p'}$. As in [1], 20.13 and [2], 32.45, we see that $L_p$ may be viewed as a right Banach $L_{1,p'}$-module and

$$(3) \quad || g*h ||_p \leq || h ||_{1,p'} || g ||_p$$

for all $h \in L_{1,p'}$ and $g \in L_p$. Consequently, for each $f \in L_{1,p'}$, there exists precisely one bounded linear operator $W_f$ on $L_p$ such that, for all $g \in L_p$,

$$(4) \quad W_f(g) = g*f \quad \text{and} \quad || W_f || \leq || f ||_{1,p'}.$$ 

It is clear that $C_0$ is a dense subset of $L_{1,p'}$ and so, since $\{ W_f : f \in C_0 \}$ is a subset of the Banach space $\mathcal{M}_p$, we have

$$(5) \quad \{ W_f : f \in L_{1,p'} \} \subset \mathcal{M}_p.$$ 

We define the space of $p$-well tempered functions to be

$$L_{p}^w = \{ h*f : h \in L_p', f \in L_{1,p'} \}.$$ 

The closure $\mathcal{M}_p$ of the set $\{ W_f : f \in L_{p}^w \}$ in $\mathcal{M}_p$, was studied in [3]. Its Banach algebra of left multipliers can be identified with $\mathcal{M}_p$ ([3], Th. 6) and it possesses a minimal left approximate identity $\{ W_h \}$ such that $\{ h \} \subset C_0*C_0$ and

$$(6) \quad \lim_{\gamma} || W_{h_\gamma} \circ T \circ W_{h_\gamma}(g) - T(g) ||_p^r = 0$$

for each $g \in L_p^w$ and $T \in \mathcal{M}_p$ (see [3], proofs to Theorem 3 and Lemma 1).

**Lemma 1.** Let $T \in \mathcal{M}_p(L_p', || \cdot ||_p^r)$ be such that $T(g) = 0$ for all $g \in L_p^w$. Then $T = 0$.

**Proof.** Assume that $T \neq 0$. Then there exists some $h \in L_p'$ such that $T(h) \neq 0$ and some $g \in C_0$ such that $g*T(h) \neq 0$. Let $\{ h_\gamma \}$ be the net in $C_0*C_0$ which appears in (6). It follows from (6) that

$$0 = \lim_{\gamma} || W_{h_\gamma} \circ W_h \circ W_{h_\gamma}(g) - W_h(g) ||_p^r$$

$$= \lim_{\gamma} || g*h_\gamma*h_\gamma - g*h ||_p^r.$$
Note that $g * h * g * h$ is in $L_p^w$ for each $\gamma$ and so

$$
\| g * T(h) \|_p^* = \| T(g * h) \|_p^*
$$

$$
= \lim_{\gamma} \| T(g * h * g * h) \|_p^* = 0 :
$$

an absurdity. Thus, $T = 0$.

**Theorem 1.** Define $\omega : M_p \to m_r(L'_p, \| \|_p^*)$ by letting $\omega_r(f) = T(f)$ for each $T \in M_p$ and $f \in L'_p$. Then $\omega$ is a surjective, isometric, algebra isomorphism.

**Proof.** Assume false. By [4], Theorem 1, there exists some $T \in m_r(L'_p, \| \|_p^*)$ such that $T \neq 0$ and

$$
T(V(f)) = 0 \text{ for all } V \in \mathfrak{A}_p \text{ and } f \in L'_p.
$$

Since $\mathfrak{A}_p$ possesses a left minimal approximate identity, it is clear that the set $\{V(f) : f \in L'_p, V \in \mathfrak{A}_p\} \cap L^w_p$ is dense in $(L^w_p, \| \|_p^*)$. This implies that

$$
T(g) = 0 \text{ for all } g \in L^w_p.
$$

By Lemma 1, $T = 0$: an absurdity.

For each $f \in L'_p$, let

$$
\| f \|_p^* = \| f \|_p^* + \| f \|_p^*.
$$

We have used the symbol $\| \|_p$ to represent the operator norm on $M_p$. The map $\omega$ defined in Theorem 1 shows that $\| \|_p$ also is the operator norm on $M_p$ when $M_p$ is regarded as a family of operators on $(L'_p, \| \|_p^*)$. We may regard $M_p$ as a family of operators on the normed space $(L'_p, \| \|_p^*)$ and, in this case, we shall write $\| \|_p$ for the operator norm.

**Lemma 2.** For each $T \in M_p$, we have

$$
\| T \|_p \leq \| T \|.
$$

**Proof.** For $g \in L'_p$, we have

$$
\| T(g) \|_p^* = \| T(g) \|_p^* + \| T(g) \|_p^*
$$

$$
\leq \| T \| \| g \|_p^* + \| T \| \| g \|_p^* = \| T \| \| g \|_p^*.
$$

**Theorem 2.** The algebra $(L'_p, \| \|_p^*)$ is a Banach algebra. The set $L^w_p$ is a closed two-sided ideal in $(L'_p, \| \|_p^*)$. 
Proof. From Lemma 2, we have

\[ ||f \ast g||_p = ||W_{\rho}(f) \ast g||_p \leq ||W_{\rho}|| \cdot ||f||_p \leq ||W_{\rho}|| \cdot ||f||_p \]

for all \( f \) and \( g \) in \( L_{\rho} \). Hence \( (L_{\rho}, ||||_p) \) is a normed algebra.

Let \( \{f_n\} \) be a Cauchy sequence in \( (L_{\rho}, ||||_p) \). There exists a function \( f \in L_{\rho} \) and a bounded linear operator \( W \) on \( L_{\rho} \) such that

\[ \lim_{n} ||f_n - f|| = 0 = \lim_{n} ||Wf_n - W||. \]

For all \( g \in C_0 \) such that \( ||g|| \leq 1 \), we have

\[ ||g \ast f||_p = \lim ||g \ast f_n||_p \leq \lim ||f_n||_p \cdot ||g||_p \leq \lim ||f_n||_p. \]

This implies via (1) that \( f \) is in \( L_{\rho} \). For all \( h \in C_0 \), we have

\[ W(h) = \lim W_{f_n}(h) = \lim h \ast f_n = h \ast f = W_f(h), \]

all the limits being taken in \( L_{\rho} \). Since \( C_0 \) is dense in \( L_{\rho} \), this yields that \( W = W_f \). We have shown that

\[ \lim_{n} ||f_n - f||_p = 0. \]

Thus, \( (L_{\rho}, ||||_p) \) is complete.

Evidently \( (L_{\rho}, ||||_p) \) is a right \( L_{1,\rho^*} \)-module and so by [2], 32.22, \( L_{1,\rho^*} \sim L_{1,\rho^*} \) is a closed linear subspace. But this is just \( L_{\rho^*} \).

That \( L_{\rho^*} \) is a left ideal of \( L_{\rho} \) is clear. Let \( g \) and \( h \) be in \( L_{\rho^*} \) and \( L_{\rho} \) respectively. Choose the net \( \{h_r\} \) so that (6) holds. We have

\[ 0 = \lim ||W_{h_r} \circ W_h \circ W_{h_r}(g) - W_h(g)||_p \]

\[ = \lim ||g \ast h_r \ast h - h \ast h||_p. \]

From Lemma 2 of [3] we see that the nets \( \{W_{h_r}\} \) and \( \{W_{h \ast h_r}\} \) converge to the identity operator and to \( W_1 \), respectively, in the strong operator topology (as operators on \( L_{\rho} \)). Consequently,

\[ \lim ||g \ast h_r \ast h - g \ast h||_p \]

\[ \leq \lim ||g \ast h_r \ast h - g \ast h||_p + \lim ||g \ast h - g \ast h||_p \]

\[ \leq \lim ||g \ast h_r - g||_p \cdot ||h||_p + \lim ||g \ast h - g||_p \]

\[ \leq \lim ||W_{h_r}(g) - g||_p \cdot ||h||_p + \lim ||W_{h \ast h_r} - W_h(g)||_p = 0. \]

Thus, we have proved
\[ \lim_r \| g*h_r*h*hr - g*h \|_p = 0 \]

and so, since each \( g*h_r*h*hr \) is in the closed set \( L_p^w \), it follows that \( g*h \) is there as well. This shows that \( L_p^w \) is a right ideal.

**COROLLARY 1.** The subspace \( L_p^w \) of \( L_p \) is \( \mathcal{M}_p \)-invariant.

*Proof.* Let \( T \) be in \( \mathcal{M}_p \) and \( f \in L_p^w \). It follows from Lemmas 1 and 2 of [3] that there exists a net \( \{f_\alpha\} \) in \( L_p^w \) such that

\[ \lim_\alpha \| T(f) - W_{f_\alpha}(f) \|_p = 0 = \lim_\alpha \| T(f) - W_{f_\alpha}(f) \|_p . \]

But this just means

\[ \lim_\alpha \| T(f) - f*f_\alpha \|_p = 0 = \lim_\alpha \| T(f) - f*f_\alpha \|_p \]

and so

\[ \lim_\alpha \| T(f) - f*f_\alpha \|_p = 0 . \]

But, by Theorem 2, each \( f*f_\alpha \) is in \( L_p^w \) and so \( T(f) \) is as well.

**COROLLARY 2.** The Banach algebra \( \mathcal{M}_p \) is linearly isometric to \( m_r(L_p^w, \| \cdot \|_p) \).

*Proof.* It is known that \( \mathcal{M}_p \) is linearly isometric to \( m_r(\mathcal{A}_p, \| \cdot \|) \).

Each element of \( m_r(L_p^w, \| \cdot \|_p) \) clearly may be identified with an element of \( m_r(\mathcal{A}_p, \| \cdot \|) \). Thus, to prove this corollary, it will suffice to show that each element of \( m_r(\mathcal{A}_p, \| \cdot \|) \) can be identified with an element of \( m_r(L_p^w, \| \cdot \|_p) \). But this follows from Corollary 1.

**LEMMA 3.** Let \( T \in m_r(L_p^w, \| \cdot \|_p) \) be such that \( T(g) = 0 \) for all \( g \in L_p^w \). Then \( T = 0 \).

*Proof.* Repeat the proof for Lemma 1, noticing that, as in the proof to Theorem 2,

\[ \lim_r \| g*h_r*h*hr - g*h \|_p = 0 . \]

It follows from Lemma 2 that the natural restriction mapping of \( \mathcal{M}_p \) into \( m_r(L_p^w, \| \cdot \|_p) \) is a norm non-increasing algebra isomorphism. There arise natural questions:

(i) when is the mapping onto?
(ii) when is the mapping a homeomorphism?
(iii) when is the mapping an isometry?
Question (iii) clearly implies (ii).

**PROPOSITION 1.** The restriction mapping of $\mathcal{M}_p$ into $m_r(L_p', || ||_p)$ is surjective if and only if it is a homeomorphism.

*Proof.* Let $\Psi$ denote the restriction mapping. If $\Psi$ is onto, the open mapping theorem implies that it is a homeomorphism.

Now suppose that $\Psi$ is a homeomorphism. Let $T$ be an element of $m_r(L_p', || ||_p)$. In view of Lemma 3, $T$ is completely determined by its restriction to $L_p'$. Thus, $T$ may be identified with a multiplier on $\{\Psi(W_f): f \in L_p^w\}$, and so with a multiplier on its closure $\Psi(\mathcal{M}_p)$ in $\Psi(\mathcal{M}_p)$ as well. It follows that $T$ may be identified with a multiplier on $\mathcal{M}_p$, which, in view of [3], Theorem 6, may be identified with some $V \in \mathcal{M}_p$. It follows that $\Psi(V) = T$. Hence, $\Psi$ is surjective.

When $p = 1$, then $L_p^t = L_p^w = L_p$ and $|| ||_1 = || ||_p = 1/2 || ||_p$. When $p = 2$, we have the following:

**THEOREM 3.** The algebra $m_r(L_2, || ||_2)$ is linearly isometric and isomorphic with $\mathcal{M}_2$.

*Proof.* In view of the fact that $\mathcal{M}_2$ is a $C^*$-algebra, it follows from [5], 4.8.4 that $|| T* || \leq || T* || \cdot || T ||$ for all $T \in \mathcal{M}_2$. But Lemma 2 implies

$$|| T* || \leq || T* || = || T || \quad \text{and} \quad || T || \leq || T ||$$

for $T \in \mathcal{M}_2$ and so $|| T || = || T ||$. Thus, $\Psi$ is an isometry and Theorem 3 now follows from Proposition 1.

**References**


Received July 24, 1972.

Washington State University
Wm. R. Allaway, *On finding the distribution function for an orthogonal polynomial set* .......................................................... 305
Eric Amar, *Sur un théorème de Mooney relatif aux fonctions analytiques bornées* .... 311
Robert Morgan Brooks, *Analytic structure in the spectrum of a natural system* ..... 315
Bahattin Cengiz, *On extremely regular function spaces* ................................ 335
Paul Frazier Duvall, Jr. and Jim Maxwell, *Tame $Z^2$-actions on $E^n$* .............. 349
Allen Roy Freedman, *On the additivity theorem for n-dimensional asymptotic density* ........................................................ 357
John Griffin and Kelly Denis McKennon, *Multipliers and the group $L_p$-algebras* . 365
Charles Lemuel Hagopian, *Characterizations of $\lambda$ connected plane continua* ...... 371
Jon Craig Helton, *Bounds for products of interval functions* .......................... 377
Ikuko Kayashima, *On relations between Nörlund and Riesz means* ................. 391
Everett Lee Lady, *Slender rings and modules* .......................................... 397
Shozo Matsuura, *On the Lu Qi-Keng conjecture and the Bergman representative domains* ....................................................... 407
Stephen H. McCleary, *The lattice-ordered group of automorphisms of an $\alpha$-set* ............ 417
Stephen H. McCleary, $o-2$-transitive ordered permutation groups ...................... 425
Stephen H. McCleary, $o$-primitive ordered permutation groups. II ..................... 431
Richard Rochberg, *Almost isometries of Banach spaces and moduli of planar domains* .......................... 445
R. F. Rossa, *Radical properties involving one-sided ideals* .............................. 467
Robert A. Rubin, *On exact localization* .................................................. 473
S. Sribala, *On $\Sigma$-inverse semigroups* .................................................. 483
H. M. (Hari Mohan) Srivastava, *On the Konhauser sets of biorthogonal polynomials suggested by the Laguerre polynomials* ........... 489
Stuart A. Steinberg, *Rings of quotients of rings without nilpotent elements* .......... 493
Daniel Mullane Sunday, *The self-equivalences of an $H$-space* ........................ 507
W. J. Thron and Richard Hawks Warren, *On the lattice of proximities of Čech compatible with a given closure space* ......................... 519
Frank Uhlig, *The number of vectors jointly annihilated by two real quadratic forms determines the inertia of matrices in the associated pencil* ........... 537
Frank Uhlig, *On the maximal number of linearly independent real vectors annihilated simultaneously by two real quadratic forms* ................. 543
Frank Uhlig, *Definite and semidefinite matrices in a real symmetric matrix pencil* .... 561
Arnold Lewis Villone, *Self-adjoint extensions of symmetric differential operators* . 569
Cary Webb, *Tensor and direct products* ................................................... 579
James Victor Whittaker, *On normal subgroups of differentiable homeomorphisms* .......................................................... 595
Jerome L. Paul, *Addendum to: “Sequences of homeomorphisms which converge to homeomorphisms”* ........................................... 615
David E. Fields, *Correction to: “Dimension theory in power series rings”* .......... 616
Peter Michael Curran, *Correction to: “Cohomology of finitely presented groups”* .... 617
Billy E. Rhoades, *Correction to: “Commutants of some Hausdorff matrices”* ........ 617
Charles W. Trigg, *Corrections to: “Versum sequences in the binary system”* ......... 619