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MULTIPLIERS AND THE GROUP L_p -ALGEBRAS

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Let G be a locally compact group, p a number in $[1, \infty[$, and L_p the usual L_p -space with respect to left Haar measure on G . The space L_p^t consists of those functions f in L_p^t such that $g*f$ is well-defined and in L_p for each g in L_p . Since each function in L_p^t may be identified with a linear operator on L_p which, as it turns out, is bounded; the operator norm may be super-imposed on L_p^t and, under this norm $\| \cdot \|_p^t, L_p^t$ is a normed algebra. The family of right multipliers (i.e., bounded linear operators which commute with left multiplication operators) on any normed algebra A will be written as $\mathfrak{m}_r(A)$ and the family of left multipliers as $\mathfrak{m}_l(A)$. The family of all bounded linear operators on L_p which commute with left translations will be written as \mathfrak{M}_p .

It was shown in a previous issue of this journal that the Banach algebra \mathfrak{M}_p is linearly isomorphic to the normed algebra $\mathfrak{m}_r(L_p^t)$, whenever the group G is either Abelian or compact. This fact is shown, in the present paper, to hold for general locally compact G . The norm $\| \cdot \|_p^t$ is defective in that, unless $p = 1$, $(L_p^t, \| \cdot \|_p^t)$ is never complete.

An attempt will be made in the sequel to supply this deficiency by the introduction of a second norm $\| \cdot \|_p^t$ on L_p^t under which L_p^t is always a Banach algebra. It will be seen that, for $p=2$ (and of course for $p=1$), the Banach algebra $\mathfrak{m}_r(L_p^t, \| \cdot \|_p^t)$ is linearly isometric to \mathfrak{M}_p .

Let G be a fixed, but arbitrary, locally compact topological group with left Haar measure λ . Write C_{00} for the family of continuous, complex-valued functions on G with compact support.

Let p be a fixed, but arbitrary, number in $[1, \infty[$ and write $\| \cdot \|_p$ for the norm on the Banach space $L_p = L_p(G, \lambda)$. The group L_p -algebra L_p^t is the set

$$\{f \in L_p : g*f \in L_p \text{ for all } g \in L_p\}.$$

A function $f \in L_p$ is said to be p -tempered and, as shown in [3], the number

$$(1) \quad \|f\|_p^t = \sup \{\|g*f\|_p : g \in C_{00} \|g\|_p \leq 1\}$$

is finite. Conversely, if $\|f\|_p^t$ is finite for some $f \in L_p$, then—as proved in [3]— f is p -tempered and there exists precisely one operator W_f in \mathfrak{M}_p such that

$$\|W_f\| = \|f\|_p^t \quad \text{and} \quad W_f(g) = g*f$$

for all $g \in L_p$.

Let Δ be the modular function for G and let

$$L_{1,p'} = \{f\Delta^{1/p'} : f \in L_1\} \quad (p' = p/(p-1))$$

which is linearly isometric to L_1 when it bears the norm $\| \cdot \|_{1,p'}$ defined by

$$(2) \quad \|h\|_{1,p'} = \int_G |h| \Delta^{-1/p'} d\lambda$$

for each $h \in L_{1,p'}$. As in [1], 20.13 and [2], 32.45, we see that L_p may be viewed as a right Banach $L_{1,p'}$ -module and

$$(3) \quad \|g * h\|_p \leq \|h\|_{1,p'} \|g\|_p$$

for all $h \in L_{1,p'}$ and $g \in L_p$. Consequently, for each $f \in L_{1,p'}$, there exists precisely one bounded linear operator W_f on L_p such that, for all $g \in L_p$,

$$(4) \quad W_f(g) = g * f \quad \text{and} \quad \|W_f\| \leq \|f\|_{1,p'}$$

It is clear that C_{00} is a dense subset of $L_{1,p'}$ and so, since $\{W_f : f \in C_{00}\}$ is a subset of the Banach space \mathfrak{M}_p , we have

$$(5) \quad \{W_f : f \in L_{1,p'}\} \subset \mathfrak{M}_p$$

We define the space of p -well tempered functions to be

$$L_p^{wt} = \{h * f : h \in L_p^t, f \in L_{1,p'}\}$$

The closure \mathfrak{A}_p of the set $\{W_f : f \in L_p^{wt}\}$ in \mathfrak{M}_p was studied in [3]. Its Banach algebra of left multipliers can be identified with \mathfrak{M}_p ([3], Th. 6) and it possesses a minimal left approximate identity $\{W_{h_\gamma}\}$ such that $\{h_\gamma\} \subset C_{00} * C_{00}$ and

$$(6) \quad \lim_{\gamma} \|W_{h_\gamma} \circ T \circ W_{h_\gamma}(g) - T(g)\|_p^t = 0$$

for each $g \in L_p^{wt}$ and $T \in \mathfrak{M}_p$ (see [3], proofs to Theorem 3 and Lemma 1).

LEMMA 1. *Let $T \in m_r(L_p^t, \| \cdot \|_p^t)$ be such that $T(g) = 0$ for all $g \in L_p^{wt}$. Then $T = 0$.*

Proof. Assume that $T \neq 0$. Then there exists some $h \in L_p^t$ such that $T(h) \neq 0$ and some $g \in C_{00}$ such that $g * T(h) \neq 0$. Let $\{h_\gamma\}$ be the net in $C_{00} * C_{00}$ which appears in (6). It follows from (6) that

$$\begin{aligned} 0 &= \lim_{\gamma} \|W_{h_\gamma} \circ W_h \circ W_{h_\gamma}(g) - W_h(g)\|_p^t \\ &= \lim_{\gamma} \|g * h_\gamma * h * h_\gamma - g * h\|_p^t. \end{aligned}$$

Note that $g * h_\gamma * h * h_\gamma$ is in L_p^{wt} for each γ and so

$$\begin{aligned} \|g * T(h)\|_p^t &= \|T(g * h)\|_p^t \\ &= \lim_\gamma \|T(g * h_\gamma * h * h_\gamma)\|_p^t = 0 : \end{aligned}$$

an absurdity. Thus, $T = 0$.

THEOREM 1. *Define $\omega | \mathfrak{M}_p \rightarrow m_r(L_p^t, \| \cdot \|_p^t)$ by letting $\omega_\tau(f) = T(f)$ for each $T \in \mathfrak{M}_p$ and $f \in L_p^t$. Then ω is a surjective, isometric, algebra isomorphism.*

Proof. Assume false. By [4], Theorem 1, there exists some $T \in m_r(L_p^t, \| \cdot \|_p^t)$ such that $T \neq 0$ and

$$T(V(f)) = 0 \quad \text{for all } V \in \mathfrak{A}_p \quad \text{and} \quad f \in L_p^t .$$

Since \mathfrak{A}_p possesses a left minimal approximate identity, it is clear that the set $\{V(f) : f \in L_p^t, V \in \mathfrak{A}_p\} \cap L_p^{wt}$ is dense in $(L_p^{wt}, \| \cdot \|_p^t)$. This implies that

$$T(g) = 0 \quad \text{for all } g \in L_p^{wt} .$$

By Lemma 1, $T = 0$: an absurdity.

For each $f \in L_p^t$, let

$$(7) \quad ||| f |||_p^t = \|f\|_p^t + \|f\|_p .$$

We have used the symbol $||| \cdot |||$ to represent the operator norm on \mathfrak{M}_p . The map ω defined in Theorem 1 shows that $||| \cdot |||$ also is the operator norm on \mathfrak{M}_p when \mathfrak{M}_p is regarded as a family of operators on $(L_p^t, \| \cdot \|_p^t)$. We may regard \mathfrak{M}_p as a family of operators on the normed space $(L_p^t, ||| \cdot |||_p^t)$ and, in this case, we shall write $||| \cdot |||$ for the operator norm.

LEMMA 2. *For each $T \in \mathfrak{M}_p$, we have*

$$||| T ||| \leq \|T\| .$$

Proof. For $g \in L_p^t$, we have

$$\begin{aligned} ||| T(g) |||_p^t &= \|T(g)\|_p^t + \|T(g)\|_p \\ &\leq \|T\| \cdot \|g\|_p^t + \|T\| \cdot \|g\|_p = \|T\| \cdot |||g|||_p^t . \end{aligned}$$

THEOREM 2. *The algebra $(L_p^t, ||| \cdot |||_p^t)$ is a Banach algebra. The set L_p^{wt} is a closed two-sided ideal in $(L_p^t, ||| \cdot |||_p^t)$.*

Proof. From Lemma 2, we have

$$\begin{aligned} ||| f * g |||_p^t &= ||| W_g(f) |||_p^t \leq ||| W_g ||| \cdot ||| f |||_p^t \leq ||| W_g ||| \cdot ||| f |||_p^t \\ &= ||| g |||_p^t \cdot ||| f |||_p^t \leq ||| g |||_p^t \cdot ||| f |||_p^t \end{aligned}$$

for all f and g in L_p^t . Hence $(L_p^t, ||| \cdot |||_p^t)$ is a normed algebra.

Let $\{f_n\}$ be a Cauchy sequence in $(L_p^t, ||| \cdot |||_p^t)$. There exists a function $f \in L_p$ and a bounded linear operator W on L_p such that

$$\lim_n ||| f_n - f ||| = 0 = \lim_n ||| W_{f_n} - W ||| .$$

For all $g \in C_{00}$ such that $||| g ||| \leq 1$, we have

$$||| g * f |||_p = \lim ||| g * f_n |||_p \leq \overline{\lim} ||| f_n |||_p^t ||| g |||_p \leq \overline{\lim} ||| f_n |||_p^t .$$

This implies via (1) that f is in L_p^t . For all $h \in C_{00}$, we have

$$W(h) = \lim_n W_{f_n}(h) = \lim_n h * f_n = h * f = W_f(h) ,$$

all the limits being taken in L_p . Since C_{00} is dense in L_p , this yields that $W = W_f$. We have shown that

$$\lim_n ||| f_n - f |||_p^t = 0 .$$

Thus, $(L_p^t, ||| \cdot |||_p^t)$ is complete.

Evidently $(L_p^t, ||| \cdot |||_p^t)$ is a right $L_{1,p}$ -module and so by [2], 32.22, $L_p^{t*} L_{1,p}$ is a closed linear subspace. But this is just L_p^{wt} .

That L_p^{wt} is a left ideal of L_p^t is clear. Let g and h be in L_p^{wt} and L_p^t respectively. Choose the net $\{h_\gamma\}$ so that (6) holds. We have

$$\begin{aligned} 0 &= \lim_n ||| W_{h_\gamma} \circ W_h \circ W_{h_\gamma}(g) - W_h(g) |||_p^t \\ &= \lim ||| g * h_\gamma * h * h_\gamma - h * h |||_p^t . \end{aligned}$$

From Lemma 2 of [3] we see that the nets $\{W_{h_\gamma}\}$ and $\{W_{h * h_\gamma}\}$ converge to the identity operator and to W_h , respectively, in the strong operator topology (as operators on L_p). Consequently,

$$\begin{aligned} &\overline{\lim}_\gamma ||| g * h_\gamma * h * h_\gamma - g * h |||_p \\ &\leq \overline{\lim}_\gamma ||| g * h_\gamma * h * h_\gamma - g * h * h_\gamma |||_p + \overline{\lim}_\gamma ||| g * h * h_\gamma - g * h |||_p \\ &\leq \overline{\lim}_\gamma ||| g * h_\gamma - g ||| ||| h * h_\gamma |||_p^t + \overline{\lim}_\gamma ||| g * h * h_\gamma - g * h |||_p \\ &\leq \overline{\lim}_\gamma ||| W_{h_\gamma}(g) - g |||_p ||| h |||_p^t + \overline{\lim}_\gamma ||| W_{h * h_\gamma} - W_h(g) |||_p = 0 . \end{aligned}$$

Thus, we have proved

$$\lim_r ||| g * h_r * h * h_r - g * h |||_p^t = 0$$

and so, since each $g * h_r * h * h_r$ is in the closed set L_p^{wt} , it follows that $g * h$ is there as well. This shows that L_p^{wt} is a right ideal.

COROLLARY 1. *The subspace L_p^{wt} of L_p is \mathfrak{M}_p -invariant.*

Proof. Let T be in \mathfrak{M}_p and $f \in L_p^{wt}$. It follows from Lemmas 1 and 2 of [3] that there exists a net $\{f_\alpha\}$ in L_p^{wt} such that

$$\lim_\alpha || T(f) - W_{f_\alpha}(f) || = 0 = \lim_\alpha || T(f) - W_{f_\alpha}(f) ||_p.$$

But this just means

$$\lim_\alpha || T(f) - f * f_\alpha ||_p^t = 0 = \lim_\alpha || T(f) - f * f_\alpha ||_p$$

and so

$$\lim_\alpha ||| T(f) - f * f_\alpha |||_p^t = 0.$$

But, by Theorem 2, each $f * f_\alpha$ is in L_p^{wt} and so $T(f)$ is as well.

COROLLARY 2. *The Banach algebra \mathfrak{M}_p is linearly isometric to $m_r(L_p^{wt}, || \cdot ||_p^t)$.*

Proof. It is known that \mathfrak{M}_p is linearly isometric to $m_1(\mathfrak{A}_p, || \cdot ||)$. Each element of $m_r(L_p^{wt}, || \cdot ||_p^t)$ clearly may be identified with an element of $m_r(\mathfrak{A}_p, || \cdot ||)$. Thus, to prove this corollary, it will suffice to show that each element of $m_1(\mathfrak{A}_p, || \cdot ||)$ can be identified with an element of $m_r(L_p^{wt}, || \cdot ||_p^t)$. But this follows from Corollary 1.

LEMMA 3. *Let $T \in m_r(L_p^t, ||| \cdot |||_p^t)$ be such that $T(g) = 0$ for all $g \in L_p^{wt}$. Then $T = 0$.*

Proof. Repeat the proof for Lemma 1, noticing that, as in the proof to Theorem 2,

$$\lim_r ||| g * h_r * h * h_r - g * h |||_p^t = 0.$$

It follows from Lemma 2 that the natural restriction mapping of \mathfrak{M}_p into $m_r(L_p^t, ||| \cdot |||_p^t)$ is a norm non-increasing algebra isomorphism. There arise natural questions:

- (i) when is the mapping onto?
- (ii) when is the mapping a homeomorphism?
- (iii) when is the mapping an isometry?

Question (iii) clearly implies (ii).

PROPOSITION 1. *The restriction mapping of \mathfrak{M}_p into $m_r(L_p^t, ||| |||_p^t)$ is surjective if and only if it is a homeomorphism.*

Proof. Let Ψ denote the restriction mapping. If Ψ is onto, the open mapping theorem implies that it is a homeomorphism.

Now suppose that Ψ is a homeomorphism. Let T be an element of $m_r(L_p^t, ||| |||_p^t)$. In view of Lemma 3, T is completely determined by its restriction to L_p^{wt} . Thus, T may be identified with a multiplier on $\{\Psi(W_f): f \in L_p^{wt}\}$, and so with a multiplier on its closure $\Psi(\mathfrak{M}_p)$ in $\Psi(\mathfrak{M}_p)$ as well. It follows that T may be identified with a multiplier on \mathfrak{M}_p , which, in view of [3], Theorem 6, may be identified with some $V \in \mathfrak{M}_p$. It follows that $\Psi(V) = T$. Hence, Ψ is surjective.

When $p = 1$, then $L_p^t = L_p^{wt} = L_p$ and $||| |||_1 = ||| |||_1^t = 1/2 ||| |||_1^t$. When $p = 2$, we have the following:

THEOREM 3. *The algebra $m_r(L_2^t, ||| |||_2^t)$ is linearly isometric and isomorphic with \mathfrak{M}_2 .*

Proof. In view of the fact that \mathfrak{M}_2 is a C^* -algebra, it follows from [5], 4.8.4 that $||| T |||^2 \leq ||| T^* ||| \cdot ||| T |||$ for all $T \in \mathfrak{M}_2$. But Lemma 2 implies

$$||| T^* ||| \leq ||| T^* ||| = ||| T ||| \quad \text{and} \quad ||| T ||| \leq ||| T |||$$

for $T \in \mathfrak{M}_2$ and so $||| T ||| = ||| T |||$. Thus, Ψ is an isometry and Theorem 3 now follows from Proposition 1.

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