MULTIPLIERS AND THE GROUP $L_p$-ALGEBRAS

JOHN GRIFFIN AND KELLY DENIS McKENNON
Let $G$ be a locally compact group, $p$ a number in $[1, \infty[$, and $L_p$ the usual $L_p$-space with respect to left Haar measure on $G$. The space $L_p'$ consists of those functions $f$ in $L_p'$ such that $g*f$ is well-defined and in $L_p$ for each $g$ in $L_p$. Since each function in $L_p'$ may be identified with a linear operator on $L_p$ which, as it turns out, is bounded; the operator norm may be super-imposed on $L_p'$ and, under this norm $\| \|_p$, $L_p'$ is a normed algebra. The family of right multipliers (i.e., bounded linear operators which commute with left multiplication operators) on any normed algebra $A$ will be written as $m_r(A)$ and the family of left multipliers as $m_l(A)$. The family of all bounded linear operators on $L_p$ which commute with left translations will be written as $\mathcal{M}_p$.

It was shown in a previous issue of this journal that the Banach algebra $\mathcal{M}_p$ is linearly isomorphic to the normed algebra $m_r(L_p')$, whenever the group $G$ is either Abelian or compact. This fact is shown, in the present paper, to hold for general locally compact $G$. The norm $\| \|_p$ is defective in that, unless $p = 1$, $(L_p', \| \|_p)$ is never complete.

An attempt will be made in the sequel to supply this deficiency by the introduction of a second norm $\| \|_p$ on $L_p'$ under which $L_p'$ is always a Banach algebra. It will be seen that, for $p = 2$ (and of course for $p = 1$), the Banach algebra $m_r(L_p', \| \|_p)$ is linearly isometric to $\mathcal{M}_p$. 

Let $G$ be a fixed, but arbitrary, locally compact topological group with left Haar measure $\lambda$. Write $C_0$ for the family of continuous, complex-valued functions on $G$ with compact support.

Let $p$ be a fixed, but arbitrary, number in $[1, \infty[$ and write $\| \|_p$ for the norm on the Banach space $L_p = L_p(G, \lambda)$. The group $L_p$-algebra $L_p'$ is the set

$$
\{ f \in L_p : g*f \in L_p \quad \text{for all} \quad g \in L_p \}.
$$

A function $f \in L_p$ is said to be $p$-tempered and, as shown in [3], the number

$$
\| f \|_p' = \sup \{ \| g*f \|_p : g \in C_0, \| g \|_p \leq 1 \}
$$

is finite. Conversely, if $\| f \|_p'$ is finite for some $f \in L_p$, then—as proved in [3]—$f$ is $p$-tempered and there exists precisely one operator $W_f$ in $\mathcal{M}_p$ such that

$$
\| W_f \| = \| f \|_p' \quad \text{and} \quad W_f(g) = g*f
$$
for all $g \in L_p$.

Let $\Delta$ be the modular function for $G$ and let

$$L_{1,p'} = \{ f^{1/p'} : f \in L_1 \} \quad (p' = p/(p - 1))$$

which is linearly isometric to $L_1$ when it bears the norm $\| \|_{1,p'}$ defined by

$$\| h \|_{1,p'} = \int_G |h| \Delta^{-1/p'} d\lambda$$

for each $h \in L_{1,p'}$. As in [1], 20.13 and [2], 32.45, we see that $L_p$ may be viewed as a right Banach $L_{1,p'}$-module and

$$\| g*h \|_p \leq \| h \|_{1,p'} \| g \|_p$$

for all $h \in L_{1,p'}$ and $g \in L_p$. Consequently, for each $f \in L_{1,p'}$, there exists precisely one bounded linear operator $W_f$ on $L_p$ such that, for all $g \in L_p$,

$$W_f(g) = g*f \quad \text{and} \quad \| W_f \| \leq \| f \|_{1,p'} .$$

It is clear that $C_0$ is a dense subset of $L_{1,p'}$ and so, since $\{ W_f : f \in C_0 \}$ is a subset of the Banach space $\mathcal{M}_p$, we have

$$\{ W_f : f \in L_{1,p'} \} \subset \mathcal{M}_p .$$

We define the space of $p$-well tempered functions to be

$$L^{wt}_p = \{ h*f : h \in L^{t}_p, f \in L_{1,p'} \} .$$

The closure $\mathcal{M}_p$ of the set $\{ W_f : f \in L^{wt}_p \}$ in $\mathcal{M}_p$ was studied in [3]. Its Banach algebra of left multipliers can be identified with $\mathcal{M}_p$ ([3], Th. 6) and it possesses a minimal left approximate identity $\{ W_{h_r} \}$ such that $\{ h_r \} \subset C_0 \ast C_0$ and

$$\lim \| W_{h_r} \circ T \circ W_{h_r}(g) - T(g) \|_p = 0$$

for each $g \in L^{wt}_p$ and $T \in \mathcal{M}_p$ (see [3], proofs to Theorem 3 and Lemma 1).

**Lemma 1.** Let $T \in m_r(L^{t}_p, \| \|_p^r)$ be such that $T(g) = 0$ for all $g \in L^{wt}_p$. Then $T = 0$.

**Proof.** Assume that $T \neq 0$. Then there exists some $h \in L_p$ such that $T(h) \neq 0$ and some $g \in C_0$ such that $g*T(h) \neq 0$. Let $\{ h_r \}$ be the net in $C_0 \ast C_0$ which appears in (6). It follows from (6) that

$$0 = \lim \| W_{h_r} \circ W_h \circ W_{h_r}(g) - W_h(g) \|_p^r$$

$$= \lim \| g*h_r^*h*h_r - g*h \|_p$$.
Note that \( g \ast h_r \ast h \ast h_r \) is in \( L^w_p \) for each \( \gamma \) and so
\[
\| g \ast T(h) \|_p^\gamma = \| T(g \ast h) \|_p^\gamma \\
= \lim_{r} \| T(g \ast h_r \ast h \ast h_r) \|_p^\gamma = 0
\]
an absurdity. Thus, \( T = 0 \).

**THEOREM 1.** Define \( \omega : \mathcal{M}_p \to \mathcal{M}_p(L^t_p, \| \cdot \|_p) \) by letting \( \omega_T(f) = T(f) \) for each \( T \in \mathcal{M}_p \) and \( f \in L^t_p \). Then \( \omega \) is a surjective, isometric, algebra isomorphism.

**Proof.** Assume false. By [4], Theorem 1, there exists some \( T \in \mathcal{M}_p(L^t_p, \| \cdot \|_p) \) such that \( T \neq 0 \) and
\[
T(V(f)) = 0 \quad \text{for all} \quad V \in \mathcal{A}_p \quad \text{and} \quad f \in L^t_p.
\]
Since \( \mathcal{A}_p \) possesses a left minimal approximate identity, it is clear that the set \( \{ V(f) : f \in L^t_p, \quad V \in \mathcal{A}_p \} \cap L^w_p \) is dense in \( (L^w_p, \| \cdot \|_p) \). This implies that
\[
T(g) = 0 \quad \text{for all} \quad g \in L^w_p.
\]
By Lemma 1, \( T = 0 \): an absurdity.

For each \( f \in L^t_p \), let
\[
(7) \quad \| f \|_p^\gamma = \| f \|_p^\gamma + \| f \|_p.
\]
We have used the symbol \( \| \cdot \|_p \) to represent the operator norm on \( \mathcal{M}_p \). The map \( \omega \) defined in Theorem 1 shows that \( \| \cdot \|_p \) also is the operator norm on \( \mathcal{M}_p \) when \( \mathcal{M}_p \) is regarded as a family of operators on \( (L^t_p, \| \cdot \|_p) \). We may regard \( \mathcal{M}_p \) as a family of operators on the normed space \( (L^t_p, \| \cdot \|_p) \) and, in this case, we shall write \( \| \cdot \| \) for the operator norm.

**LEMMA 2.** For each \( T \in \mathcal{M}_p \), we have
\[
\| T \| \leq \| T \|.
\]

**Proof.** For \( g \in L^t_p \), we have
\[
\| T(g) \|_p^\gamma = \| T(g) \|_p^\gamma + \| T(g) \|_p \\
\leq \| T \| \cdot \| g \|_p^\gamma + \| T \| \cdot \| g \|_p = \| T \| \cdot \| g \|_p^\gamma.
\]

**THEOREM 2.** The algebra \( (L^t_p, \| \cdot \|_p) \) is a Banach algebra. The set \( L^w_p \) is a closed two-sided ideal in \( (L^t_p, \| \cdot \|_p) \).
Proof. From Lemma 2, we have
\[ ||f^*g||_p = ||W_\rho(f)||_p \leq ||W_\rho|| \cdot ||f||_p \leq ||W_\rho|| \cdot ||f||_p \]
for all \( f \) and \( g \) in \( L^*_p \). Hence \((L^*_p, ||||_p^*)\) is a normed algebra.

Let \( \{f_n\} \) be a Cauchy sequence in \((L^*_p, ||||_p^*)\). There exists a function \( f \in L_p \) and a bounded linear operator \( W \) on \( L_p \) such that
\[ \lim_n ||f_n - f|| = 0 = \lim_n ||Wf_n - W|| . \]

For all \( g \in C_0 \) such that \( ||g|| \leq 1 \), we have
\[ ||g*f||_p = \lim ||g*f_n||_p \leq \lim ||f_n||_p \cdot ||g||_p \leq \lim ||f_n||_p^* . \]
This implies via (1) that \( f \) is in \( L^*_p \). For all \( h \in C_0 \), we have
\[ W(h) = \lim_n Wf_n(h) = \lim_n h*f_n = h*f = W_f(h) , \]
all the limits being taken in \( L_p \). Since \( C_0 \) is dense in \( L_p \), this yields that \( W = W_f \). We have shown that
\[ \lim_n ||f_n - f||_p^* = 0 . \]

Thus, \((L^*_p, ||||_p^*)\) is complete.

Evidently \((L^*_p, ||||_p^*)\) is a right \( L_{1,p}'-\)module and so by [2], 32.22, \( L_{p}^* \) is a closed linear subspace. But this is just \( L_{p}^* \).

That \( L_{p}^* \) is a left ideal of \( L_p \) is clear. Let \( g \) and \( h \) be in \( L_{p}^* \) and \( L_p \), respectively. Choose the net \( \{h_t\} \) so that (6) holds. We have
\[ 0 = \lim_n ||W_{h_t}^\circ W_{h_t}(g) - W_h(g)||_p^* \]
\[ = \lim ||g*_{h_t}h*h_t - h*h||_p^* . \]
From Lemma 2 of [3] we see that the nets \( \{W_{h_t}^\circ\} \) and \( \{W_{h*h_t}\} \) converge to the identity operator and to \( W_h \), respectively, in the strong operator topology (as operators on \( L_p \)). Consequently,
\[ \lim ||g*_{h_t}h*h_t - g*h||_p \]
\[ \leq \lim ||g*_{h_t}h*h_t - g*h*h_t||_p + \lim ||g*h*h_t - g*h||_p \]
\[ \leq \lim ||g*_{h_t} - g|| \cdot ||h*h_t||_p^* + \lim ||g*h*h_t - g*h||_p \]
\[ \leq \lim ||W_{h_t}(g) - g||_p \cdot ||h||_p^* + \lim ||W_{h*h_t} - W_h(g)||_p = 0 . \]
Thus, we have proved
\[
\lim_{r} \| g * h_r * h_r - g * h \|_p' = 0
\]

and so, since each \( g * h_r * h_r \) is in the closed set \( L_p^{wt} \), it follows that \( g * h \) is there as well. This shows that \( L_p^{wt} \) is a right ideal.

**COROLLARY 1.** The subspace \( L_p^{wt} \) of \( L_p \) is \( \mathcal{M}_p \)-invariant.

*Proof.* Let \( T \) be in \( \mathcal{M}_p \) and \( f \in L_p^{wt} \). It follows from Lemmas 1 and 2 of [3] that there exists a net \( \{ f_\alpha \} \) in \( L_p^{wt} \) such that
\[
\lim_{\alpha} \| T(f) - W_{f_\alpha}(f) \| = 0 = \lim_{\alpha} \| T(f) - W_{f_\alpha}(f) \|_p.
\]
But this just means
\[
\lim_{\alpha} \| T(f) - f*f_\alpha \|_p' = 0 = \lim_{\alpha} \| T(f) - f*f_\alpha \|_p
\]
and so
\[
\lim_{\alpha} \| T(f) - f*f_\alpha \|_p' = 0.
\]
But, by Theorem 2, each \( f*f_\alpha \) is in \( L_p^{wt} \) and so \( T(f) \) is as well.

**COROLLARY 2.** The Banach algebra \( \mathcal{M}_p \) is linearly isometric to \( m_r(L_p^{wt}, \| \|_p') \).

*Proof.* It is known that \( \mathcal{M}_p \) is linearly isometric to \( m_t(\mathcal{M}_p, \| \|) \). Each element of \( m_r(L_p^{wt}, \| \|_p') \) clearly may be identified with an element of \( m_r(\mathcal{M}_p, \| \|) \). Thus, to prove this corollary, it will suffice to show that each element of \( m_t(\mathcal{M}_p, \| \|) \) can be identified with an element of \( m_r(L_p^{wt}, \| \|_p') \). But this follows from Corollary 1.

**LEMMA 3.** Let \( T \in m_r(L_p', \| \|_p') \) be such that \( T(g) = 0 \) for all \( g \in L_p^{wt} \). Then \( T = 0 \).

*Proof.* Repeat the proof for Lemma 1, noticing that, as in the proof to Theorem 2,
\[
\lim_{r} \| g * h_r * h_r - g * h \|_p' = 0.
\]
It follows from Lemma 2 that the natural restriction mapping of \( \mathcal{M}_p \) into \( m_r(L_p', \| \|_p') \) is a norm non-increasing algebra isomorphism. There arise natural questions:

(i) when is the mapping onto?
(ii) when is the mapping a homeomorphism?
(iii) when is the mapping an isometry?
Question (iii) clearly implies (ii).

**Proposition 1.** The restriction mapping of \( \mathcal{M}_p \) into \( m_r(L'_p, || ||'_p) \) is surjective if and only if it is a homeomorphism.

**Proof.** Let \( \Psi \) denote the restriction mapping. If \( \Psi \) is onto, the open mapping theorem implies that it is a homeomorphism.

Now suppose that \( \Psi \) is a homeomorphism. Let \( T \) be an element of \( m_r(L'_p, || ||'_p) \). In view of Lemma 3, \( T \) is completely determined by its restriction to \( L'_p \). Thus, \( T \) may be identified with a multiplier on \( \{ \Psi(W_f) : f \in L'_p \} \), and so with a multiplier on its closure \( \Psi(\mathcal{M}_p) \) as well. It follows that \( T \) may be identified with a multiplier on \( \mathcal{M}_p \), which, in view of [3], Theorem 6, may be identified with some \( V \in \mathcal{M}_p \). It follows that \( \Psi(V) = T \). Hence, \( \Psi \) is surjective.

When \( p = 1 \), then \( L'_p = L'_p L_1 = L_1 \) and \( || ||'_1 = || ||_1 = 1/2 || ||'_p \). When \( p = 2 \), we have the following:

**Theorem 3.** The algebra \( m_r(L'_2, || ||'_2) \) is linearly isometric and isomorphic with \( \mathcal{M}_2 \).

**Proof.** In view of the fact that \( \mathcal{M}_2 \) is a \( C^* \)-algebra, it follows from [5], 4.8.4 that \( || T ||^2 \leq || T^* || \cdot || T || \) for all \( T \in \mathcal{M}_2 \). But Lemma 2 implies

\[
|| T^* || \leq || T^* || = || T || \quad \text{and} \quad || T || \leq || T ||
\]

for \( T \in \mathcal{M}_2 \) and so \( || T || = || T || \). Thus, \( \Psi \) is an isometry and Theorem 3 now follows from Proposition 1.

**References**


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