

Pacific Journal of Mathematics

CHARACTERIZATIONS OF λ CONNECTED PLANE CONTINUA

CHARLES LEMUEL HAGOPIAN

CHARACTERIZATIONS OF λ CONNECTED PLANE CONTINUA

CHARLES L. HAGOPIAN

A continuum M is said to be λ connected if any two of its points can be joined by a hereditarily decomposable continuum in M . Here we characterize λ connected plane continua in terms of Jones' functions K and L .

A nondegenerate metric space that is both compact and connected is called a *continuum*. A continuum M is said to be *aposyndetic at a point p of M with respect to a point q of M* if there exists an open set U and a continuum H in M such that $p \in U \subset H \subset M - \{q\}$.

In [1], F. Burton Jones defines the functions K and L on a continuum M into the set of subsets of M as follows:

For each point x of M , the set $K(x)$ ($L(x)$) consists of all points y of M such that M is not aposyndetic at x (y) with respect to y (x).

Note that for each point x of M , the set $L(x)$ is connected and closed in M [1, Th. 3]. For any point x of M , the set $K(x)$ is closed [1, Th. 2] but may fail to be connected [2, Ex. 4], [3].

Suppose that M is a plane continuum. In this paper it is proved that the following three statements are equivalent.

- I. M is λ connected.
- II. For each point x of M , the set $K(x)$ does not contain an indecomposable continuum.
- III. For each point x of M , every continuum in $L(x)$ is decomposable.

Throughout this paper E^2 is the Euclidean plane. For a given set S in E^2 , we denote the closure and the boundary of S relative to E^2 by $\text{Cl } S$ and $\text{Bd } S$ respectively.

DEFINITION. Let M be a continuum in E^2 . A subcontinuum L of M is said to be a *link* in M if L is either the boundary of a complementary domain of M or the limit of a convergent sequence of complementary domains of M .

It is known that a plane continuum is λ connected if and only if each of its links is hereditarily decomposable [5, Th. 2].

THEOREM 1. *Suppose that a continuum M in E^2 contains an indecomposable continuum I , that disjoint circular regions V and Z in E^2 meet I , that a point x belongs to $M - \text{Cl } (V \cup Z)$, and that ε is a positive real number. Then there exist continua H and F in I , arc-segments R and T in V , and a point y of $I \cap Z$ such that (1)*

$H \cup F \cup R \cup T$ separates y from x in E^2 , and (2) if D is the y -component of $E^2 - (H \cup F \cup R \cup T)$, then each point of D is within ε of I .

Proof. Define p and q to be points of $V \cap I$ that belong to distinct composants of I . Let $\{P_n\}$ and $\{Q_n\}$ be monotone descending sequences of circular regions in E^2 centered on and converging to p and q respectively such that $\text{Cl } P_1 \cap \text{Cl } Q_1 = \emptyset$ and $\text{Cl } (P_1 \cup Q_1)$ is in V .

Suppose that for each positive integer n , only finitely many disjoint continua in $I - (P_n \cup Q_n)$ intersect $\text{Bd } P_n$, $\text{Bd } Q_n$, and Z . Since I has uncountably many composants, there exists a component C of I such that for each n , no continuum in $C - (P_n \cup Q_n)$ meets $\text{Bd } P_n$, $\text{Bd } Q_n$, and Z . It follows that for each n , there is a continuum L_n in $C - (P_n \cup Q_n \cup Z)$ that meets both $\text{Bd } P_n$ and $\text{Bd } Q_n$. The limit of $\{L_n\}$ is a continuum in $I - Z$ that contains $\{p, q\}$. But since p and q belong to different composants of I and Z intersects I , this is a contradiction. Hence for some integer n , there exists an infinite collection W of disjoint continua in $I - (P_n \cup Q_n)$ such that each element of W meets $\text{Bd } P_n$, $\text{Bd } Q_n$, and Z .

There exists a sequence of distinct continua $\{H_i\}$ and two sequences of disjoint arc-segments $\{R_i\}$ and $\{T_i\}$ such that for each i ,

- (1) H_i is an element of W ,
- (2) R_i and T_i are in $\text{Bd } P_n$ and $\text{Bd } Q_n$ respectively,
- (3) R_i and T_i each meets H_{2i} and no other element of $\{H_i\}$, and each has one endpoint in H_{2i-1} and the other endpoint in H_{2i+1} .

For each positive integer i , let y_i be a point of $H_{2i} \cap Z$ and define D_i to be the complementary domain of $H_{2i-1} \cup H_{2i+1} \cup R_i \cup T_i$ that contains y_i . Note that the elements of the sequence $\{D_i\}$ are disjoint domains in $E^2 - \text{Cl } (P_n \cup Q_n)$. Since the union of the continuum $I \cup \text{Cl } (P_n \cup Q_n)$ with its bounded complementary domains is a compact subset of E^2 , for some i , every point of D_i is within ε of I and $H_{2i-1} \cup H_{2i+1} \cup R_i \cup T_i$ separates y_i from x in E^2 .

THEOREM 2. *If M is a λ connected continuum in E^2 , then for each point x of M , every continuum in the set $K(x)$ is decomposable.*

Proof. Assume that for some point x of M , the set $K(x)$ contains an indecomposable continuum I . We shall prove that this assumption implies the existence of a link in M that contains I ; this will contradict the hypothesis of this theorem [5, Th. 2].

Let v and z be points of $M - \{x\}$ that belong to distinct composants of I . Define $\{V_i\}$ and $\{Z_i\}$ to be monotone descending sequences of circular regions in E^2 centered on and converging to v and z respectively such that $\text{Cl } V_1 \cap \text{Cl } Z_1 = \emptyset$ and $\text{Cl } (V_1 \cup Z_1)$ is in $E^2 - \{x\}$.

First we show that for each positive integer i , there exists an

arc A_i in $E^2 - M$ that goes from $\text{Bd } V_i$ to $\text{Bd } Z_i$ such that each point of A_i is within i^{-1} of I . By Theorem 1, for any given positive integer i , there exist continua H and F in I , arc-segments R and T in V_i , and a point y of $I \cap Z_i$ such that $H \cup F \cup R \cup T$ separates y from x in E^2 and each point of D (the y -component of $E^2 - (H \cup F \cup R \cup T)$) is within i^{-1} of I . Let U be a circular region containing x in E^2 whose closure misses $H \cup F \cup R \cup T$. Let G be a circular region containing y in E^2 whose closure is in $D \cap Z_i$. Since M is not aposyndetic at x with respect to y , the component of $M - G$ that contains x is not open relative to M at x . Hence there exist two components X and Y of $M - G$ that meet U . It follows that a simple closed curve J in $(E^2 - M) \cup G$ separates X from Y in E^2 [6, Th. 13, p. 170]. Note that J must intersect both G and U [6, Th. 50, p. 18]. Since $J \cap (M - G) = \emptyset$ and $H \cup F \cup R \cup T$ separates G from U in E^2 , there is an arc-segment B in $(J \cap D) - M$ that has one endpoint in $\text{Bd } G$ and the other endpoint in $R \cup T$. We define A_i to be an arc in $B - (V_i \cup Z_i)$ that goes from $\text{Bd } V_i$ to $\text{Bd } Z_i$. Since A_i is in D , each of its points is within i^{-1} of I .

Note that since v and z do not belong to the same component of I , the limit of each subsequence of $\{A_i\}$ is I . For each i , let Q_i be the complementary domain of M that contains A_i . If $\{Q_i\}$ does not have infinitely many distinct elements, then for some i , the link $\text{Bd } Q_i$ in M contains I . Suppose that $\{Q_i\}$ has infinitely many distinct elements. Then some subsequence of $\{Q_i\}$ converges to a link in M [6, Th. 59, p. 24]. It follows that a link in M contains I . This contradicts the fact that M is λ connected [5, Th. 2]. Hence for each point x of M , every continuum in $K(x)$ is decomposable.

THEOREM 3. *Suppose that M is a continuum in E^2 and for each point x of M , every continuum in $K(x)$ is decomposable. Then for each point x of M , every continuum in $L(x)$ is decomposable.*

Proof. Assume that for some point x of M , there is an indecomposable continuum I in $L(x)$. We shall prove that from this assumption it follows that M is not aposyndetic at any point of I with respect to any other point of I . Hence for each point z of I , the set $K(z)$ in M contains I . This will contradict our hypothesis.

Suppose there exists a continuum E in M that does not contain I whose interior relative to M contains a point of I . There exist mutually exclusive circular regions V and Z in E^2 such that

- (1) x does not belong to $\text{Cl}(V \cup Z)$,
- (2) V and Z each meets I ,
- (3) E and V are disjoint,
- (4) $M \cap Z$ is contained in E .

According to Theorem 1, there exist continua H and F in I , arc-segments R and T in V , and a point y of $I \cap Z$ such that $H \cup F \cup R \cup T$ separates y from x in E^2 . Define D to be the y -component of $E^2 - (H \cup F \cup R \cup T)$. There exists a circular region G in E^2 containing y such that $\text{Cl } G$ is in $D \cap Z$. Let U be a circular region in E^2 containing x whose closure misses $H \cup F \cup R \cup T$.

Since M is not aposyndetic at y with respect to x , the y -component of $M - U$ is not open relative to M at y . Hence $\text{Bd } G - M$ contains an arc-segment A whose endpoints, p and q , lie in different components of $M - U$. There exists a simple closed curve J in $(E^2 - M) \cup U$ that separates p from q in E^2 such that $J \cap A$ is connected. Let B denote the component of $J - U$ that contains $J \cap A$. Since $H \cup F \cup R \cup T$ separates G from U in E^2 and B does not intersect $H \cup F$, it follows that both components of $B - A$ meet $R \cup T$. Evidently $B \cup V$ separates p from q in E^2 [6, Th. 32, p. 181]. But since E is a continuum in $E^2 - (B \cup V)$ that contains $\{p, q\}$, this is a contradiction. Hence each subcontinuum of M that contains a point of I in its interior relative to M contains I . This implies that for any point z of I , the set $K(z)$ in M contains I , which contradicts the hypothesis of this theorem. Hence for each point x of M , every continuum in $L(x)$ is decomposable.

THEOREM 4. *Suppose that for each point x of a plane continuum M , every continuum in $L(x)$ is decomposable. Then M is λ connected.*

Proof. Assume that M is not λ connected. It follows that some link in M contains an indecomposable continuum I [5, Th. 2]. By Theorem 1 in [4], each subcontinuum of M that contains a nonempty open subset of I contains I . But this implies that for each point x of I , the set $L(x)$ contains I , which is impossible. Hence M is λ connected.

THEOREM 5. *Suppose that M is a plane continuum. The following three statements are equivalent.*

- I. M is λ connected.
- II. For each point x of M , every continuum in the set $K(x)$ is decomposable.
- III. For each point x of M , every continuum in $L(x)$ is decomposable.

Proof. This follows directly from Theorems 2, 3, and 4.

REFERENCES

1. F. B. Jones, *Concerning non-aposyndetic continua*, Amer. J. Math., **70** (1948), 403-413.

2. C. L. Hagopian, *Concerning arcwise connectedness and the existence of simple closed curves in plane continua*, Trans. Amer. Math. Soc., **147** (1970), 389-402.
3. ———, *A cut point theorem for plane continua*, Duke Math. J., **38** (1971), 509-512.
4. ———, *λ connected plane continua*, Trans. Amer. Math. Soc., **191** (1974).
5. ———, *Planar λ connected continua*, Proc. Amer. Math. Soc., **39** (1973), 190-194.
6. R. L. Moore, *Foundations of point set theory*, rev. ed., Amer. Math. Soc. Colloq. Publ., vol. 13, Amer. Math. Soc., Providence, R.I., 1962.

Received November 13, 1972.

CALIFORNIA STATE UNIVERSITY, SACRAMENTO

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

RICHARD ARENS (Managing Editor)

University of California
Los Angeles, California 90024

J. DUGUNDJI*

Department of Mathematics
University of Southern California
Los Angeles, California 90007

R. A. BEAUMONT

University of Washington
Seattle, Washington 98105

D. GILBARG AND J. MILGRAM

Stanford University
Stanford, California 94305

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
NAVAL WEAPONS CENTER

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. 39. All other communications to the editors should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$48.00 a year (6 Vols., 12 issues). Special rate: \$24.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

* C. R. DePrima California Institute of Technology, Pasadena, CA 91109, will replace J. Dugundji until August 1974.

Copyright © 1973 by
Pacific Journal of Mathematics
All Rights Reserved

Pacific Journal of Mathematics

Vol. 49, No. 2

June, 1973

Wm. R. Allaway, <i>On finding the distribution function for an orthogonal polynomial set</i>	305
Eric Amar, <i>Sur un théorème de Mooney relatif aux fonctions analytiques bornées</i>	311
Robert Morgan Brooks, <i>Analytic structure in the spectrum of a natural system</i>	315
Bahattin Cengiz, <i>On extremely regular function spaces</i>	335
Kwang-nan Chow and Moses Glasner, <i>Atoms on the Royden boundary</i>	339
Paul Frazier Duvall, Jr. and Jim Maxwell, <i>Tame Z^2-actions on E^n</i>	349
Allen Roy Freedman, <i>On the additivity theorem for n-dimensional asymptotic density</i>	357
John Griffin and Kelly Denis McKennon, <i>Multipliers and the group L_p-algebras</i>	365
Charles Lemuel Hagopian, <i>Characterizations of λ connected plane continua</i>	371
Jon Craig Helton, <i>Bounds for products of interval functions</i>	377
Ikuko Kayashima, <i>On relations between Nörlund and Riesz means</i>	391
Everett Lee Lady, <i>Slender rings and modules</i>	397
Shozo Matsuura, <i>On the Lu Qi-Keng conjecture and the Bergman representative domains</i>	407
Stephen H. McCleary, <i>The lattice-ordered group of automorphisms of an α-set</i>	417
Stephen H. McCleary, <i>$o-2$-transitive ordered permutation groups</i>	425
Stephen H. McCleary, <i>o-primitive ordered permutation groups. II</i>	431
Richard Rochberg, <i>Almost isometries of Banach spaces and moduli of planar domains</i>	445
R. F. Rossa, <i>Radical properties involving one-sided ideals</i>	467
Robert A. Rubin, <i>On exact localization</i>	473
S. Sribala, <i>On Σ-inverse semigroups</i>	483
H. M. (Hari Mohan) Srivastava, <i>On the Konhauser sets of biorthogonal polynomials suggested by the Laguerre polynomials</i>	489
Stuart A. Steinberg, <i>Rings of quotients of rings without nilpotent elements</i>	493
Daniel Mullane Sunday, <i>The self-equivalences of an H-space</i>	507
W. J. Thron and Richard Hawks Warren, <i>On the lattice of proximities of Čech compatible with a given closure space</i>	519
Frank Uhlig, <i>The number of vectors jointly annihilated by two real quadratic forms determines the inertia of matrices in the associated pencil</i>	537
Frank Uhlig, <i>On the maximal number of linearly independent real vectors annihilated simultaneously by two real quadratic forms</i>	543
Frank Uhlig, <i>Definite and semidefinite matrices in a real symmetric matrix pencil</i> ...	561
Arnold Lewis Villone, <i>Self-adjoint extensions of symmetric differential operators</i>	569
Cary Webb, <i>Tensor and direct products</i>	579
James Victor Whittaker, <i>On normal subgroups of differentiable homeomorphisms</i>	595
Jerome L. Paul, <i>Addendum to: "Sequences of homeomorphisms which converge to homeomorphisms"</i>	615
David E. Fields, <i>Correction to: "Dimension theory in power series rings"</i>	616
Peter Michael Curran, <i>Correction to: "Cohomology of finitely presented groups"</i>	617
Billy E. Rhoades, <i>Correction to: "Commutants of some Hausdorff matrices"</i>	617
Charles W. Trigg, <i>Corrections to: "Versum sequences in the binary system"</i>	619