CHARACTERIZATIONS OF $\lambda$ CONNECTED PLANE CONTINUA

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A continuum $M$ is said to be $\lambda$ connected if any two of its points can be joined by a hereditarily decomposable continuum in $M$. Here we characterize $\lambda$ connected plane continua in terms of Jones' functions $K$ and $L$.

A nondegenerate metric space that is both compact and connected is called a continuum. A continuum $M$ is said to be aposyndetic at a point $p$ of $M$ with respect to a point $q$ of $M$ if there exists an open set $U$ and a continuum $H$ in $M$ such that $p \in U \subset H \subset M - \{q\}$.

In [1], F. Burton Jones defines the functions $K$ and $L$ on a continuum $M$ into the set of subsets of $M$ as follows:

For each point $x$ of $M$, the set $K(x)$ ($L(x)$) consists of all points $y$ of $M$ such that $M$ is not aposyndetic at $x$ ($y$) with respect to $y$ ($x$).

Note that for each point $x$ of $M$, the set $L(x)$ is connected and closed in $M$ [1, Th. 3]. For any point $x$ of $M$, the set $K(x)$ is closed [1, Th. 2] but may fail to be connected [2, Ex. 4], [3].

Suppose that $M$ is a plane continuum. In this paper it is proved that the following three statements are equivalent.

I. $M$ is $\lambda$ connected.

II. For each point $x$ of $M$, the set $K(x)$ does not contain an indecomposable continuum.

III. For each point $x$ of $M$, every continuum in $L(x)$ is decomposable.

Throughout this paper $E^2$ is the Euclidean plane. For a given set $S$ in $E^2$, we denote the closure and the boundary of $S$ relative to $E^2$ by $\text{Cl} S$ and $\text{Bd} S$ respectively.

DEFINITION. Let $M$ be a continuum in $E^2$. A subcontinuum $L$ of $M$ is said to be a link in $M$ if $L$ is either the boundary of a complementary domain of $M$ or the limit of a convergent sequence of complementary domains of $M$.

It is known that a plane continuum is $\lambda$ connected if and only if each of its links is hereditarily decomposable [5, Th. 2].

THEOREM 1. Suppose that a continuum $M$ in $E^2$ contains an indecomposable continuum $I$, that disjoint circular regions $V$ and $Z$ in $E^2$ meet $I$, that a point $x$ belongs to $M - \text{Cl} (V \cup Z)$, and that $\varepsilon$ is a positive real number. Then there exist continua $H$ and $F$ in $I$, arc-segments $R$ and $T$ in $V$, and a point $y$ of $I \cap Z$ such that (1)
\(H \cup F \cup R \cup T\) separates \(y\) from \(x\) in \(E^2\), and (2) if \(D\) is the \(y\)-component of \(E^2 - (H \cup F \cup R \cup T)\), then each point of \(D\) is within \(\varepsilon\) of \(I\).

**Proof.** Define \(p\) and \(q\) to be points of \(V \cap I\) that belong to distinct composants of \(I\). Let \(\{P_n\}\) and \(\{Q_n\}\) be monotone descending sequences of circular regions in \(E^2\) centered on and converging to \(p\) and \(q\) respectively such that \(\mathrm{Cl} \ P_1 \cap \mathrm{Cl} \ Q_1 = \emptyset\) and \(\mathrm{Cl} \ (P \cup Q)\) is in \(V\).

Suppose that for each positive integer \(n\), only finitely many disjoint continua in \(I - (P_n \cup Q_n)\) intersect \(\mathrm{Bd} \ P_n, \mathrm{Bd} \ Q_n,\) and \(Z\). Since \(I\) has uncountably many composants, there exists a composant \(C\) of \(I\) such that for each \(n\), no continuum in \(C - (P_n \cup Q_n)\) meets \(\mathrm{Bd} \ P_n, \mathrm{Bd} \ Q_n,\) and \(Z\). It follows that for each \(n\), there is a continuum \(L_n\) in \(C - (P_n \cup Q_n \cup Z)\) that meets both \(\mathrm{Bd} \ P_n\) and \(\mathrm{Bd} \ Q_n\). The limit of \(\{L_n\}\) is a continuum in \(I - Z\) that contains \(\{p, q\}\). But since \(p\) and \(q\) belong to different composants of \(I\) and \(Z\) intersects \(I\), this is a contradiction. Hence for some integer \(n\), there exists an infinite collection \(W\) of disjoint continua in \(I - (P_n \cup Q_n)\) such that each element of \(W\) meets \(\mathrm{Bd} \ P_n, \mathrm{Bd} \ Q_n,\) and \(Z\).

There exists a sequence of distinct continua \(\{H_i\}\) and two sequences of disjoint arc-segments \(\{R_i\}\) and \(\{T_i\}\) such that for each \(i\),

1. \(H_i\) is an element of \(W\),
2. \(R_i\) and \(T_i\) are in \(\mathrm{Bd} \ P_n\) and \(\mathrm{Bd} \ Q_n\) respectively,
3. \(R_i\) and \(T_i\) each meets \(H_{2i}\) and no other element of \(\{H_i\}\), and each has one endpoint in \(H_{2i-1}\) and the other endpoint in \(H_{2i+1}\).

For each positive integer \(i\), let \(y_i\) be a point of \(H_{2i} \cap Z\) and define \(D_i\) to be the complementary domain of \(H_{2i-1} \cup H_{2i+1} \cup R_i \cup T_i\) that contains \(y_i\). Note that the elements of the sequence \(\{D_i\}\) are disjoint domains in \(E^2 - \mathrm{Cl} \ (P_n \cup Q_n)\). Since the union of the continuum \(I \cup \mathrm{Cl} \ (P_n \cup Q_n)\) with its bounded complementary domains is a compact subset of \(E^2\), for some \(i\), every point of \(D_i\) is within \(\varepsilon\) of \(I\) and \(H_{2i-1} \cup H_{2i+1} \cup R_i \cup T_i\) separates \(y_i\) from \(x\) in \(E^2\).

**Theorem 2.** If \(M\) is a \(\lambda\) connected continuum in \(E^2\), then for each point \(x\) of \(M\), every continuum in the set \(K(x)\) is decomposable.

**Proof.** Assume that for some point \(x\) of \(M\), the set \(K(x)\) contains an indecomposable continuum \(I\). We shall prove that this assumption implies the existence of a link in \(M\) that contains \(I\); this will contradict the hypothesis of this theorem [5, Th. 2].

Let \(v\) and \(z\) be points of \(M - \{x\}\) that belong to distinct composants of \(I\). Define \(\{V_i\}\) and \(\{Z_i\}\) to be monotone descending sequences of circular regions in \(E^2\) centered on and converging to \(v\) and \(z\) respectively such that \(\mathrm{Cl} \ V_1 \cap \mathrm{Cl} \ Z_i = \emptyset\) and \(\mathrm{Cl} \ (V_1 \cup Z_i)\) is in \(E^2 - \{x\}\).

First we show that for each positive integer \(i\), there exists an
arc $A_i$ in $E^2 - M$ that goes from $\text{Bd } V_i$ to $\text{Bd } Z_i$ such that each point of $A_i$ is within $i^{-1}$ of $I$. By Theorem 1, for any given positive integer $i$, there exist continua $H$ and $F$ in $I$, arc-segments $R$ and $T$ in $V_i$, and a point $y$ of $I \cap Z_i$ such that $H \cup F \cup R \cup T$ separates $y$ from $x$ in $E^2$ and each point of $D$ (the $y$-component of $E^2 - (H \cup F \cup R \cup T)$) is within $i^{-1}$ of $I$. Let $U$ be a circular region containing $x$ in $E^2$ whose closure misses $H \cup F \cup R \cup T$. Let $G$ be a circular region containing $y$ in $E^2$ whose closure is in $D \cap Z_i$. Since $M$ is not aposyndetic at $x$ with respect to $y$, the component of $M - G$ that contains $x$ is not open relative to $M$ at $x$. Hence there exist two components $X$ and $Y$ of $M - G$ that meet $U$. It follows that a simple closed curve $J$ in $(E^2 - M) \cup G$ separates $X$ from $Y$ in $E^2$ [6, Th. 13, p. 170]. Note that $J$ must intersect both $G$ and $U$ [6, Th. 50, p. 18]. Since $J \cap (M - G) = \emptyset$ and $H \cup F \cup R \cup T$ separates $G$ from $U$ in $E^2$, there is an arc-segment $B$ in $(J \cap D) - M$ that has one endpoint in $\text{Bd } G$ and the other endpoint in $R \cup T$. We define $A_i$ to be an arc in $B - (V_i \cup Z_i)$ that goes from $\text{Bd } V_i$ to $\text{Bd } Z_i$. Since $A_i$ is in $D$, each of its points is within $i^{-1}$ of $I$.

Note that since $v$ and $z$ do not belong to the same composant of $I$, the limit of each subsequence of $\{A_i\}$ is $I$. For each $i$, let $Q_i$ be the complementary domain of $M$ that contains $A_i$. If $\{Q_i\}$ does not have infinitely many distinct elements, then for some $i$, the link $\text{Bd } Q_i$ in $M$ contains $I$. Suppose that $\{Q_i\}$ has infinitely many distinct elements. Then some subsequence of $\{Q_i\}$ converges to a link in $M$ [6, Th. 59, p. 24]. It follows that a link in $M$ contains $I$. This contradicts the fact that $M$ is $\lambda$ connected [5, Th. 2]. Hence for each point $x$ of $M$, every continuum in $K(x)$ is decomposable.

**Theorem 3.** Suppose that $M$ is a continuum in $E^2$ and for each point $x$ of $M$, every continuum in $K(x)$ is decomposable. Then for each point $x$ of $M$, every continuum in $L(x)$ is decomposable.

**Proof.** Assume that for some point $x$ of $M$, there is an indecomposable continuum $I$ in $L(x)$. We shall prove that from this assumption it follows that $M$ is not aposyndetic at any point of $I$ with respect to any other point of $I$. Hence for each point $z$ of $I$, the set $K(z)$ in $M$ contains $I$. This will contradict our hypothesis.

Suppose there exists a continuum $E$ in $M$ that does not contain $I$ whose interior relative to $M$ contains a point of $I$. There exist mutually exclusive circular regions $V$ and $Z$ in $E^2$ such that

1. $x$ does not belong to $\text{Cl } (V \cup Z)$,
2. $V$ and $Z$ each meets $I$,
3. $E$ and $V$ are disjoint,
4. $M \cap Z$ is contained in $E$. 
According to Theorem 1, there exist continua $H$ and $F$ in $I$, arc-segments $R$ and $T$ in $V$, and a point $y$ of $I \cap Z$ such that $H \cup F \cup R \cup T$ separates $y$ from $x$ in $E^2$. Define $D$ to be the $y$-component of $E^2 - (H \cup F \cup R \cup T)$. There exists a circular region $G$ in $E^2$ containing $y$ such that $\text{Cl } G$ is in $D \cap Z$. Let $U$ be a circular region in $E^2$ containing $x$ whose closure misses $H \cup F \cup R \cup T$.

Since $M$ is not aposyndetic at $y$ with respect to $x$, the $y$-component of $M - U$ is not open relative to $M$ at $y$. Hence $\text{Bd } G - M$ contains an arc-segment $A$ whose endpoints, $p$ and $q$, lie in different components of $M - U$. There exists a simple closed curve $J$ in $(E^2 - M) \cup U$ that separates $p$ from $q$ in $E^2$ such that $J \cap A$ is connected. Let $B$ denote the component of $J - U$ that contains $J \cap A$. Since $H \cup F \cup R \cup T$ separates $G$ from $U$ in $E^2$ and $B$ does not intersect $H \cup F$, it follows that both components of $B - A$ meet $R \cup T$. Evidently $B \cup V$ separates $p$ from $q$ in $E^2$ [6, Th. 32, p. 181]. But since $E$ is a continuum in $E^2 - (B \cup V)$ that contains $\{p, q\}$, this is a contradiction. Hence each subcontinuum of $M$ that contains a point of $I$ in its interior relative to $M$ contains $I$. This implies that for any point $z$ of $I$, the set $K(z)$ in $M$ contains $I$, which contradicts the hypothesis of this theorem. Hence for each point $x$ of $M$, every continuum in $L(x)$ is decomposable.

**Theorem 4.** Suppose that for each point $x$ of a plane continuum $M$, every continuum in $L(x)$ is decomposable. Then $M$ is $\lambda$ connected.

**Proof.** Assume that $M$ is not $\lambda$ connected. It follows that some link in $M$ contains an indecomposable continuum $I$ [5, Th. 2]. By Theorem 1 in [4], each subcontinuum of $M$ that contains a nonempty open subset of $I$ contains $I$. But this implies that for each point $x$ of $I$, the set $L(x)$ contains $I$, which is impossible. Hence $M$ is $\lambda$ connected.

**Theorem 5.** Suppose that $M$ is a plane continuum. The following three statements are equivalent.

I. $M$ is $\lambda$ connected.

II. For each point $x$ of $M$, every continuum in the set $K(x)$ is decomposable.

III. For each point $x$ of $M$, every continuum in $L(x)$ is decomposable.

**Proof.** This follows directly from Theorems 2, 3, and 4.

**References**


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