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**CHARACTERIZATIONS OF  $\lambda$  CONNECTED PLANE CONTINUA**

CHARLES LEMUEL HAGOPIAN

## CHARACTERIZATIONS OF $\lambda$ CONNECTED PLANE CONTINUA

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**A continuum  $M$  is said to be  $\lambda$  connected if any two of its points can be joined by a hereditarily decomposable continuum in  $M$ . Here we characterize  $\lambda$  connected plane continua in terms of Jones' functions  $K$  and  $L$ .**

A nondegenerate metric space that is both compact and connected is called a *continuum*. A continuum  $M$  is said to be *aposyndetic* at a point  $p$  of  $M$  with respect to a point  $q$  of  $M$  if there exists an open set  $U$  and a continuum  $H$  in  $M$  such that  $p \in U \subset H \subset M - \{q\}$ .

In [1], F. Burton Jones defines the functions  $K$  and  $L$  on a continuum  $M$  into the set of subsets of  $M$  as follows:

For each point  $x$  of  $M$ , the set  $K(x)$  ( $L(x)$ ) consists of all points  $y$  of  $M$  such that  $M$  is not aposyndetic at  $x$  ( $y$ ) with respect to  $y$  ( $x$ ).

Note that for each point  $x$  of  $M$ , the set  $L(x)$  is connected and closed in  $M$  [1, Th. 3]. For any point  $x$  of  $M$ , the set  $K(x)$  is closed [1, Th. 2] but may fail to be connected [2, Ex. 4], [3].

Suppose that  $M$  is a plane continuum. In this paper it is proved that the following three statements are equivalent.

- I.  $M$  is  $\lambda$  connected.
- II. For each point  $x$  of  $M$ , the set  $K(x)$  does not contain an indecomposable continuum.
- III. For each point  $x$  of  $M$ , every continuum in  $L(x)$  is decomposable.

Throughout this paper  $E^2$  is the Euclidean plane. For a given set  $S$  in  $E^2$ , we denote the closure and the boundary of  $S$  relative to  $E^2$  by  $\text{Cl } S$  and  $\text{Bd } S$  respectively.

**DEFINITION.** Let  $M$  be a continuum in  $E^2$ . A subcontinuum  $L$  of  $M$  is said to be a *link* in  $M$  if  $L$  is either the boundary of a complementary domain of  $M$  or the limit of a convergent sequence of complementary domains of  $M$ .

It is known that a plane continuum is  $\lambda$  connected if and only if each of its links is hereditarily decomposable [5, Th. 2].

**THEOREM 1.** *Suppose that a continuum  $M$  in  $E^2$  contains an indecomposable continuum  $I$ , that disjoint circular regions  $V$  and  $Z$  in  $E^2$  meet  $I$ , that a point  $x$  belongs to  $M - \text{Cl}(V \cup Z)$ , and that  $\varepsilon$  is a positive real number. Then there exist continua  $H$  and  $F$  in  $I$ , arc-segments  $R$  and  $T$  in  $V$ , and a point  $y$  of  $I \cap Z$  such that (1)*

$H \cup F \cup R \cup T$  separates  $y$  from  $x$  in  $E^2$ , and (2) if  $D$  is the  $y$ -component of  $E^2 - (H \cup F \cup R \cup T)$ , then each point of  $D$  is within  $\varepsilon$  of  $I$ .

*Proof.* Define  $p$  and  $q$  to be points of  $V \cap I$  that belong to distinct composants of  $I$ . Let  $\{P_n\}$  and  $\{Q_n\}$  be monotone descending sequences of circular regions in  $E^2$  centered on and converging to  $p$  and  $q$  respectively such that  $\text{Cl } P_1 \cap \text{Cl } Q_1 = \emptyset$  and  $\text{Cl } (P_1 \cup Q_1)$  is in  $V$ .

Suppose that for each positive integer  $n$ , only finitely many disjoint continua in  $I - (P_n \cup Q_n)$  intersect  $\text{Bd } P_n$ ,  $\text{Bd } Q_n$ , and  $Z$ . Since  $I$  has uncountably many composants, there exists a component  $C$  of  $I$  such that for each  $n$ , no continuum in  $C - (P_n \cup Q_n)$  meets  $\text{Bd } P_n$ ,  $\text{Bd } Q_n$ , and  $Z$ . It follows that for each  $n$ , there is a continuum  $L_n$  in  $C - (P_n \cup Q_n \cup Z)$  that meets both  $\text{Bd } P_n$  and  $\text{Bd } Q_n$ . The limit of  $\{L_n\}$  is a continuum in  $I - Z$  that contains  $\{p, q\}$ . But since  $p$  and  $q$  belong to different composants of  $I$  and  $Z$  intersects  $I$ , this is a contradiction. Hence for some integer  $n$ , there exists an infinite collection  $W$  of disjoint continua in  $I - (P_n \cup Q_n)$  such that each element of  $W$  meets  $\text{Bd } P_n$ ,  $\text{Bd } Q_n$ , and  $Z$ .

There exists a sequence of distinct continua  $\{H_i\}$  and two sequences of disjoint arc-segments  $\{R_i\}$  and  $\{T_i\}$  such that for each  $i$ ,

(1)  $H_i$  is an element of  $W$ ,

(2)  $R_i$  and  $T_i$  are in  $\text{Bd } P_n$  and  $\text{Bd } Q_n$  respectively,

(3)  $R_i$  and  $T_i$  each meets  $H_{2i}$  and no other element of  $\{H_i\}$ , and each has one endpoint in  $H_{2i-1}$  and the other endpoint in  $H_{2i+1}$ .

For each positive integer  $i$ , let  $y_i$  be a point of  $H_{2i} \cap Z$  and define  $D_i$  to be the complementary domain of  $H_{2i-1} \cup H_{2i+1} \cup R_i \cup T_i$  that contains  $y_i$ . Note that the elements of the sequence  $\{D_i\}$  are disjoint domains in  $E^2 - \text{Cl } (P_n \cup Q_n)$ . Since the union of the continuum  $I \cup \text{Cl } (P_n \cup Q_n)$  with its bounded complementary domains is a compact subset of  $E^2$ , for some  $i$ , every point of  $D_i$  is within  $\varepsilon$  of  $I$  and  $H_{2i-1} \cup H_{2i+1} \cup R_i \cup T_i$  separates  $y_i$  from  $x$  in  $E^2$ .

**THEOREM 2.** *If  $M$  is a  $\lambda$  connected continuum in  $E^2$ , then for each point  $x$  of  $M$ , every continuum in the set  $K(x)$  is decomposable.*

*Proof.* Assume that for some point  $x$  of  $M$ , the set  $K(x)$  contains an indecomposable continuum  $I$ . We shall prove that this assumption implies the existence of a link in  $M$  that contains  $I$ ; this will contradict the hypothesis of this theorem [5, Th. 2].

Let  $v$  and  $z$  be points of  $M - \{x\}$  that belong to distinct composants of  $I$ . Define  $\{V_i\}$  and  $\{Z_i\}$  to be monotone descending sequences of circular regions in  $E^2$  centered on and converging to  $v$  and  $z$  respectively such that  $\text{Cl } V_1 \cap \text{Cl } Z_1 = \emptyset$  and  $\text{Cl } (V_1 \cup Z_1)$  is in  $E^2 - \{x\}$ .

First we show that for each positive integer  $i$ , there exists an

arc  $A_i$  in  $E^2 - M$  that goes from  $\text{Bd } V_i$  to  $\text{Bd } Z_i$  such that each point of  $A_i$  is within  $i^{-1}$  of  $I$ . By Theorem 1, for any given positive integer  $i$ , there exist continua  $H$  and  $F$  in  $I$ , arc-segments  $R$  and  $T$  in  $V_i$ , and a point  $y$  of  $I \cap Z_i$  such that  $H \cup F \cup R \cup T$  separates  $y$  from  $x$  in  $E^2$  and each point of  $D$  (the  $y$ -component of  $E^2 - (H \cup F \cup R \cup T)$ ) is within  $i^{-1}$  of  $I$ . Let  $U$  be a circular region containing  $x$  in  $E^2$  whose closure misses  $H \cup F \cup R \cup T$ . Let  $G$  be a circular region containing  $y$  in  $E^2$  whose closure is in  $D \cap Z_i$ . Since  $M$  is not aposyndetic at  $x$  with respect to  $y$ , the component of  $M - G$  that contains  $x$  is not open relative to  $M$  at  $x$ . Hence there exist two components  $X$  and  $Y$  of  $M - G$  that meet  $U$ . It follows that a simple closed curve  $J$  in  $(E^2 - M) \cup G$  separates  $X$  from  $Y$  in  $E^2$  [6, Th. 13, p. 170]. Note that  $J$  must intersect both  $G$  and  $U$  [6, Th. 50, p. 18]. Since  $J \cap (M - G) = \emptyset$  and  $H \cup F \cup R \cup T$  separates  $G$  from  $U$  in  $E^2$ , there is an arc-segment  $B$  in  $(J \cap D) - M$  that has one endpoint in  $\text{Bd } G$  and the other endpoint in  $R \cup T$ . We define  $A_i$  to be an arc in  $B - (V_i \cup Z_i)$  that goes from  $\text{Bd } V_i$  to  $\text{Bd } Z_i$ . Since  $A_i$  is in  $D$ , each of its points is within  $i^{-1}$  of  $I$ .

Note that since  $v$  and  $z$  do not belong to the same component of  $I$ , the limit of each subsequence of  $\{A_i\}$  is  $I$ . For each  $i$ , let  $Q_i$  be the complementary domain of  $M$  that contains  $A_i$ . If  $\{Q_i\}$  does not have infinitely many distinct elements, then for some  $i$ , the link  $\text{Bd } Q_i$  in  $M$  contains  $I$ . Suppose that  $\{Q_i\}$  has infinitely many distinct elements. Then some subsequence of  $\{Q_i\}$  converges to a link in  $M$  [6, Th. 59, p. 24]. It follows that a link in  $M$  contains  $I$ . This contradicts the fact that  $M$  is  $\lambda$  connected [5, Th. 2]. Hence for each point  $x$  of  $M$ , every continuum in  $K(x)$  is decomposable.

**THEOREM 3.** *Suppose that  $M$  is a continuum in  $E^2$  and for each point  $x$  of  $M$ , every continuum in  $K(x)$  is decomposable. Then for each point  $x$  of  $M$ , every continuum in  $L(x)$  is decomposable.*

*Proof.* Assume that for some point  $x$  of  $M$ , there is an indecomposable continuum  $I$  in  $L(x)$ . We shall prove that from this assumption it follows that  $M$  is not aposyndetic at any point of  $I$  with respect to any other point of  $I$ . Hence for each point  $z$  of  $I$ , the set  $K(z)$  in  $M$  contains  $I$ . This will contradict our hypothesis.

Suppose there exists a continuum  $E$  in  $M$  that does not contain  $I$  whose interior relative to  $M$  contains a point of  $I$ . There exist mutually exclusive circular regions  $V$  and  $Z$  in  $E^2$  such that

- (1)  $x$  does not belong to  $\text{Cl}(V \cup Z)$ ,
- (2)  $V$  and  $Z$  each meets  $I$ ,
- (3)  $E$  and  $V$  are disjoint,
- (4)  $M \cap Z$  is contained in  $E$ .

According to Theorem 1, there exist continua  $H$  and  $F$  in  $I$ , arc-segments  $R$  and  $T$  in  $V$ , and a point  $y$  of  $I \cap Z$  such that  $H \cup F \cup R \cup T$  separates  $y$  from  $x$  in  $E^2$ . Define  $D$  to be the  $y$ -component of  $E^2 - (H \cup F \cup R \cup T)$ . There exists a circular region  $G$  in  $E^2$  containing  $y$  such that  $\text{Cl } G$  is in  $D \cap Z$ . Let  $U$  be a circular region in  $E^2$  containing  $x$  whose closure misses  $H \cup F \cup R \cup T$ .

Since  $M$  is not aposyndetic at  $y$  with respect to  $x$ , the  $y$ -component of  $M - U$  is not open relative to  $M$  at  $y$ . Hence  $\text{Bd } G - M$  contains an arc-segment  $A$  whose endpoints,  $p$  and  $q$ , lie in different components of  $M - U$ . There exists a simple closed curve  $J$  in  $(E^2 - M) \cup U$  that separates  $p$  from  $q$  in  $E^2$  such that  $J \cap A$  is connected. Let  $B$  denote the component of  $J - U$  that contains  $J \cap A$ . Since  $H \cup F \cup R \cup T$  separates  $G$  from  $U$  in  $E^2$  and  $B$  does not intersect  $H \cup F$ , it follows that both components of  $B - A$  meet  $R \cup T$ . Evidently  $B \cup V$  separates  $p$  from  $q$  in  $E^2$  [6, Th. 32, p. 181]. But since  $E$  is a continuum in  $E^2 - (B \cup V)$  that contains  $\{p, q\}$ , this is a contradiction. Hence each subcontinuum of  $M$  that contains a point of  $I$  in its interior relative to  $M$  contains  $I$ . This implies that for any point  $z$  of  $I$ , the set  $K(z)$  in  $M$  contains  $I$ , which contradicts the hypothesis of this theorem. Hence for each point  $x$  of  $M$ , every continuum in  $L(x)$  is decomposable.

**THEOREM 4.** *Suppose that for each point  $x$  of a plane continuum  $M$ , every continuum in  $L(x)$  is decomposable. Then  $M$  is  $\lambda$  connected.*

*Proof.* Assume that  $M$  is not  $\lambda$  connected. It follows that some link in  $M$  contains an indecomposable continuum  $I$  [5, Th. 2]. By Theorem 1 in [4], each subcontinuum of  $M$  that contains a nonempty open subset of  $I$  contains  $I$ . But this implies that for each point  $x$  of  $I$ , the set  $L(x)$  contains  $I$ , which is impossible. Hence  $M$  is  $\lambda$  connected.

**THEOREM 5.** *Suppose that  $M$  is a plane continuum. The following three statements are equivalent.*

- I.  $M$  is  $\lambda$  connected.
- II. For each point  $x$  of  $M$ , every continuum in the set  $K(x)$  is decomposable.
- III. For each point  $x$  of  $M$ , every continuum in  $L(x)$  is decomposable.

*Proof.* This follows directly from Theorems 2, 3, and 4.

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