

# Pacific Journal of Mathematics

**ON RELATIONS BETWEEN NÖRLUND AND RIESZ MEANS**

IKUKO KAYASHIMA

## ON RELATIONS BETWEEN NÖRLUND AND RIESZ MEANS

IKUKO KAYASHIMA

**Several results on relations between (absolute) Nörlund summability and (absolute) Riesz summability are known. Among them, Dikshit gives sufficient conditions for  $|\bar{N}, q_n| \subseteq |N, p_n|$  when the sequence  $\{p_n\}$  is nonincreasing. The purpose of this paper is to give sufficient conditions for  $|N, p_n| \subseteq |\bar{N}, q_n|$  or  $|\bar{N}, q_n| \subseteq |N, p_n|$  when  $\{p_n\}$  is monotone. The results obtained here are also absolute summability analogues of Ishiguro's theorems and Kuttner and Rhoades' theorems which state the inclusion relations between  $(N, p_n)$  and  $(\bar{N}, p_n)$  summability.**

1. Let  $\{p_n\}$  be a sequence such that  $p_n > 0, P_n = \sum_{k=0}^n p_k \neq 0$ . A series  $\sum_{n=0}^{\infty} a_n$  with its partial sum  $s_n$  is said to be summable  $(N, p_n)$  to sum  $s$ , if  $t_n = \sum_{k=0}^n p_{n-k} s_k / P_n \rightarrow s$  as  $n \rightarrow \infty$ , and summable  $(\bar{N}, p_n)$  to sum  $s$ , if  $u_n = \sum_{k=0}^n p_k s_k / P_n \rightarrow s$  as  $n \rightarrow \infty$ . It is said to be absolutely summable  $(N, p_n)$ , or summable  $|N, p_n|$ , if  $\sum |t_n - t_{n+1}| < \infty$ , and absolutely summable  $(\bar{N}, p_n)$ , or summable  $|\bar{N}, p_n|$ , if  $\sum |u_n - u_{n+1}| < \infty$ . Given two summability methods  $A$  and  $B$ , we write  $(A) \subseteq (B)$  if each series summable  $A$  is summable  $B$ . Throughout this paper, we write for a sequence  $\{b_n\}$

$$b_{-n} = 0 (n \geq 1), \Delta b_n = b_n - b_{n+1},$$

and for a double sequence  $\{c_{mn}\}$

$$\Delta_n(c_{mn}) = c_{mn} - c_{m, n+1},$$

and  $K$  denotes an absolute constant, not necessarily the same at each occurrence.

On inclusion relations between two summability, the following results are known.

**THEOREM A.** [1] *If the sequence  $\{p_n\}$  is nonincreasing,  $Q_n \rightarrow +\infty$  and  $Q_n/q_{n+1} = O(P_{n+1})$ , where  $q_n > 0$  and  $Q_n = \sum_{k=0}^n q_k \neq 0$ , then  $|\bar{N}, q_n| \subseteq |N, p_n|$ .*

**THEOREM B.** [2] *If  $\{p_n\}$  is the nondecreasing sequence such that  $P_n \rightarrow +\infty$  and  $p_n = o(P_n)$ , then  $(\bar{N}, p_n) \subseteq (N, p_n)$ .*

**THEOREM C.** [3] *If  $\{p_n\}$  is the nonincreasing sequence such that  $P_n \rightarrow +\infty$ , then  $(N, p_n) \subseteq (\bar{N}, p_n)$ .*

REMARK. Kuttner and Rhoades' theorem [3, Theorem 2] is more precise than Theorem C, but we refer to it in the above form.

THEOREM D. [3] *If  $\{p_n\}$  is the nonincreasing sequence such that  $p_n \geq K > 0$ , then  $(N, p_n)$  and  $(\bar{N}, p_n)$  are equivalent.*

The purpose of this paper is to prove the following theorems.

THEOREM 1. *If  $\{p_n\}$  and  $\{q_n\}$  are positive and nondecreasing sequences and if  $\{p_{n+1}/p_n\}$  is nonincreasing, then  $| \bar{N}, q_n | \subseteq | N, p_n |$ .*

This theorem deals with the case in which  $\{p_n\}$  is nondecreasing, while theorem A deals with the case in which  $\{p_n\}$  is nonincreasing. In this Theorem, if we put  $p_n = q_n$ , then we obtain the following

COROLLARY 1. *If  $\{p_n\}$  is the nondecreasing sequence such that  $\{p_{n+1}/p_n\}$  is nonincreasing, then  $| \bar{N}, p_n | \subseteq | N, p_n |$ .*

This is an absolute summability analogue of Theorem B.

THEOREM 2. *If  $\{p_n\}$  and  $\{q_n\}$  are positive and nonincreasing sequences and if  $\{p_{n+1}/p_n\}$  is nondecreasing, then  $| N, p_n | \subseteq | \bar{N}, q_n |$ .*

In this theorem, if we put  $p_n = q_n$ , we obtain the following

COROLLARY 2. *If  $\{p_n\}$  is the nonincreasing sequence such that  $\{p_{n+1}/p_n\}$  is nondecreasing, then  $| N, p_n | \subseteq | \bar{N}, p_n |$ .*

This is an absolute summability analogue of Theorem C.

THEOREM 3. *If  $\{p_n\}$  is the nonincreasing sequence such that  $p_n \geq K > 0$ , then  $| \bar{N}, p_n | \subseteq | N, p_n |$ .*

Combining Theorem 3 and Corollary 2 we have the following

COROLLARY 3. *If  $\{p_n\}$  is the nonincreasing sequence such that  $\{p_{n+1}/p_n\}$  is nondecreasing and  $p_n \geq K > 0$ , then  $| N, p_n |$  and  $| \bar{N}, p_n |$  are equivalent.*

This is an absolute summability analogue of Theorem D. Theorems 1-3 are proved in §§3-5, respectively.

The author takes this opportunity of expressing her heartfelt thanks to Professor H. Hirokawa for his kind encouragement and valuable suggestions in the preparation of this paper.

2. We require the following lemmas.

LEMMA 1. Let  $y_n = \sum_{k=0}^n c_{nk}x_k$ . If

(i)  $\sum_{j=0}^n |c_{nj}| \leq K < \infty$  for all  $n$ , and

(ii)  $\sum_{j=k}^n (c_{nj} - c_{n-1,j}) \geq 0$  for  $k = 0, 1, 2, \dots, n$ ,

then  $\sum_{n=0}^{\infty} |\Delta y_n| < \infty$  whenever  $\sum_{n=0}^{\infty} |\Delta x_n| < \infty$ .

This is easily proved by the method analogous to that of the proof of McFadden's theorem [4, Theorem (2.12)].

LEMMA 2. For  $m, n = 0, 1, 2, \dots$ ,

$$\sum_{k=0}^m Q_k \Delta_k \left( \frac{p_{n-k}}{q_k} \right) = P_n - P_{n-m-1} - \frac{Q_m}{q_{m+1}} p_{n-m-1}.$$

This is Lemma 2 in [1].

LEMMA 3. If  $\{p_n\}$  is the nondecreasing sequence such that  $\{p_{n+1}/p_n\}$  is nonincreasing and  $p_{n-k}/P_n < p_{n-k-1}/P_{n-1}$ , then

$$k \left( \frac{p_{n-k}}{P_n} - \frac{p_{n-k-1}}{P_{n-1}} \right) \geq \sum_{m=0}^{k-1} \left( \frac{p_{n-m}}{P_n} - \frac{p_{n-m-1}}{P_{n-1}} \right).$$

This is due to McFadden (see [4, p. 178]).

3. Proof of Theorem 1. Let us write

$$t_n = \frac{1}{P_n} \sum_{k=0}^n p_{n-k} s_k \quad \text{and} \quad u_n = \frac{1}{Q_n} \sum_{k=0}^n q_k s_k.$$

By Abel's transformation, we have

$$\begin{aligned} t_n &= \frac{1}{P_n} \sum_{k=0}^{n-1} Q_k u_k \Delta_k \left( \frac{p_{n-k}}{q_k} \right) + \frac{p_0 Q_n u_n}{P_n q_n} \\ &= \sum_{k=0}^n a_{nk} u_k, \end{aligned}$$

where

$$a_{nk} = \frac{Q_k}{P_n} \Delta_k \left( \frac{p_{n-k}}{q_k} \right).$$

To prove theorem, we must verify that the conditions of Lemma 1 with  $\{c_{nk}\}$  replaced by  $\{a_{nk}\}$  are satisfied. Since  $\{p_n\}$  and  $\{q_n\}$  are positive and nondecreasing,  $a_{nk} \geq 0$ . And if  $s_k = 1$  for all  $k$ , then  $t_n = 1, u_n = 1$  for all  $n$ .

Hence,

$$\sum_{j=0}^n |a_{nj}| = \sum_{j=0}^n a_{nj} = 1.$$

Therefore, it is sufficient to show that

$$P \equiv \sum_{j=k}^n (a_{nj} - a_{n-1,j}) \geq 0 \quad \text{for } k = 0, 1, 2, \dots, n.$$

By Lemma 2, we have

$$\begin{aligned} P &= \frac{1}{P_n} \left( \sum_{j=0}^n - \sum_{j=0}^{k-1} \right) Q_j \Delta_j \left( \frac{p_{n-j}}{q_j} \right) - \frac{1}{P_{n-1}} \left( \sum_{j=0}^{n-1} - \sum_{j=0}^{k-1} \right) Q_j \Delta_j \left( \frac{p_{n-j-1}}{q_j} \right) \\ &= \frac{1}{P_n} \left( P_{n-k} + \frac{Q_{k-1}}{q_k} p_{n-k} \right) - \frac{1}{P_{n-1}} \left( P_{n-k-1} + \frac{Q_{k-1}}{q_k} p_{n-k-1} \right) \\ &= \frac{P_{n-k}}{P_n} - \frac{P_{n-k-1}}{P_{n-1}} + \frac{Q_{k-1}}{q_k} \left( \frac{p_{n-k}}{P_n} - \frac{p_{n-k-1}}{P_{n-1}} \right). \end{aligned}$$

Since  $\{p_{n+1}/p_n\}$  is nonincreasing, it is easily deducible that

$$\frac{P_{n-k}}{P_n} - \frac{P_{n-k-1}}{P_{n-1}} \geq 0.$$

Thus, if  $p_{n-k}/P_n - p_{n-k-1}/P_{n-1} \geq 0$ ,  $P$  is nonnegative. Suppose on the other hand that

$$\frac{p_{n-k}}{P_n} - \frac{p_{n-k-1}}{P_{n-1}} < 0.$$

Since  $\{q_n\}$  is nondecreasing,

$$Q_{k-1} \leq kq_{k-1} \leq kq_k.$$

Hence,  $Q_{k-1}/q_k \leq k$ .

Thus, we have, by Lemma 3,

$$\begin{aligned} P &\geq \frac{P_{n-k}}{P_n} - \frac{P_{n-k-1}}{P_{n-1}} + k \left( \frac{p_{n-k}}{P_n} - \frac{p_{n-k-1}}{P_{n-1}} \right) \\ &\geq \frac{P_{n-k}}{P_n} - \frac{P_{n-k-1}}{P_{n-1}} + \sum_{m=0}^{k-1} \left( \frac{p_{n-m}}{P_n} - \frac{p_{n-m-1}}{P_{n-1}} \right) \\ &= \frac{P_{n-k}}{P_n} - \frac{P_{n-k-1}}{P_{n-1}} + \frac{P_n - P_{n-k}}{P_n} - \frac{P_{n-1} - P_{n-k-1}}{P_{n-1}} \\ &= 0. \end{aligned}$$

This completes the proof of Theorem 1.

4. *Proof of Theorem 2.* Under the conditions of  $\{p_n\}$ , using McFadden's theorem [4, Theorem (2.28)], we see that  $|N, p_n| \subseteq |C, 1|$ . Hence we need only verify, under the conditions of theorem, that

$$|C, 1| \subseteq |\bar{N}, q_n|.$$

Let us write

$$\sigma_n = \frac{1}{n+1} \sum_{k=0}^n s_k \quad \text{and} \quad u_n = \frac{1}{Q_n} \sum_{k=0}^n q_k s_k.$$

By Abel's transformation, we have

$$\begin{aligned} u_n &= \frac{1}{Q_n} \sum_{k=0}^{n-1} (k+1) \sigma_k \Delta q_k + \frac{(n+1)q_n \sigma_n}{Q_n} \\ &= \sum_{k=0}^n b_{nk} \sigma_k, \end{aligned}$$

where

$$\begin{aligned} b_{nk} &= \frac{(k+1) \Delta q_k}{Q_n} && \text{for } 0 \leq k < n, \\ &= \frac{(n+1)q_n}{Q_n} && \text{for } k = n. \end{aligned}$$

For our purposes, it is sufficient to show that the conditions of Lemma 1 with  $\{c_{nk}\}$  replaced by  $\{b_{nk}\}$  are satisfied.

Since  $\{q_n\}$  is positive and nonincreasing,  $b_{nk} \geq 0$ . And if  $s_k = 1$  for all  $k$ , then  $\sigma_n = 1, u_n = 1$  for all  $n$ .

Hence,

$$\sum_{j=0}^n |b_{nj}| = \sum_{j=0}^n b_{nj} = 1.$$

Therefore, it is sufficient to show that

$$Q \equiv \sum_{j=k}^n (b_{nj} - b_{n-1,j}) \geq 0 \quad \text{for } k = 0, 1, 2, \dots, n.$$

For  $0 \leq k \leq n-1$ , we have

$$\begin{aligned} Q &= \frac{1}{Q_n} \sum_{j=k}^{n-1} (j+1) \Delta q_j + \frac{(n+1)q_n}{Q_n} \\ &\quad - \frac{1}{Q_{n-1}} \sum_{j=k}^{n-2} (j+1) \Delta q_j - \frac{nq_{n-1}}{Q_{n-1}} \\ &= \frac{1}{Q_n} \{Q_n - (Q_{k-1} - kq_k)\} - \frac{1}{Q_{n-1}} \{Q_{n-1} - (Q_{k-1} - kq_k)\} \\ &= (kq_k - Q_{k-1}) \left( \frac{1}{Q_n} - \frac{1}{Q_{n-1}} \right). \end{aligned}$$

Since  $\{q_n\}$  is positive and nonincreasing,

$$Q_{k-1} \geq kq_{k-1} \geq kq_k \quad \text{and} \quad \frac{1}{Q_n} \leq \frac{1}{Q_{n-1}}.$$

Therefore, we have  $Q \geq 0$ . For  $k = n$ , since  $b_{nn} \geq 0$ , we have  $Q = b_{nn} \geq 0$ . Hence, we have  $Q \geq 0$  for  $k = 0, 1, 2, \dots, n$ .

This completes the proof of Theorem 2.

5. *Proof of Theorem 3.* Consider Theorem A for  $p_n = q_n$ . Then, by our assumption,

$$0 \leq \frac{P_n}{P_{n+1}p_{n+1}} \leq \frac{1}{p_{n+1}} \leq \frac{1}{K}.$$

Therefore, we have  $P_n/p_{n+1} = O(P_{n+1})$ .

Thus, using our assumption, we see that the conditions of Theorem A are satisfied for  $p_n = q_n$ .

Hence, by Theorem A, we have  $|\bar{N}, p_n| \subseteq |N, p_n|$ .

#### REFERENCES

1. G. D. Dikshit, *On inclusion relations between Riesz and Nörlund means*, Indian J. Math., **7** (1965), 73-81.
2. K. Ishiguro, *The relation between  $(N, p_n)$  and  $(\bar{N}, p_n)$  summability*, Proc. Japan Acad., **41** (1965), 120-122.
3. B. Kuttner and B. E. Rhoades, *Relations between  $(N, p_n)$  and  $(\bar{N}, p_n)$  summability*, Proc. Edinburgh Math. Soc., (2) **16** (1968), 109-116.
4. L. McFadden, *Absolute Nörlund summability*, Duke Math. J., **9** (1942), 168-207.

Received August 30, 1972.

CHIBA UNIVERSITY, CHIBA, JAPAN

# PACIFIC JOURNAL OF MATHEMATICS

## EDITORS

RICHARD ARENS (Managing Editor)

University of California  
Los Angeles, California 90024

J. DUGUNDJI\*

Department of Mathematics  
University of Southern California  
Los Angeles, California 90007

R. A. BEAUMONT

University of Washington  
Seattle, Washington 98105

D. GILBARG AND J. MILGRAM

Stanford University  
Stanford, California 94305

## ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

## SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA  
CALIFORNIA INSTITUTE OF TECHNOLOGY  
UNIVERSITY OF CALIFORNIA  
MONTANA STATE UNIVERSITY  
UNIVERSITY OF NEVADA  
NEW MEXICO STATE UNIVERSITY  
OREGON STATE UNIVERSITY  
UNIVERSITY OF OREGON  
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA  
STANFORD UNIVERSITY  
UNIVERSITY OF TOKYO  
UNIVERSITY OF UTAH  
WASHINGTON STATE UNIVERSITY  
UNIVERSITY OF WASHINGTON  
\* \* \*  
AMERICAN MATHEMATICAL SOCIETY  
NAVAL WEAPONS CENTER

---

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

---

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. 39. All other communications to the editors should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

---

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$48.00 a year (6 Vols., 12 issues). Special rate: \$24.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

\* C. R. DePrima California Institute of Technology, Pasadena, CA 91109, will replace J. Dugundji until August 1974.

Copyright © 1973 by  
Pacific Journal of Mathematics  
All Rights Reserved



# Pacific Journal of Mathematics

Vol. 49, No. 2

June, 1973

Wm. R. Allaway, <i>On finding the distribution function for an orthogonal polynomial set</i> .....	305
Eric Amar, <i>Sur un théorème de Mooney relatif aux fonctions analytiques bornées</i> .....	311
Robert Morgan Brooks, <i>Analytic structure in the spectrum of a natural system</i> .....	315
Bahattin Cengiz, <i>On extremely regular function spaces</i> .....	335
Kwang-nan Chow and Moses Glasner, <i>Atoms on the Royden boundary</i> .....	339
Paul Frazier Duvall, Jr. and Jim Maxwell, <i>Tame <math>Z^2</math>-actions on <math>E^n</math></i> .....	349
Allen Roy Freedman, <i>On the additivity theorem for <math>n</math>-dimensional asymptotic density</i> .....	357
John Griffin and Kelly Denis McKennon, <i>Multipliers and the group <math>L_p</math>-algebras</i> .....	365
Charles Lemuel Hagopian, <i>Characterizations of <math>\lambda</math> connected plane continua</i> .....	371
Jon Craig Helton, <i>Bounds for products of interval functions</i> .....	377
Ikuko Kayashima, <i>On relations between Nörlund and Riesz means</i> .....	391
Everett Lee Lady, <i>Slender rings and modules</i> .....	397
Shozo Matsuura, <i>On the Lu Qi-Keng conjecture and the Bergman representative domains</i> .....	407
Stephen H. McCleary, <i>The lattice-ordered group of automorphisms of an <math>\alpha</math>-set</i> .....	417
Stephen H. McCleary, <i><math>o</math>-2-transitive ordered permutation groups</i> .....	425
Stephen H. McCleary, <i><math>o</math>-primitive ordered permutation groups. II</i> .....	431
Richard Rochberg, <i>Almost isometries of Banach spaces and moduli of planar domains</i> .....	445
R. F. Rossa, <i>Radical properties involving one-sided ideals</i> .....	467
Robert A. Rubin, <i>On exact localization</i> .....	473
S. Sribala, <i>On <math>\Sigma</math>-inverse semigroups</i> .....	483
H. M. (Hari Mohan) Srivastava, <i>On the Konhauser sets of biorthogonal polynomials suggested by the Laguerre polynomials</i> .....	489
Stuart A. Steinberg, <i>Rings of quotients of rings without nilpotent elements</i> .....	493
Daniel Mullane Sunday, <i>The self-equivalences of an <math>H</math>-space</i> .....	507
W. J. Thron and Richard Hawks Warren, <i>On the lattice of proximities of Čech compatible with a given closure space</i> .....	519
Frank Uhlig, <i>The number of vectors jointly annihilated by two real quadratic forms determines the inertia of matrices in the associated pencil</i> .....	537
Frank Uhlig, <i>On the maximal number of linearly independent real vectors annihilated simultaneously by two real quadratic forms</i> .....	543
Frank Uhlig, <i>Definite and semidefinite matrices in a real symmetric matrix pencil</i> ...	561
Arnold Lewis Villone, <i>Self-adjoint extensions of symmetric differential operators</i> .....	569
Cary Webb, <i>Tensor and direct products</i> .....	579
James Victor Whittaker, <i>On normal subgroups of differentiable homeomorphisms</i> .....	595
Jerome L. Paul, <i>Addendum to: "Sequences of homeomorphisms which converge to homeomorphisms"</i> .....	615
David E. Fields, <i>Correction to: "Dimension theory in power series rings"</i> .....	616
Peter Michael Curran, <i>Correction to: "Cohomology of finitely presented groups"</i> .....	617
Billy E. Rhoades, <i>Correction to: "Commutants of some Hausdorff matrices"</i> .....	617
Charles W. Trigg, <i>Corrections to: "Versum sequences in the binary system"</i> .....	619