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Several results on relations between (absolute) Nörlund summability and (absolute) Riesz summability are known. Among them, Dikshit gives sufficient conditions for $|\bar{N}, q_n| \subseteqq |N, p_n|$ when the sequence $\{p_n\}$ is nonincreasing. The purpose of this paper is to give sufficient conditions for $|N, p_n| \subseteqq |\bar{N}, q_n|$ or $|\bar{N}, q_n| \subseteqq |N, p_n|$ when $\{p_n\}$ is monotone. The results obtained here are also absolute summability analogues of Ishiguro's theorems and Kuttner and Rhoades' theorems which state the inclusion relations between (N, p_n) and $(\bar{N}, p_n)$ summability.

1. Let $\{p_n\}$ be a sequence such that $p_n > 0$, $P_n = \sum_{k=0}^{n} p_k \neq 0$. A series $\sum_{n=0}^{\infty} a_n$ with its partial sum s_n is said to be summable (N, p_n) to sum s, if $t_n = \sum_{k=0}^{n} p_{n-k}s_k/P_n \rightarrow s$ as $n \rightarrow \infty$, and summable $(\bar{N}, p_n)$ to sum s, if $u_n = \sum_{k=0}^{n} p_k s_k/P_n \rightarrow s$ as $n \rightarrow \infty$. It is said to be absolutely summable (N, p_n), or summable $|N, p_n|$, if $\Sigma |t_n - t_{n+1}| < \infty$, and absolutely summable $(\bar{N}, p_n)$, or summable $|\bar{N}, p_n|$, if $\Sigma |u_n - u_{n+1}| < \infty$. Given two summability methods A and B, we write $(A) \subseteqq (B)$ if each series summable A is summable B. Throughout this paper, we write for a sequence $\{b_n\}$

$$b_{-n} = 0 (n \geqq 1), \ \Delta b_n = b_n - b_{n+1} ,$$

and for a double sequence $\{c_{mn}\}$

$$\Delta_n(c_{mn}) = c_{mn} - c_{m,n+1} ,$$

and K denotes an absolute constant, not necessarily the same at each occurrence.

On inclusion relations between two summability, the following results are known.

THEOREM A. [1] *If the sequence $\{p_n\}$ is nonincreasing, $Q_n \rightarrow +\infty$ and $Q_n/q_{n+1} = O(P_{n+1})$, where $q_n > 0$ and $Q_n = \sum_{k=0}^{n} q_k \neq 0$, then $|\bar{N}, q_n| \subseteqq |N, p_n|$.*

THEOREM B. [2] *If $\{p_n\}$ is the nondecreasing sequence such that $P_n \rightarrow +\infty$ and $p_n = o(P_n)$, then $(\bar{N}, p_n) \subseteqq (N, p_n)$.*

THEOREM C. [3] *If $\{p_n\}$ is the nonincreasing sequence such that $P_n \rightarrow +\infty$, then $(N, p_n) \subseteqq (\bar{N}, p_n)$.*

REMARK. Kuttner and Rhoades' theorem [3, Theorem 2] is more precise than Theorem C, but we refer to it in the above form.

THEOREM D. [3] *If* $\{p_n\}$ *is the nonincreasing sequence such that* $p_n \geqq K > 0$, *then* (N, p_n) *and* $(\bar{N}, p_n)$ *are equivalent.*

The purpose of this paper is to prove the following theorems.

THEOREM 1. *If* $\{p_n\}$ *and* $\{q_n\}$ *are positive and nondecreasing sequences and if* $\{p_{n+1}/p_n\}$ *is nonincreasing, then* $|\bar{N}, q_n| \subseteqq |N, p_n|$.

This theorem deals with the case in which $\{p_n\}$ is nondecreasing, while theorem A deals with the case in which $\{p_n\}$ is nonincreasing. In this Theorem, if we put $p_n = q_n$, then we obtain the following

COROLLARY 1. *If* $\{p_n\}$ *is the nondecreasing sequence such that* $\{p_{n+1}/p_n\}$ *is nonincreasing, then* $|\bar{N}, p_n| \subseteqq |N, p_n|$.

This is an absolute summability analogue of Theorem B.

THEOREM 2. *If* $\{p_n\}$ *and* $\{q_n\}$ *are positive and nonincreasing sequences and if* $\{p_{n+1}/p_n\}$ *is nondecreasing, then* $|N, p_n| \subseteqq |\bar{N}, q_n|$.

In this theorem, if we put $p_n = q_n$, we obtain the following

COROLLARY 2. *If* $\{p_n\}$ *is the nonincreasing sequence such that* $\{p_{n+1}/p_n\}$ *is nondecreasing, then* $|N, p_n| \subseteqq |\bar{N}, p_n|$.

This is an absolute summability analogue of Theorem C.

THEOREM 3. *If* $\{p_n\}$ *is the nonincreasing sequence such that* $p_n \geqq K > 0$, *then* $|\bar{N}, p_n| \subseteqq |N, p_n|$.

Combining Theorem 3 and Corollary 2 we have the following

COROLLARY 3. *If* $\{p_n\}$ *is the nonincreasing sequence such that* $\{p_{n+1}/p_n\}$ *is nondecreasing and* $p_n \geqq K > 0$, *then* $|N, p_n|$ *and* $|\bar{N}, p_n|$ *are equivalent.*

This is an absolute summability analogue of Theorem D. Theorems 1-3 are proved in §§3-5, respectively.

The author takes this opportunity of expressing her heartfelt thanks to Professor H. Hirokawa for his kind encouragement and valuable suggestions in the preparation of this paper.

2. We require the following lemmas.

LEMMA 1. *Let* $y_n = \sum_{k=0}^{n} c_{nk} x_k$. *If*
(i) $\sum_{j=0}^{n} |c_{nj}| \leqq K < \infty$ *for all* n, *and*
(ii) $\sum_{j=k}^{n} (c_{nj} - c_{n-1,j}) \geqq 0$ *for* $k = 0, 1, 2, \cdots, n$,
then $\sum_{n=0}^{\infty} |\Delta y_n| < \infty$ *whenever* $\sum_{n=0}^{\infty} |\Delta x_n| < \infty$.

This is easily proved by the method analogous to that of the proof of McFadden's theorem [4, Theorem (2.12)].

LEMMA 2. *For* $m, n = 0, 1, 2, \cdots$,

$$\sum_{k=0}^{m} Q_k \Delta_k\left(\frac{p_{n-k}}{q_k}\right) = P_n - P_{n-m-1} - \frac{Q_m}{q_{m+1}} p_{n-m-1} .$$

This is Lemma 2 in [1].

LEMMA 3. *If* $\{p_n\}$ *is the nondecreasing sequence such that* $\{p_{n+1}/p_n\}$ *is nonincreasing and* $p_{n-k}/P_n < p_{n-k-1}/P_{n-1}$, *then*

$$k\left(\frac{p_{n-k}}{P_n} - \frac{p_{n-k-1}}{P_{n-1}}\right) \geqq \sum_{m=0}^{k-1}\left(\frac{p_{n-m}}{P_n} - \frac{p_{n-m-1}}{P_{n-1}}\right) .$$

This is due to McFadden (see [4, p. 178]).

3. *Proof of Theorem* 1. Let us write

$$t_n = \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} s_k \quad \text{and} \quad u_n = \frac{1}{Q_n} \sum_{k=0}^{n} q_k s_k .$$

By Abel's transformation, we have

$$\begin{aligned} t_n &= \frac{1}{P_n} \sum_{k=0}^{n-1} Q_k u_k \Delta_k\left(\frac{p_{n-k}}{q_k}\right) + \frac{p_0 Q_n u_n}{P_n q_n} \\ &= \sum_{k=0}^{n} a_{nk} u_k , \end{aligned}$$

where

$$a_{nk} = \frac{Q_k}{P_n} \Delta_k\left(\frac{p_{n-k}}{q_k}\right) .$$

To prove theorem, we must verify that the conditions of Lemma 1 with $\{c_{nk}\}$ replaced by $\{a_{nk}\}$ are satisfied. Since $\{p_n\}$ and $\{q_n\}$ are positive and nondecreasing, $a_{nk} \geqq 0$. And if $s_k = 1$ for all k, then $t_n = 1$, $u_n = 1$ for all n.
Hence,

$$\sum_{j=0}^{n} |a_{nj}| = \sum_{j=0}^{n} a_{nj} = 1 .$$

Therefore, it is sufficient to show that

$$P \equiv \sum_{j=k}^{n} (a_{nj} - a_{n-1,j}) \geqq 0 \qquad \text{for } k = 0, 1, 2, \cdots, n .$$

By Lemma 2, we have

$$\begin{aligned} P &= \frac{1}{P_n}\Big(\sum_{j=0}^{n} - \sum_{j=0}^{k-1}\Big)Q_j\Delta_j\Big(\frac{p_{n-j}}{q_j}\Big) - \frac{1}{P_{n-1}}\Big(\sum_{j=0}^{n-1} - \sum_{j=0}^{k-1}\Big)Q_j\Delta_j\Big(\frac{p_{n-j-1}}{q_j}\Big) \\ &= \frac{1}{P_n}\Big(P_{n-k} + \frac{Q_{k-1}}{q_k}p_{n-k}\Big) - \frac{1}{P_{n-1}}\Big(P_{n-k-1} + \frac{Q_{k-1}}{q_k}p_{n-k-1}\Big) \\ &= \frac{P_{n-k}}{P_n} - \frac{P_{n-k-1}}{P_{n-1}} + \frac{Q_{k-1}}{q_k}\Big(\frac{p_{n-k}}{P_n} - \frac{p_{n-k-1}}{P_{n-1}}\Big) . \end{aligned}$$

Since $\{p_{n+1}/p_n\}$ is nonincreasing, it is easily deducible that

$$\frac{P_{n-k}}{P_n} - \frac{P_{n-k-1}}{P_{n-1}} \geqq 0 .$$

Thus, if $p_{n-k}/P_n - p_{n-k-1}/P_{n-1} \geqq 0$, P is nonnegative. Suppose on the other hand that

$$\frac{p_{n-k}}{P_n} - \frac{p_{n-k-1}}{P_{n-1}} < 0 .$$

Since $\{q_n\}$ is nondecreasing,

$$Q_{k-1} \leqq kq_{k-1} \leqq kq_k .$$

Hence, $Q_{k-1}/q_k \leqq k$.
Thus, we have, by Lemma 3,

$$\begin{aligned} P &\geqq \frac{P_{n-k}}{P_n} - \frac{P_{n-k-1}}{P_{n-1}} + k\Big(\frac{p_{n-k}}{P_n} - \frac{p_{n-k-1}}{P_{n-1}}\Big) \\ &\geqq \frac{P_{n-k}}{P_n} - \frac{P_{n-k-1}}{P_{n-1}} + \sum_{m=0}^{k-1}\Big(\frac{p_{n-m}}{P_n} - \frac{p_{n-m-1}}{P_{n-1}}\Big) \\ &= \frac{P_{n-k}}{P_n} - \frac{P_{n-k-1}}{P_{n-1}} + \frac{P_n - P_{n-k}}{P_n} - \frac{P_{n-1} - P_{n-k-1}}{P_{n-1}} \\ &= 0 . \end{aligned}$$

This completes the proof of Theorem 1.

4. *Proof of Theorem* 2. Under the conditions of $\{p_n\}$, using McFadden's theorem [4, Theorem (2.28)], we see that $|N, p_n| \subseteqq |C, 1|$. Hence we need only verify, under the conditions of theorem, that

$|C, 1| \subseteqq |\bar{N}, q_n|$.

Let us write

$$\sigma_n = \frac{1}{n+1} \sum_{k=0}^{n} s_k \quad \text{and} \quad u_n = \frac{1}{Q_n} \sum_{k=0}^{n} q_k s_k .$$

By Abel's transformation, we have

$$\begin{aligned} u_n &= \frac{1}{Q_n} \sum_{k=0}^{n-1} (k+1)\sigma_k \Delta q_k + \frac{(n+1)q_n \sigma_n}{Q_n} \\ &= \sum_{k=0}^{n} b_{nk} \sigma_k , \end{aligned}$$

where

$$\begin{aligned} b_{nk} &= \frac{(k+1)\Delta q_k}{Q_n} \qquad \text{for } 0 \leqq k < n , \\ &= \frac{(n+1)q_n}{Q_n} \qquad \text{for } k = n . \end{aligned}$$

For our purposes, it is sufficient to show that the conditions of Lemma 1 with $\{c_{nk}\}$ replaced by $\{b_{nk}\}$ are satisfied.

Since $\{q_n\}$ is positive and nonincreasing, $b_{nk} \geqq 0$. And if $s_k = 1$ for all k, then $\sigma_n = 1$, $u_n = 1$ for all n.
Hence,

$$\sum_{j=0}^{n} |b_{nj}| = \sum_{j=0}^{n} b_{nj} = 1 .$$

Therefore, it is sufficient to show that

$$Q \equiv \sum_{j=k}^{n} (b_{nj} - b_{n-1,j}) \geqq 0 \qquad \text{for } k = 0, 1, 2, \cdots, n .$$

For $0 \leqq k \leqq n-1$, we have

$$\begin{aligned} Q &= \frac{1}{Q_n} \sum_{j=k}^{n-1} (j+1)\Delta q_j + \frac{(n+1)q_n}{Q_n} \\ &\quad - \frac{1}{Q_{n-1}} \sum_{j=k}^{n-2} (j+1)\Delta q_j - \frac{n q_{n-1}}{Q_{n-1}} \\ &= \frac{1}{Q_n}\{Q_n - (Q_{k-1} - kq_k)\} - \frac{1}{Q_{n-1}}\{Q_{n-1} - (Q_{k-1} - kq_k)\} \\ &= (kq_k - Q_{k-1})\left(\frac{1}{Q_n} - \frac{1}{Q_{n-1}}\right) . \end{aligned}$$

Since $\{q_n\}$ is positive and nonincreasing,

$$Q_{k-1} \geqq kq_{k-1} \geqq kq_k \quad \text{and} \quad \frac{1}{Q_n} \leqq \frac{1}{Q_{n-1}} .$$

Therefore, we have $Q \geqq 0$. For $k = n$, since $b_{nn} \geqq 0$, we have $Q = b_{nn} \geqq 0$. Hence, we have $Q \geqq 0$ for $k = 0, 1, 2, \cdots, n$.

This completes the proof of Theorem 2.

5. *Proof of Theorem* 3. Consider Theorem A for $p_n = q_n$. Then, by our assumption,

$$0 \leqq \frac{P_n}{P_{n+1}p_{n+1}} \leqq \frac{1}{p_{n+1}} \leqq \frac{1}{K}.$$

Therefore, we have $P_n/p_{n+1} = O(P_{n+1})$.
Thus, using our assumption, we see that the conditions of Theorem A are satisfied for $p_n = q_n$.

Hence, by Theorem A, we have $|\bar{N}, p_n| \subseteqq |N, p_n|$.

References

1. G. D. Dikshit, *On inclusion ralations between Riesz and Nörlund means*, Indian J. Math., **7** (1965), 73-81.
2. K. Ishiguro, *The relation between* (N, p_n) *and* $(\bar{N}, p_n)$ *summability*, Proc. Japan Acad., **41** (1965), 120-122.
3. B. Kuttner and B. E. Rhoades, *Relations between* (N, p_n) *and* $(\bar{N}, p_n)$ *summability*, Proc. Edinburgh Math. Soc., (2) **16** (1968), 109-116.
4. L. McFadden, *Absolute Nörlund summability*, Duke Math. J., **9** (1942), 168-207.

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