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The group of all automorphisms of a chain Ω forms a lattice-ordered group $A(\Omega)$ under the pointwise order. It is well known that if G is the symmetric group on $\aleph$ elements ($\aleph \neq 6$), then every automorphism of G is inner. Here it is shown that if Ω is an α-set, every l-automorphism of $A(\Omega)$ (preserving also the lattice structure) is inner. This is accomplished by means of an investigation of the orbits $\bar{\omega}A(\Omega)$ of Dedekind cuts $\bar{\omega}$ of Ω.

The same conjecture for arbitrary chains Ω has been investigated in [6], [4], and [8]. Lloyd proved in [6] that when Ω is the chain of rational numbers (i.e., the 0-set), or is Dedekind complete, every l-automorphism of $A(\Omega)$ is inner. He also stated this conclusion for α-sets in general, but a lacuna in his proof has been pointed out by C. Holland.

2. o-2-transitive groups $A(\Omega)$. An automorphism of a chain Ω is simply a permutation g of Ω which preserves order in the sense that $\omega < \tau$ if and only if $\omega g < \tau g$. The group $A(\Omega)$ of all automorphisms of Ω forms a lattice-ordered group (l-group) when ordered pointwise, i.e., $f \leqq g$ if and only if $\omega f \leqq \omega g$ for all $\omega \in \Omega$. We identify each $g \in A(\Omega)$ with its unique extension to $\bar{\Omega}$, the conditional completion by Dedekind cuts of Ω, and thus consider $A(\Omega)$ as an l-subgroup of $A(\bar{\Omega})$, i.e., as a subgroup which is also a sublattice.

An l-subgroup G of $A(\Omega)$ is *o-2-transitive* if for all $\beta, \gamma, \sigma, \tau \in \Omega$ with $\beta < \gamma$ and $\sigma < \tau$, there exists $g \in G$ such that $\beta g = \sigma$ and $\gamma g = \tau$. Ω is *o-2-homogeneous* if $A(\Omega)$ is o-2-transitive. (To avoid pathology, we assume throughout that Ω contains more than two points.) Corollary 16 of [8] states, for the special case in which Ω is o-2-homogeneous, that every l-automorphism ψ of $A(\Omega)$ is induced by an inner automorphism π of the larger group $A(\bar{\Omega})$, say $\pi: h \rightarrow f^{-1}hf$, f a fixed element of $A(\bar{\Omega})$; and that Ωf is an orbit $\bar{\omega}A$ of $A(\Omega)$, for some $\bar{\omega} \in \bar{\Omega}$. Thus, as was essentially obtained by Lloyd in [6] by methods different from those in [8], we have

THEOREM 1 (Lloyd). *If Ω is o-2-homogeneous, then every l-automorphism of $A(\Omega)$ is inner, provided that no orbit $\bar{\omega}A(\Omega)$, $\bar{\omega} \in \bar{\Omega} \backslash \Omega$, is o-isomorphic to Ω.*

It may be that the proviso that no orbit $\bar{\omega}A(\Omega)$, $\bar{\omega} \in \bar{\Omega}\backslash\Omega$, be o-isomorphic to Ω is satisfied by every o-2-homogeneous Ω; this is an open question.[1] We shall find at any rate that the proviso holds when Ω is an α-set.

For any $\bar{\omega} \in \bar{\Omega}$, Ω o-2-homogeneous, the orbit $\bar{\omega}A(\Omega)$ is dense in $\bar{\Omega}$. For $g \in A(\Omega)$, form $\hat{g} \in A(\bar{\omega}A(\Omega))$ by first extending g to $\bar{\Omega}$ and then restricting to $\bar{\omega}A(\Omega)$. The map $g \rightarrow \hat{g}$ is an l-isomorphism of $A(\Omega)$ *into* $A(\bar{\omega}A(\Omega))$. We shall write $(A(\Omega), \bar{\omega}A(\Omega))$ when considering $A(\Omega)$ to act on $\bar{\omega}A(\Omega)$, and shall say that $(A(\Omega), \bar{\omega}A(\Omega))$ is *entire* if the l-isomorphism is *onto* $A(\bar{\omega}A(\Omega))$.

PROPOSITION 2. *Suppose that $A(\Omega)$ is o-2-transitive on Ω, and let $\bar{\omega} \in \bar{\Omega}\backslash\Omega$. Then $A(\Omega)$ is also o-2-transitive on $\bar{\omega}A(\Omega)$.*

Proof. Let $\bar{\beta}, \bar{\gamma}, \bar{\sigma}, \bar{\tau} \in \bar{\omega}A(\Omega)$, with $\bar{\beta} < \bar{\gamma}$ and $\bar{\sigma} < \bar{\tau}$. Since $A(\Omega)$ is o-2-transitive on Ω, we can pick $f \in A(\Omega)$ such that $\bar{\beta}f \leqq \bar{\sigma}$ and $\bar{\gamma}f \geqq \bar{\tau}$. Since $\bar{\sigma}, \bar{\beta}$, and $\bar{\beta}f$ all lie in the same orbit of $(A(\Omega), \bar{\Omega})$, we can pick $1 \leqq g \in A(\Omega)$ such that $\bar{\beta}fg = \bar{\sigma}$; then $\bar{\gamma}fg \geqq \bar{\gamma}f \geqq \bar{\tau}$. Letting $r = fg \in A(\Omega)$, we have $\bar{\beta}r = \bar{\sigma}$ and $\bar{\gamma}r \geqq \bar{\tau}$. Similarly, there exists $s \in A(\Omega)$ such that $\bar{\gamma}s = \bar{\tau}$ and $\bar{\beta}s \geqq \bar{\sigma}$. Letting $t = r \wedge s$, we have $\bar{\beta}t = \bar{\sigma}$ and $\bar{\gamma}t = \bar{\tau}$. Hence $A(\Omega)$ is o-2-transitive on $\bar{\omega}A(\Omega)$.

3. Characters of points and holes of Ω. By a *hole* in Ω we shall mean an $\bar{\omega} \in \bar{\Omega}\backslash\Omega$. We now give some terminology from [2, pp. 142–4], assuming for convenience that Ω is o-2-homogeneous (and thus dense in itself). An ordinal number ω_β is *regular* if it is an initial ordinal and all of its cofinal subsets have cardinality $\aleph_\beta$. We say that the point or hole $\bar{\omega}$ has *character* $c_{\beta\gamma}$ if ω_β is the unique regular ordinal which is o-isomorphic to a cofinal subset of $\{\sigma \in \Omega \mid \sigma < \bar{\omega}\}$ (or equivalently, if $\aleph_\beta$ is the smallest cardinality of any cofinal subset of $\{\sigma \in \Omega \mid \sigma < \bar{\omega}\}$), and dually for ω_γ. Since orbits $\bar{\tau}A(\Omega)$ are dense in $\bar{\Omega}$, we can when convenient consider instead cofinal subsets of $\{\sigma \in \bar{\tau}A(\Omega) \mid \sigma < \bar{\omega}\}$. Of course, all elements of the orbit $\bar{\omega}A(\Omega)$ have the same character as $\bar{\omega}$; and one such orbit is Ω, so that all points have the same character.

PROPOSITION 3. *Let Ω be o-2-homogeneous. Suppose there exists a hole $\bar{\omega}$ having the same character as the points in Ω, and suppose that the orbit $\bar{\omega}A(\Omega)$ contains all holes of this character. Then $(A(\Omega), \bar{\omega}A(\Omega))$ is entire.*

Proof. If $\bar{\tau} \in \bar{\Omega}$, $h \in A(\bar{\omega}A(\Omega))$, then $\bar{\tau}$ and $\bar{\tau}h$ must have the same character. Since Ω consists of *all* $\bar{\tau} \in \bar{\Omega}\backslash\bar{\omega}A(\Omega)$ whose character is that of the points of Ω, we must have $\Omega h = \Omega$. The proposition follows.

The reader can prove the following rather easy proposition himself,

[1] C. Holland has recently discovered an o-2-homogeneous chain Ω for which the proviso fails.

or he can refer to the proof of Theorem 5.

PROPOSITION 4. *Let Ω be o-2-homogeneous. If there exists a hole $\bar{\omega}$ of character c_{00}, then $\bar{\omega}A(\Omega)$ is the set of all holes of character c_{00}. Hence if the points of Ω have character c_{00}, $(A(\Omega), \bar{\omega}A(\Omega))$ is entire.*

4. α-sets. If Γ and Δ are subsets of a chain Ω, we write $\Gamma < \Delta$ if and only if $\gamma < \delta$ for all $\gamma \in \Gamma$, $\delta \in \Delta$. Let α be an ordinal number. An *α-set* is a chain Ω of cardinality $\aleph_\alpha$ in which for any two (possibly empty) subsets $\Gamma < \Delta$ of cardinality less than $\aleph_\alpha$, there exists $\omega \in \Omega$ such that $\Gamma < \omega < \Delta$. If ω_α is a regular ordinal, then (assuming the generalized continuum hypothesis) there exists an α-set, and it is unique up to o-isomorphism [2, pp. 179–181]. It is easy to deduce from the definition of an α-set (or see [2, p. 179], which is not so easy) that if Ω is an α-set, its points have character $c_{\alpha\alpha}$ (so that Ω is o-2-homogeneous); that each hole has character $c_{\alpha\beta}$ or $c_{\beta\alpha}$ for some $\beta \leqq \alpha$ with ω_β regular; and that each of these characters actually is the character of some hole. (Holes of a given nonsymmetric character can be obtained as limits of monotone transfinite sequences of points of Ω. For $c_{\alpha\alpha}$ holes, see Proposition 6.)

THEOREM 5. *Let Ω be an α-set. Then every orbit of $(A(\Omega), \bar{\Omega})$ consists of the set of all holes of a given character (except for Ω, which consists of points).*

Proof. We must show that any two holes of the same character lie in the same orbit of $A(\Omega)$. By duality, it suffices to show that for any two $c_{\beta\alpha}$ holes $\bar{\tau}_1$ and $\bar{\tau}_2$ ($\beta \leqq \alpha$), the two sets $\Gamma_i = \{\sigma \in \Omega \mid \sigma < \bar{\tau}_i\}$, $i = 1, 2$, are o-isomorphic. Pick in Γ_i a strictly increasing cofinal sequence $\{\beta_i^n \mid n \in \omega_\beta\}$ indexed by ω_β. For each limit ordinal $\pi < \omega_\beta$, let $\bar{\gamma}_i^\pi = \sup\{\beta_i^n \mid n < \pi\} \in \bar{\Omega}$. Since ω_β is an initial number, any such $\bar{\gamma}_i^\pi$ has "left" character less than β, and hence is a hole with "right" character equal to α. Hence each $\Delta_i^\pi = \{\sigma \in \Omega \mid \bar{\gamma}_i^\pi < \sigma < \beta_i^\pi\}$ is an α-set. Also, each $\Delta_i^0 = \{\sigma \in \Omega \mid \sigma < \beta_i^0\}$ is an α-set, and for each ordinal $\lambda < \omega_\beta$, each $\Delta_i^{\lambda+1} = \{\sigma \in \Omega \mid \beta_i^\lambda \leqq \sigma < \beta_i^{\lambda+1}\}$ is an α-set. Hence for each $\mu < \omega_\beta$, Δ_1^μ is o-isomorphic to Δ_2^μ. It is now easy to show that Γ_1 and Γ_2 are o-isomorphic.

The following result, which was pointed out to the author by Andrew Glass, can also be established by splicing together suitable α-sets.

PROPOSITION 6. *Let Ω be an α-set, let Γ and Δ be subsets of cardinality less than $\aleph_\alpha$, and let φ be an o-isomorphism from Γ onto Δ. Then φ can be extended to an automorphism Ω.*

PROPOSITION 7. *Let Ω be an α-set. Then each orbit $\bar{\omega}A(\Omega)$ has cardinality $\aleph_\alpha$ except for the orbit of $c_{\alpha\alpha}$ holes, which has cardinality $2^{\aleph_\alpha}$.*

Proof. By definition, card $(\Omega) = \aleph_\alpha$. By [1, Theorem 13. 23], card $(\bar{\Omega}) = 2^{\aleph_\alpha}$. The number of distinct hole characters is no greater than $\aleph_\alpha$. For any character $c_{\beta\alpha}$ or $c_{\alpha\beta}$ with $\beta < \alpha$, the number of holes of that character is of cardinality $\aleph \leqq \aleph_\alpha$; and since the orbit of such holes is dense in $\bar{\Omega}$, $2^{\aleph} \geqq \text{card}\,(\bar{\Omega}) = 2^{\aleph_\alpha}$, so that $\aleph = \aleph_\alpha$. Hence $\{\bar{\omega} \in \bar{\Omega} \mid \bar{\omega}$ is not a $c_{\alpha\alpha}$ hole$\}$ has cardinality $\aleph_\alpha$. Since card $(\bar{\Omega}) = 2^{\aleph_\alpha}$, the number of $c_{\alpha\alpha}$ holes is also $2^{\aleph_\alpha}$.

COROLLARY 8. *No two orbits of $(A(\Omega), \bar{\Omega})$, Ω an α-set, are o-isomorphic.*

Proof. As mentioned after the definition of character, the character of $\bar{\omega}$ can be determined via the set $\bar{\omega}A(\Omega)$. Hence if $\bar{\omega}$ has character $c_{\alpha\beta}$ as determined by Ω, the points of the chain $\bar{\omega}A(\Omega)$ have character $c_{\alpha\beta}$ as determined by the chain $\bar{\omega}A(\Omega)$. Hence no two orbits associated with distinct characters can be o-isomorphic. Finally, Ω and the orbit of $c_{\alpha\alpha}$ holes cannot be o-isomorphic because they are of different cardinalities.

In view of Theorem 1, we have

MAIN COROLLARY 9. *Every l-automorphism of the l-group $A(\Omega)$, Ω an α-set, is inner.*

Since every chain can be o-embedded in a sufficiently large α-set [2, p. 181], we have

COROLLARY 10. *Every chain can be embedded in a chain Ω such that every l-automorphism of $A(\Omega)$ is inner.*

Since every l-group can be embedded in some $A(\Omega)$, Ω an α-set [3, Theorem 4], we also have

COROLLARY 11. *Every l-group can be embedded in an l-group all of whose l-automorphisms are inner.*

5. **Representations.** By a *representation* of an l-group G we mean l-isomorphism of G into some $A(\Sigma)$. In §2, o-2-transitive $A(\Omega)$'s were canonically represented as l-subgroups of $A(\bar{\omega}A(\Omega))$, $\bar{\omega} \in \bar{\Omega}$, and we identified $A(\Omega)$ with its image. Here we shall find that these constitute all the "nice" representations of $A(\Omega)$.

If G_i is an l-subgroup of $A(\Omega_i)$, $i = 1, 2$, an *o-isomorphism* from (G_1, Ω_1) onto (G_2, Ω_2) consists of an o-isomorphism ψ from Ω_1 onto Ω_2 and an l-isomorphism θ from G_1 onto G_2 such that $(\omega g)\psi = (\omega\psi)(g\theta)$ for all $\omega \in \Omega_1$, $g \in G_1$. In [4], Holland defined a transitive l-subgroup G of $A(\Omega)$ to be *weakly o-primitive* if G is faithful on $\bar{\omega}G$, $\bar{\omega} \in \bar{\Omega}$, *only* when $\bar{\omega}G$ is dense in $\bar{\Omega}$. (For other formulations of the condition, see [4].) As a special case of [4, Theorem 7], we have

THEOREM 12 (Holland). *Suppose that $A(\Omega)$ is o-2-transitive and let θ be a representation of $A(\Omega)$ as a weakly o-primitive l-subgroup of some $A(\Sigma)$. Then there is an o-isomorphism ψ from some $\bar{\omega}A(\Omega)$, $\bar{\omega} \in \bar{\Omega}$, onto Σ which, together with θ, furnishes an o-isomorphism from $(A(\Omega), \bar{\omega}A(\Omega))$ onto $((A(\Omega))\theta, \Sigma)$. In particular, the collection of $(A(\Omega), \bar{\omega}A(\Omega))$'s, $\bar{\omega} \in \bar{\Omega}$, constitute (up to o-isomorphism) all weakly o-primitive representations of $A(\Omega)$.*

A representation θ of an l-group G is *complete* if it preserves arbitrary suprema and infima that exist in G, or equivalently, if $G\theta$ is a *complete* l-subgroup of $A(\Sigma)$ in the sense that arbitrary suprema (infima) that exist in $G\theta$ are also suprema (infima) in $A(\Sigma)$.

THEOREM 13. *Theorem 12 remains valid if one considers complete transitive representations instead of weakly o-primitive representations.*

Proof. First we show that each $(A(\Omega), \bar{\omega}A(\Omega))$, $\bar{\omega} \in \bar{\Omega}$, is indeed complete. For [8, Theorem 1] states that the stabilizer subgroup $A(\Omega)_{\bar{\omega}} = \{g \in A(\Omega) \mid \bar{\omega}g = \bar{\omega}\}$ is closed under arbitrary suprema and infima that exist in $A(\Omega)$, so that for $(A(\Omega), \bar{\omega}A(\Omega))$ the stabilizer subgroups of *points* are closed; and [7, Theorem 7] states that for transitive l-subgroups, this latter condition is equivalent to completeness.

Now let θ be any complete transitive representation of $A(\Omega)$ in some $A(\Sigma)$. Pick any $\sigma \in \Sigma$. Since $(A(\Omega))\theta$ is a complete subgroup of $A(\Sigma)$, the stabilizer subgroup $A(\Sigma)_\sigma$ is a closed prime subgroup of $A(\Sigma)$ (by [8, Theorem 1] again); while by [8, Theorem 11], *every* proper closed prime subgroup of $A(\Omega)$ is $A(\Omega)_{\bar{\omega}}$ for some $\bar{\omega} \in \bar{\Omega}$. Hence for some $\bar{\omega} \in \bar{\Omega}$, $(A(\Omega)_{\bar{\omega}})\theta = A(\Sigma)_\sigma$. Thus (see, for example, the proof of Lemma 14 of [4]) there exists an o-isomorphism ψ from $\bar{\omega}A(\Omega)$ onto Σ which, together with θ, furnishes an o-isomorphism from $(A(\Omega), \bar{\omega}A(\Omega))$ onto $((A(\Omega))\theta, \Sigma)$.

Unfortunately, there are generally other (neither weakly o-primitive nor complete) transitive representations of $A(\Omega)$, as is seen by the argument given in [4, p. 433] for Ω the reals.

In general there seems to be no guarantee that $(A(\Omega), \bar{\omega}A(\Omega))$'s will be nonisomorphic for distinct orbits of $A(\Omega)$, but by Corollary

8 we have

THEOREM 14. *Let Ω be an α-set. Then the $(A(\Omega), \bar{\omega}A(\Omega))$'s are nonisomorphic for distinct orbits of $A(\Omega)$, and they constitute (up to o-isomorphism) all weakly o-primitive (alternately, all complete transitive) representations of $A(\Omega)$.*

THEOREM 15. *Let Ω be an α-set, and let Γ be the orbit of holes of character $c_{\alpha\alpha}$. If $\Delta = \Gamma$, or if $\Delta = \Omega$, then $(A(\Omega), \Delta)$ is entire, and Δ possesses an anti-automorphism. If $\Delta = \bar{\omega}A(\Omega)$, where $\bar{\omega}$ is a hole of character $c_{\beta\alpha}$ or $c_{\alpha\beta}$ ($\beta < \alpha$, ω_β regular), then $(A(\Omega), \Delta)$ is not entire, and the points of Δ are nonsymmetric, so that not even the intervals of the o-2-homogeneous chain Δ possess anti-automorphisms.*

Proof. Proposition 3 and Theorem 5 establish that $(A(\Omega), \Gamma)$ is entire. Now let $\Delta = \bar{\omega}A(\Omega)$, $\bar{\omega}$ nonsymmetric. Pick any $\beta \in \Omega$ and any $c_{\alpha\alpha}$ hole $\bar{\gamma}$. Then $L(\beta) = \{\delta \in \Omega \mid \delta < \beta\}$ and $U(\beta) = \{\delta \in \Omega \mid \delta > \beta\}$ are α-sets, and similarly for $\bar{\gamma}$. By the uniqueness of α-sets, there exist o-isomorphisms f of $L(\beta)$ onto $L(\bar{\gamma})$ and g of $U(\beta)$ onto $U(\bar{\gamma})$. Define a map h by setting $\lambda h = \lambda f$ if $\lambda \in L(\beta)$, and $\lambda h = \lambda g$ if $\lambda \in U(\beta)$. Since Δ is the set of all holes of a given character, $h \in A(\Delta)$, and by construction $\beta h = \bar{\gamma}$. Hence Ω and Γ lie in the same orbit of $A(\Delta)$, so that $(A(\Omega), \Delta) \neq (A(\Delta), \Delta)$.

Reversing the ordering of an α-set yields an α-set, so by the uniqueness of α-sets, Ω has an anti-automorphism, and it induces an anti-automorphism of Γ. Nonsymmetric holes have been discussed above.

COROLLARY 16. *$\Pi = \Omega \cup \Gamma$ is o-2-homogeneous. The orbits of $A(\Pi)$ are, besides Π itself, precisely the orbits $\bar{\omega}A(\Omega)$ of $A(\Omega)$ for nonsymmetric $\bar{\omega}$. For each orbit Δ, $(A(\Pi), \Delta)$ is entire; and $(A(\Pi), \bar{\Omega})$ is entire. The representations $(A(\Pi), \Delta)$ constitute all the weakly o-primitive (alternately, complete transitive) representations of $A(\Pi)$. All l-automorphisms of $A(\Pi)$ are inner.*

Proof. If $\alpha = 0$, so that Π is the reals, the conclusion (well known except for part about complete transitive representations) follows from Theorems 12, 13, and 1. Now suppose that $\alpha > 0$ and let $\Delta = \bar{\omega}A(\Omega)$, $\bar{\omega}$ nonsymmetric. By the proof of the theorem, all of Π lies in the same orbit Λ of $A(\Delta)$. Since Π consists of all holes in Δ of character $c_{\alpha\alpha}$, $\Pi = \Lambda$. By Proposition 2, Π is o-2-homogeneous. Since $A(\Omega) \subset A(\Pi)$ and Δ consists of all elements of $\bar{\Omega}$ of a given character, Δ is also an orbit of $A(\Pi)$. Since we have already established that Π is an orbit of $A(\Delta)$, $(A(\Pi), \Delta)$ is entire. Also, Π is the set of all points of $\bar{\Omega}$ of character $c_{\alpha\alpha}$, so $(A(\Pi), \bar{\Omega})$ is entire. (This extension of terminology to the nonhomogeneous chain $\bar{\Omega}$ causes no difficulties.) For the rest, apply Theorems 12, 13, and 1.

COROLLARY 17. *If Ω is an α-set, then $A(\Omega)$ is self-normalizing in $A(\bar{\Omega})$.*

Proof. If $g(A(\Omega))g^{-1} = A(\Omega)$ for $g \in A(\bar{\Omega})$, then Ωg must be a union of orbits of $A(\Omega)$. This implies that $\Omega g = \Omega$ (by the proof of Corollary 8), so that $g \in A(\Omega)$.

We say that a chain Ω (without a greatest element) has *initial character* c_β if $\aleph_\beta$ is the smallest cardinality of any coinitial subset of Ω; and dually for *final character*. In the definition of an α-set, permitting Γ or Δ to be empty forces both of these characters to be c_α.

PROPOSITION 18. *Let $\aleph_\alpha$, $\aleph_\beta$, and $\aleph_\gamma$ be regular cardinals, with $\beta, \gamma \leqq \alpha + 1$. Then there exists a chain Ω, unique up to o-isomorphism, such that for any two nonempty subsets $\Gamma < \Delta$ of cardinality less than $\aleph_\alpha$, there exists $\omega \in \Omega$ such that $\Gamma < \omega < \Delta$, and having initial character c_β and final character c_γ. (If β or $\gamma = \alpha + 1$, cardinality $\aleph_\alpha$ is required only for intervals of Ω, not for Ω itself.) Ω satisfies all of the results proved in this paper for α-sets, except for the anti-automorphisms of Theorem 15.*

Proof. Let Σ be an α-set. To obtain final character c_β, $\beta < \alpha$, let $\bar{\sigma}$ be a hole of character $c_{\beta\alpha}$ and delete $\{\sigma \in \Sigma \mid \sigma > \bar{\sigma}\}$. To obtain final character $c_{\alpha+1}$, use $\overleftarrow{\Sigma \times \omega_{\alpha+1}}$, ordered lexicographically from the right. Similar considerations regarding the initial character establish the existence of Ω. Uniqueness is proved in the manner of the proof of Theorem 5. The proofs of the results about α-sets require no change.

Let $L(\Omega) = \{g \in A(\Omega) \mid$ there exists $\sigma \in \Omega$ such that $\omega g = \omega$ for all $\omega \leqq \sigma\}$, an l-ideal of $A(\Omega)$; let $U(\Omega)$ be the dual; and let $B(\Omega) = L(\Omega) \cap U(\Omega)$. If Ω is o-2-homogeneous, these three l-ideals are proper and distinct, and even $B(\Omega)$ is o-2-transitive and has the same orbits as $A(\Omega)$. If we pick any one of these three types of l-ideals and substitute it for $A(\Omega)$ throughout the paper, all results remain true except that in Theorem 1 and Corollary 9 the l-automorphism of the ideal need not be inner, but merely induced by an inner automorphism of $A(\Omega)$. The proofs require only minor changes.

Finally, if Ω is an α-set, it is not the case that all *group* automorphisms of $A(\Omega)$ are inner. For let f be an anti-automorphism of the chain Ω. Then $g \rightarrow f^{-1}gf$ is a group automorphism of $A(\Omega)$, and since it interchanges $L(\Omega)$ and $U(\Omega)$, it is not inner. Its restriction to $B(\Omega)$ is a group automorphism of $B(\Omega)$ which can easily be shown not to be inner. Are there group automorphisms of $L(\Omega)$ and $U(\Omega)$ which are not inner?

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