Pacific Journal of Mathematics

$o-2$-TRANSITIVE ORDERED PERMUTATION GROUPS

STEPHEN H. MCCLEARY

Vol. 49, No. 2 June 1973

PACIFIC JOURNAL OF MATHEMATICS
Vol. 49, No. 2, 1973

O-2-TRANSITIVE ORDERED PERMUTATION GROUPS

STEPHEN H. MCCLEARY

The group of all automorphisms of a chain Ω forms a lattice-ordered group $A(\Omega)$ under the pointwise order. Let G be an l-subgroup of $A(\Omega)$ which is o-2-transitive, i.e., for any $\beta < \gamma$ and $\sigma < \tau$, there exists $g \in G$ such that $\beta g = \sigma$ and $\gamma g = \tau$. It is shown that G is a complete subgroup of $A(\Omega)$ if and only if G is completely distributive if and only if G contains an element $\neq 1$ of bounded support. There is a discussion of the pathological groups in which these conditions are absent.

1. The dichotomy among o-2-transitive groups. The group $A(\Omega)$ or order-preserving permutations (automorphisms) of a chain Ω becomes a lattice-ordered group (l-group) when ordered pointwise, i.e., $f \leqq g$ if and only if and only if $\beta f \leqq \beta g$ for all $\beta \in \Omega$. We assume throughout this paper that (G, Ω) is an l-permutation group, i.e., that G is an l-subgroup of $A(\Omega)$ (simultaneously a subgroup and a sublattice).

Let $\bar{\Omega}$ be the completion by Dedekind cuts (without end points) of Ω. Each $g \in G$ can be extended uniquely to an order-preserving permutation (o-permutation) of $\bar{\Omega}$, which will also be denoted by g. For $\bar{\omega} \in \bar{\Omega}$, let $G_{\bar{\omega}}$ be the *stabilizer subgroup* $\{g \in G \mid \bar{\omega} g = \bar{\omega}\}$. $G_{\bar{\omega}}$ is a *prime* subgroup of G (i.e., a convex l-subgroup of G such that $g_1 \wedge g_2 = 1$, with $g_1, g_2 \in G$, implies $g_1 \in G_{\bar{\omega}}$ or $g_2 \in G_{\bar{\omega}}$). If G is transitive on Ω, then of course all G_α's ($\alpha \in \Omega$) are conjugate in G.

The author showed in [5, Theorem 7] that for a transitive l-subgroup G of $A(\Omega)$, the following are equivalent:

(1) G_α is a *closed* subgroup of G for one (hence every) $\alpha \in \Omega$, i.e., if $g = \bigvee_{i \in I} g_i$ with each $g_i \in G_\alpha$, then $g \in G_\alpha$.

(2) G is a *complete* subgroup of $A(\Omega)$, i.e., if in G, $g = \bigvee_{i \in I} g_i$, then also in $A(\Omega)$, $g = \bigvee_{i \in I} g_i$.

(3) Sups *in G are pointwise*, i.e., if $g = \bigvee_{i \in I} g_i$ with each $g_i \in G$, then for each $\beta \in \Omega$, βg is the sup in Ω of $\{\beta g_i \mid i \in I\}$.

Moreover, it was shown in [5, Corollary 15] that in the presence of these conditions, we have

(4) G is a *completely distributive l-group*, i.e., $\bigwedge_{i \in I} \bigvee_{k \in K} g_{ik} = \bigvee_{f \in K^I} \bigwedge_{i \in I} g_{if(i)}$ for any collection $\{g_{ik} \mid i \in I, k \in K\}$ of G for which the indicated sups and infs exist.

The *distributive radical* $D(G)$ is the intersection of the closed prime subgroups of G [1, Theorem 3.4]. $D(G) = \{1\}$ iff G is completely distributive [1, Corollary 3.8]; and at the opposite extreme, $D(G) = G$ iff G has no closed prime subgroups $\neq G$.

The *support* Supp (k) of $k \in G$ means $\{\omega \in \Omega \mid \omega k \neq \omega\}$; it is *bounded* if there exist $\beta, \gamma \in \Omega$ such that $\beta < \text{Supp}(k) < \gamma$ (i.e., $\beta < \sigma < \gamma$ for all $\sigma \in \text{Supp}(k)$). It is well known (see, for example, the proof of [2, Theorem 6]) that the elements of bounded support in an o-2-transitive l-permutation group G form an l-ideal L of G which is contained in all l-ideals $\neq \{1\}$. If G contains no element $\neq 1$ of bounded support, so that $L = \{1\}$, then with an eye on the next theorem, we shall say that G is a *pathologically* o-2-transitive group.

MAIN THEOREM 1. *Suppose that* (G, Ω) *is an* o-*2-transitive* l-*permutation group. Then conditions* (1), (2), (3), *and* (4) *are all equivalent, and they fail if and only if* G *is pathological. Moreover, in the pathological case,* G *has no proper closed prime subgroups, so that the distributive radical* $D(G) = G$.

Proof. First, assume that G has an element $\neq 1$ of bounded support. Then since G is o-2-transitive, given any nondegenerate interval Δ of Ω, G has an element $\neq 1$ with support a subset of Δ. Now suppose $g = \bigvee_{i \in I} g_i$, with $1 < g \in G \backslash G_\alpha$ and each $g_i \in G_\alpha$. Pick $1 > h \in G$ such that $\text{Supp}(h) \subseteqq (\alpha g^{-1}, \alpha)$, where the usual notation is used for intervals of Ω. Then for each $i \in I$, $g_i \leqq hg < g$ (since when $\eta \in \text{Supp}(h)$, $\eta g_i < \alpha < \alpha hg$), a contradiction. Therefore, G_α is closed, and the other conditions follow.

Now assume that G lacks elements $\neq 1$ of bounded support. We can express an arbitrary $1 < g \in G$ as $\bigvee_{i \in I} g_i$ with each $g_i \in G_\alpha$, as follows: For each $\beta \notin [\alpha g^{-1}, \alpha]$, we have either $\alpha < \beta \leqq \beta g$, or else $\beta \leqq \beta g < \alpha$, so we may use o-2-transitivity to pick $g_\beta \in G_\alpha$ such that $\beta g_\beta = \beta g$. Now $g = \bigvee (g_\beta \wedge g)$, for if $g_\beta \wedge g \leqq h < g$ for each β, then $\text{Supp}(h^{-1}g) \subseteqq [\alpha g^{-1}, \alpha]$, violating the hypothesis, since for $\beta \notin [\alpha g^{-1}, \alpha]$, we have $\beta g = \beta(g_\beta \wedge g) \leqq \beta h \leqq \beta g$. Since each $g_\beta \wedge g \in G_\alpha$, G_α is not closed in G.

It remains only to show that in the pathological case, G contains no proper closed prime subgroup, for then $D(G) = G$ and G is not completely distributive. Suppose P is such a subgroup. In [6, Corollary 4], it is shown that every closed convex l-subgroup of an l-permutation group (G, Ω) must be $\bigcap \{G_{\bar{\omega}} \mid \bar{\omega} \in \bar{\Delta}\}$ for some $\bar{\Delta} \subseteqq \bar{\Omega}$. (In [6], it is assumed that G is a complete subgroup of $A(\Omega)$, so that the $G_{\bar{\omega}}$'s will be closed, but no other use is made of completeness.) But in fact the $G_{\bar{\omega}}$'s are closed, for P is closed, and in any l-group, a prime subgroup containing a closed prime is itself closed [1, Lemma 3.3]. But it was shown above that no G_α, $\alpha \in \Omega$, is closed; and in view of the following lemma, the proof also applies to $G_{\bar{\omega}}$, $\bar{\omega} \in \Omega$.

LEMMA 2. *Let* (G, Ω) *be an* o-*2-transitive* l-*permutation group.*

Let $\bar{\omega} \in \bar{\Omega}$, *and let* $\beta, \gamma \in \Omega$ *with either* $\bar{\omega} < \beta < \gamma$, *or* $\beta < \gamma < \bar{\omega}$. *Then there exists* $g \in G_{\bar{\omega}}$ *such that* $\beta g = \gamma$.

Proof. Suppose that $\bar{\omega} < \beta < \gamma$, and pick $\alpha \in \Omega$ such that $\bar{\omega} < \alpha < \beta < \gamma$. Use o-2-transitivity to pick $k \in G$ such that $\alpha k < \bar{\omega}$ and $\beta k = \gamma$, and take g to be $k \vee 1$. The other case is similar. This concludes the proofs of the lemma and the theorem.

Incidentally, conditions (1), (2), (3), and (4) still make sense when G is any subgroup (not necessarily an l-subgroup) of $A(\Omega)$; and if G is o-2-transitive and contains an element $\neq 1$ of bounded support, these conditions hold. (The first paragraph of the proof of Theorem 1 can easily be adapted to show that sups are pointwise. From this (2) and (1) follow as in [5], and (2) implies (4).)

If $\omega g \neq \omega$, $\{\gamma \in \Omega \mid \omega g^{-n} < \gamma < \omega g^{n}$ for some integer $n\}$ is called an *interval of support* of g; and in [5], G is said to be *depressible* if for every $g \in G$ and every interval of support Δ of g, there exists $k \in G$ such that $\omega k = \omega g$ if $\omega \in \Delta$, but $\omega k = \omega$ if $\omega \notin \Delta$. Convex l-subgroups of $A(\Omega)$ are automatically depressible.

PROPOSITION 3. *Depressible o-2-transitive l-permutation groups are never pathological.*

Proof. The following lemma establishes the existence of an element having a bounded interval of support, and depressibility does the rest.

LEMMA 4. *Let* (G, Ω) *be an o-2-transitive l-permutation group. Then, for every positive integer* n, (G, Ω) *is o-n-transitive,* i.e., *if* $\beta_1 < \cdots < \beta_n$ *and* $\gamma_1 < \cdots < \gamma_n$, *there exists* $g \in G$ *such that* $\beta_i g = \gamma_i$, $i = 1, \cdots, n$.

Proof. Given $\beta_1 < \cdots < \beta_n$ and $\gamma_1 < \cdots < \gamma_n$, we may suppose by induction that there exists $h \in G$ such that $\beta_i h = \gamma_i$, $i = 1, \cdots, n-1$. If $\beta_n h \geqq \gamma_n$, we use o-2-transitivity to pick $k \in G$ such that $\beta_1 k = \gamma_{n-1}$ and $\beta_n h = \gamma_n$. Now $\beta_i(h \wedge k) = \gamma_i$, $i = 1, \cdots, n$. If $\beta_n h < \gamma_n$, a similar argument works.

2. Pathologically o-2-transitive groups. The following example of a pathological group was given by Holland in [3, p. 433]. Let Ω be the reals and let G be the l-subgroup of $A(\Omega)$ consisting of those o-permutations g of Ω for which there exists a positive integer $n = n_g$ such that $(\omega + n)g = \omega g + n$ for all $\omega \in \Omega$. Lloyd [4, p. 399] used very special properties of this example to show that G is not completely distributive (cf. Theorem 1), but is l-simple (has no proper l-ideals). Are all pathological groups l-simple? The author has been unable to settle this question, but attempts to construct additional examples of pathological groups seem to lead inevitably to some sort of periodicity sufficient to guarantee l-simplicity, as in the following modification of Holland's example.[1]

As in that example, let Ω be the reals. Now let H be the l-subgroup of $A(\Omega)$ consisting of those o-permutations h of Ω having the property that for all $\omega \in \Omega$, there exists a positive integer $n = n_{h,\omega}$ such that $(\omega + qn)h = \omega h + qn$ for all integers q. (For definiteness, let $n_{h,\omega}$ be the least positive integer having this property.) H contains the previous group G and is also pathologically o-2-transitive. H and G are not isomorphic as l-groups, for there exists $1 < z \in G$ (defined by $\omega z = \omega + 1$) such that every $g \in G$ commutes with some z^p, $p > 0$, whereas it can be shown that H contains no such element z.

Of course, still other examples can be obtained by letting Ω be the rationals (or other appropriate subgroup of the reals) and proceeding in either of the two ways already mentioned.

Lloyd's proof of the l-simplicity of the first example does not apply to the modification. However, all of the above examples can be shown to be l-simple by the following fairly similar argument, which will be phrased in terms of the modified group H. Let $\{1\} \neq L$ be an l-ideal of H. Since for every $1 < h \in H$, $\{\omega h - \omega \mid \omega \in \Omega\}$ has an upper bound, namely $0h - 0 + n_{h,0}$, we shall have $L = H$ provided there exists $\varepsilon > 0$ such that the translation $\omega \rightarrow \omega + \varepsilon$ is exceeded by some $g \in L$. Now pick $1 < k \in L$ and $\alpha \in \Omega$ such that $\alpha k > \alpha$. Let t be the translation $\omega \rightarrow \omega + (\alpha k - \alpha)$, and let p be an integer such that $p(\alpha k - \alpha) > n_{k,\alpha}$. Let $k_1 = k$, let $k_i = t^{-1}k_{i-1}t$ $(i = 2, \cdots, p)$, and let $g = k_1 \cdots k_p \in L$. The reader can verify that $(\alpha + qn_{k,\alpha})g = \alpha g + qn_{k,\alpha}$ for all integers q, so that $n_{g,\alpha} \leqq n_{k,\alpha}$. Now $\alpha + n_{g,\alpha} \leqq \alpha + n_{k,\alpha} < \alpha g$. Hence g exceeds the translation $\omega \rightarrow \omega + \varepsilon$, where $\varepsilon = \alpha g - (\alpha + n_{g,\alpha})$.

As mentioned above, it may be the case that all pathologically o-2-transitive groups are l-simple. At any rate, any proper l-ideal must itself be a pathologically o-2-transitive group, as we proceed to prove.

An *o-block* of a transitive l-permutation group (G, Ω) is a nonempty convex subset Δ of Ω having the property that for any $g \in G$, either $\Delta g = \Delta$ or Δg does not meet Δ. The *o-block system* $\tilde{\Delta}$ determined by Δ is the set of translates Δg $(g \in G)$, a partition of Ω. If Δ contains more than one point and $\Delta \neq \Omega$, $\tilde{\Delta}$ is *proper*. G is *o-primitive* if it has no proper o-block systems.

LEMMA 5. *Let (G, Ω) be a transitive l-permutation group, and let L be a proper l-ideal of G which is intransitive on Ω. Then the*

[1] Andrew Glass has recently shown that pathological groups need not be l-simple, and need not be periodic in any sense.

orbits of L form a proper o-block system of G.

Proof. The analogous statement for nonordered permutation groups is precisely Proposition 7.1 of [7]. Here the fact that L is an l-ideal forces the blocks to be convex.

THEOREM 6. *Let (G, Ω) be a (pathologically) o-2-transitive l-permutation group, and let $\{1\} \neq L$ be an l-ideal of G. Then (L, Ω) is also (pathologically) o-2-transitive.*

Proof. o-2-transitive groups are certainly o-primitive, so by the lemma, L must be transitive on Ω. Let $\alpha < \beta < \gamma$, all in Ω. Pick $1 \leqq g \in G_\alpha$ such that $\beta g = \gamma$, and pick $1 \leqq k \in L$ such that $\beta k \geqq \gamma$. Then $1 \leqq k \wedge g \leqq k \in L$, so $k \wedge g \in L_\alpha$, and $\beta(k \wedge g) = \gamma$. Hence $\{\beta \in \Omega \mid \beta > \alpha\}$ is all one orbit of G_α, from which it follows easily that L is o-2-transitive. Certainly if G contains no element $\neq 1$ of bounded support, neither does L.

REFERENCES

1. R. D. Byrd and J. T. Lloyd, *Closed subgroups and complete distributivity in lattice-ordered groups*, Math. Zeit., **101** (1967), 123-130.
2. C. Holland, *The lattice-ordered group of automorphisms of an ordered set*, Michigan Math. J., **10** (1963), 399-408.
3. ———, *Transitive lattice-ordered permutation groups*, Math. Zeit., **87** (1965), 420-433.
4. J. T. Lloyd, *Complete distributivity in certain infinite permutation groups*, Michigan Math. J., **14** (1967), 393-400.
5. S. H. McCleary, *Pointwise suprema of order-preserving permutations*, Illinois J. Math., **16** (1972), 69-75.
6. ———, *The closed prime subgroups of certain ordered permutation groups*, Pacific J. Math., **31** (1969), 745-753.
7. H. Wielandt, *Finite Permutation Groups*, Academic Press, New York, N. Y., 1964.

Received May 24, 1973.

UNIVERSITY OF GEORGIA

PACIFIC JOURNAL OF MATHEMATICS

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. **39**. All other communications to the editors should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: $48.00 a year (6 Vols., 12 issues). Special rate: $24.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

* C. R. DePrima California Institute of Technology, Pasadena, CA 91109, will replace J. Dugundji until August 1974.

Pacific Journal of Mathematics

Vol. 49, No. 2 June, 1973