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# $O$-PRIMITIVE ORDERED PERMUTATION GROUPS II

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**This paper is a sequal to** *o-primitive ordered permutation groups* **[Pacific J. Math., 40 (1972), 349–372]. There it was shown that if $A(\Omega)$ is the lattice-ordered group of all $o$-permutations of a chain $\Omega$, and if $G$ is an $l$-subgroup of $A(\Omega)$ which is periodically $o$-primitive (transitive and lacking proper convex blocks, but neither $o$-2-transitive nor regular), then the (convex) orbits of any stabilizer subgroup $G_\alpha$, $\alpha \in \Omega$, themselves form a chain $o$-isomorphic to the integers. Let $\Delta$ be any non-singleton orbit of $G_\alpha$. Here it is shown that $G_\alpha$ is faithful on $\Delta$ and that $G_\alpha \mid \Delta$ is $o$-2-transitive and contains an element $\neq 1$ of bounded support. From this it follows that all $o$-primitive groups (except for certain pathological $o$-2-transitive groups) are complete $l$-subgroups of $A(\Omega)$, and hence are completely distributive. When $G$ is "full", $G_\alpha \mid \Delta$ satisfies an important "splice" property, and $G_\alpha$ and $G$ are laterally complete. There is a detailed description of the unique full group $G$ for which $\Delta$ is an $\alpha$-set, and a listing of the other "nice" permutation group representations of $G$.**

We assume throughout that $G$ is a transitive $l$-subgroup of $A(\Omega)$ (except that the more general *coherent* subgroups of $A(\Omega)$ are discussed briefly in the last section). Thus the orbits of $G_\alpha$ are convex, and the concepts of "orbital" [6] and "orbit" coincide. Although familiarity with [6] is assumed, we recapitulate most of Main Theorem 40 for $l$-permutation groups.

THEOREM 1. *Let $(G, \Omega)$ be an o-primitive l-permutation group which is neither o-2-transitive nor regular, and let $\alpha \in \Omega$. Then the long orbits of $G_\alpha$ form a chain o-isomorphic to the integers. Let $\Delta_1 = (\Delta_1)_\alpha$ denote the first positive long orbit, and let $\Delta_{j+1}$ be the first long orbit greater than $\Delta_j$. Between $\Delta_j$ and $\Delta_{j+1}$ there lies at most one point of $\Omega$. Either there is a positive integer $n$ such that* $\sup \Delta_j \in \Omega$ *if and only if* $j \equiv 0 \pmod n$, *and we say that $G$ has* Config $(n)$; *or* $\sup \Delta_j \in \Omega$ *only when $j = 0$, and we say that $G$ has* Config $(\infty)$. *There is a unique o-permutation $z$ of $\bar{\Omega}$, $\bar{\Omega}$ the Dedekind completion (without end points) of $\Omega$, such that $\alpha z = \sup (\Delta_1)_\alpha$ for each $\alpha \in \Omega$. $z$ is the period of $G$ in the sense that it generates (as a group) the centralizer $Z_{A(\bar{\Omega})}G$; so that $(\bar{\beta} z)g = (\bar{\beta} g)z$ for all $\bar{\beta} \in \bar{\Omega}$, $g \in G$.*

This periodicity is of paramount importance, and is the key to most of the following results. The action of $g \in G$ on any $\Delta_j$ determines its action on all of $\Omega$. $(\Delta_{j+1})_\alpha$ is "one period up" from $(\Delta_j)_\alpha$ in the sense that $(\bar{\Delta}_j)_\alpha z = (\bar{\Delta}_{j+1})_\alpha$.

Full groups are those for which $G$ is the entire centralizer $Z_{A(\bar{\Omega})}z$. It is shown that $G$ is full if and only if $G_\alpha$ is the set of all $o$-permutations of $\Delta_1$ which preserve the sets $\Delta_j z^{1-j} \subseteqq \bar{\Delta}_1$; and that the full group $G$ is determined by these sets. It is shown also that the *primitive* (in the ordinary permutation group sense) $l$-permutation groups are precisely those which are $o$-2-transitive or periodically $o$-primitive with Config $(\infty)$.

**2. Representations.** A *representation* of an $l$-group $G$ is an $l$-isomorphism $\theta$ of $G$ into some $A(\Sigma)$. Here all representations will be transitive (meaning that $G\theta$ is a transitive $l$-subgroup of $A(\Sigma)$). Now let $(G, \Omega)$ be an $l$-permutation group, and let $\bar{\omega} \in \bar{\Omega}$. For $g \in G$, form $\hat{g} \in A(\bar{\omega}G)$ by first extending $g$ to $\bar{\Omega}$ and then restricting to $\bar{\omega}G$. If $G$ is faithful on $\bar{\omega}G$ (e.g., if $\bar{\omega}G$ is dense in $\bar{\Omega}$), the map $g \to \hat{g}$ is a representation of $G$ into $A(\bar{\omega}G)$. We shall identify $G$ with its image and speak of $(G, \bar{\omega}G)$. Of course, $(G, \bar{\omega}G)$ depends only on the orbit $\bar{\omega}G$ and not on the particular $\bar{\omega}$.

Holland [3] defined a transitive $l$-permutation group $(G, \Omega)$ to be *weakly o-primitive* if for every $o$-block system $\tilde{\Delta}$ of $G$ (except the system of singletons) there exists $1 \neq g \in G$ such that $\Gamma g = \Gamma$ for all $\Gamma \in \Delta$. A representation $\theta$ of an $l$-group $G$ into $A(\Sigma)$ is *complete* if it preserves arbitrary sups that exist in $G$, or equivalently, if $G\theta$ is a complete $l$-subgroup of $A(\Sigma)$ in the sense that if $g \in G$ is the sup in $G$ of $\{g_i \mid i \in I\}$, then $g$ is also the sup in $A(\Sigma)$ of $\{g_i\}$. In [8], the present author defined an $o$-2-transitive group $G$ to be *pathological* if it contains no element $\neq 1$ of bounded support. We collect some facts about representations of $o$-primitive groups.

THEOREM 2. *Let $(G, \Omega)$ be o-primitive, but not pathologically o-2-transitive. Let $\theta$ be a weakly o-primitive (alternately, a complete transitive) representation of $G$ as an l-subgroup of some $A(\Sigma)$. Then there is an o-isomorphism $\psi$ from some $\bar{\omega}G$ onto $\Sigma$ which, together with $\theta$, furnishes an o-isomorphism from $(G, \bar{\omega}G)$ onto $(G\theta, \Sigma)$. The collection of $(G, \bar{\omega}G)$'s constitute (up to o-isomorphism) all weakly o-primitive (alternately, all complete transitive) representations of $G$.*

*Proof.* At present we shall treat only weakly $o$-primitive representations; after Theorem 6, we shall return to complete representations. If $(G, \Omega)$ is $o$-2-transitive (and not pathological), the first statement is a special case of [3, Theorem 7]; and by the proof of that theorem it suffices in general to show that a prime subgroup of $G$ which moves every $\bar{\beta} \in \bar{\Omega}$ must in fact be all of $G$. If $(G, \Omega)$ is regular, this is obvious. Now suppose $(G, \Omega)$ is periodically $o$-primitive, that $P$ is a prime subgroup of $G$ moving every $\bar{\beta} \in \bar{\Omega}$, and that $1 < g \in G$. Then (as in [2, p. 329]) given any bounded interval $\Pi$ of $\Gamma$, we may take the sup of an appropriate finite collection of elements of $P$ and raise it to an appropriate power to obtain $1 < f \in L$ such that $f$ exceeds $g$ on $\Pi$. We take $\Pi$ to be of the form $(\alpha, \alpha z)$, and periodicity guarantees that $f$ exceeds $g$ on $\Omega$. Therefore $P = G$, concluding the periodically $o$-primitive case. Finally, anticipating Theorem 3, we find that every $(G, \bar{\omega}G)$ is in fact weakly $o$-primitive (indeed, primitive).

An $l$-group $G$ is *laterally complete* if every pairwise disjoint set of elements has a sup in $G$. For an $l$-permutation group $(G, \Omega)$, we formulate a much stronger property: Let $\mathscr{D}_1 = \{\Delta_{i,1} \mid i \in I\}$ be a collection of pairwise disjoint nondegenerate segments of $\Omega$ such that $\bigcup \mathscr{D}_1$ is dense in $\Omega$, with $\mathscr{D}_1$ totally ordered in the natural way; and similarly for $\mathscr{D}_2 = \{\Delta_{i,2} \mid i \in I\}$. Let $f$ be an $o$-isomorphism from $\mathscr{D}_1$ onto $\mathscr{D}_2$ such that whenever $\bar{\mu}$ is a proper Dedekind cut in $\mathscr{D}_1$ (and thus may be considered as an element of $\bar{\Omega}$), $\bar{\mu}$ and $\bar{\mu}f$ lie in the same orbit of $G$. Suppose that for each $i \in I$, there exists $g_i \in G$ such that $\Delta_{i,1}g_i = \Delta_{i,2}$. We shall say that $(G, \Omega)$ has the *splice property* if whenever these circumstances occur, there exists $g \in G$ such that $g \mid \Delta_{i,1} = g_i$ for each $i$. It is easily checked that it suffices to consider the special case in which each $g_i$ is positive ($\omega \leqq \omega g_i$ for each $\omega \in \Delta_{i,1}$).

If $(G, \Omega)$ is *entire* (i.e., if $G = A(\Omega)$), then $G$ has the splice property. Actually, $A(\Omega)$ satisfies a stronger property, whereby $\bar{\mu}$ and $\bar{\mu}f$ are required only to be both in $\Omega$ or both in $\bar{\Omega}\backslash\Omega$; but there are examples in which this stronger property does not carry over to $(A(\Omega), \bar{\omega}A(\Omega))$.

$(G, \Omega)$ is *depressible* if given any $g \in G$ and $\gamma \in \Omega$ for which $\gamma g \neq \gamma$, there exists $h \in G$ such that $\omega h = \omega g$ if $\omega$ lies in the interval of support $\operatorname{Conv}\{\gamma g^n \mid n \text{ an integer}\}$, and $\omega g = \omega$ otherwise. Clearly groups having the splice property are depressible as well as laterally complete. On the other hand, if $\Omega$ is the real numbers, and $G$ is the set of all $o$-permutations $g$ of $\Omega$ for which there exists no monotone sequence $\omega_n \to \omega$ such that $(\omega_n g - \omega)/(\omega_n - \omega) \to 0$ or $\infty$, then $G$ is a depressible, laterally complete, $o$-2-transitive $l$-permutation group, but it does not have the splice property. (When showing that $G$ is closed under product, use the fact that if $u_n v_n \to 0$ or $\infty$ $(u_n, v_n > 0)$, then some subsequence of $\{u_n\}$ or of $\{v_n\}$ approaches 0 or $\infty$, respectively.)

Holland [3] defined a transitive $l$-permutation group $(G, \Omega)$ to be locally *o-primitive* if in the totally ordered set of $o$-block systems (excluding the system of singletons), there is a smallest system $\tilde{\Delta}$. The $o$-blocks in $\tilde{\Delta}$ are called the *primitive segments* of $G$. If $\Gamma$ is a primitive segment, let $G \mid \Gamma$ denote the restriction of $G$ to $\Gamma$, i.e., $\{g \mid \Gamma : g \in G \text{ and } \Gamma g = \Gamma\}$. All $(G \mid \Gamma, \Gamma)$'s are isomorphic as $o$-permutation groups, and they are $o$-primitive. A property preserved by isomorphism and enjoyed by $(G \mid \Gamma, \Gamma)$ for one (hence every) primitive segment $\Gamma$ will be said to be enjoyed locally by $(G, \Omega)$. In view of Theorem 1, every locally $o$-primitive group must be locally $o$-2-transitive, locally regular, or locally periodically $o$-primitive.

For any $o$-primitive $(G, \Omega)$, and any $\bar{\omega} \in \bar{\Omega}$, $\bar{\omega}G$ is dense in $\bar{\Omega}$ ([3, Theorem 2]). Thus Theorem 2 leads us to

THEOREM 3. *Let $(G, \Omega)$ be a transitive l-permutation group and let $\bar{\omega} \in \bar{\Omega}$ with $\bar{\omega}G$ dense in $\bar{\Omega}$. Then*

(1) *Extension to $\bar{\Omega}$ followed by restriction to $\bar{\omega}G$ provides a canonical one-to-one correspondence between the collection of non-singleton o-blocks of $(G, \Omega)$ and that of $(G, \bar{\omega}G)$.*

(2) *The image under this correspondence of an o-block system of $(G, \Omega)$ is an o-block system of $(G, \bar{\omega}G)$.*

(3) *A canonical o-isomorphism from the tower of o-block systems of $(G, \Omega)$ onto that of $(G, \bar{\omega}G)$ is given by* (2).

(4) *Corresponding o-primitive components of $(G, \Omega)$ and $(G, \bar{\omega}G)$ are o-isomorphic, with the following exception: In the locally o-primitive case, if $\Gamma$ is the primitive segment of $(G, \Omega)$ for which $\bar{\omega} \in \bar{\Gamma}$, $\Gamma_{\bar{\omega}} = \Gamma \cap \bar{\omega}G$ is the primitive segment of $\bar{\omega}G$ containing $\bar{\omega}$. But at least $(G \mid \Gamma, \Gamma)$ and $(G \mid \Gamma_{\bar{\omega}}, \Gamma_{\bar{\omega}})$ are both pathologically (or both nonpathologically) o-2-transitive, both regular, or both periodically o-primitive (in which case both have the same period $z$).*

*Moreover, the following properties are enjoyed either by both $(G, \Omega)$ and $(G, \bar{\omega}G)$, or by neither:*

(5) *Weak o-primitivity.*

(6) *The splice property.*

(7) *Depressibility.*

(8) *Entirety on some dense orbit of $G$.*

(9) *Completeness in the entire group $A(\Omega)$, respectively $A(\bar{\omega}G)$. (See the comments preceding Theorem* 8.)

All parts of the theorem are routine except for (9), which is contained in [5, Theorem 1]. In connection with (8), we mention that examples in which entirety does not carry over to $(A(\Omega), \bar{\omega}A(\Omega))$ are to be found in [7, Theorem 15].

3. **Periodic $o$-primitivity.** Let $(G, \Omega)$ be periodically $o$-primitive with Config $(n)$, and let $I_n$ be $\{1, \cdots, n\}$ if $n$ is finite, and be the integers if $n = \infty$. Let $\Omega_k = \Omega z^{1-k} \subseteqq \bar{\Omega}(k \in I_n)$. Since $z$ centralizes $G$, $\Omega_k G = \Omega_k$ for each $k$. Let $\alpha \in \Omega$, and let the long orbits $\{\Delta_j\}$ of $G_\alpha$ be denoted as in the introduction. $\Delta_j z^{1-j} = \bar{\Delta}_1 \cap \Omega_k$, where $k \equiv j \pmod n$. By the *signature* of $G$ we shall mean the collection $\{\bar{\Delta}_1 \cap \Omega_k (= \Delta_k z^{1-k}) \mid k \in I_n\}$. [6, Theorem 54] gives three conditions which the signature must satisfy.

By an abstract $n$-signature ($n = 1, 2, \cdots, \infty$), we shall mean a collection $\{\Sigma_k \mid k \in I_n\}$, with each $\Sigma_k$ a specified subset of $\bar{\Sigma}_1$, satisfying the conditions (for the particular $n$) of [6, Theorem 54]. By an $o$-isomorphism of one such signature $\{\Sigma_k\}$ onto another $\{\Pi_k\}$ we shall mean an $o$-isomorphism $\varphi$ from $\bar{\Sigma}_1$ onto $\bar{\Pi}_1$ such that $\Sigma_k\varphi = \Pi_k$ for each $k \in I_n$. The signature defined above for $G$ is of course an $n$-signature, and by the transitivity of $G$, it is independent (up to $o$-isomorphism) of the choice of $\alpha$.

Recall that the full groups are universal in the sense that every periodically $o$-primitive group $G$ is contained in a full group having the same period $z$, namely $Z_{A(\bar{\Omega})}z$.

THEOREM 4. *For each $n = 1, 2, \cdots, \infty$, a one-to-one correspondence between the collection of (o-isomorphism classes of) full periodically o-primitive groups $(G, \Omega)$ of* Config $(n)$ *and the collection of (o-isomorphism classes of) n-signatures is given by mapping each group to its signature. When $n = 1$, the chain $\Omega$ determines the (full) group $G$ and the signature of $(G, \Omega)$.*

*Proof.* In [6, Theorem 54] we constructed from an arbitrary $n$-signature $\{\Sigma_k\}$ a full group $(G, \Omega)$ such that for an appropriate $\alpha$, $\Delta_k z^{1-k} = \Sigma_k$ for all $k \in I_n$. Hence our mapping is onto. It is also one-to-one, for the signature of $(G, \Omega)$ determines $\Omega$ and determines the period $z$ on $\Omega$ and hence on $\bar{\Omega}$; and then since $G$ is full, $G = Z_{A(\bar{\Omega})}z$. If $n = 1$, $\Sigma_1 = \Delta_1$ is a closed interval of $\Omega$. Since $G$ is $o$-primitive, $A(\Omega)$ must be $o$-primitive and thus $o$-2-transitive, so that all nondegenerate closed intervals of $\Omega$ are $o$-isomorphic. Hence $\Omega$ determines $\Delta_1$ and thus determines $G$.

LEMMA 5. *Let $(G, \Omega)$ be a periodically o-primitive l-permutation group. Let $\alpha \in \Omega$ and let $\Delta$ be any long orbit of $G_\alpha$. Then $G_\alpha$ is faithful on $\Delta$, and $(G_\alpha, \Delta)$ is a nonpathologically o-2-transitive l-permutation group.*

*Proof.* Periodicity guarantees that $G_\alpha$ is faithful on $\Delta$ and that, in view of part (4) of Theorem 3, we may assume that $\Delta$ is the first positive orbit of $G_\alpha$. Let $\Gamma = \Delta'$, the last negative orbit of $G_\alpha$. Since $(G_\alpha, \Delta)$ is transitive, it will be $o$-2-transitive if for $\beta \in \Delta$, $(G_\alpha)_\beta = G_\alpha \cap G_\beta$ is transitive on $\Pi = \{\delta \in \Delta \mid \beta < \delta\}$. Pick any $\gamma, \delta \in \Pi$, with $\gamma < \delta$. Next pick $1 \leqq h \in G_\alpha$ so that $\gamma h = \delta$. Then $\beta \leqq \beta h < \delta$, so $\beta h \in \Delta$. Now pick $1 \geqq r \in G_\delta$ so that $(\beta h)r = \beta$. Since $\alpha r \leqq \alpha < \beta < \delta$, and since both $\alpha r$ and $\alpha$ lie in the last negative orbit $\Gamma_\delta$ of $G_\delta$, we have $\alpha r, \alpha \in \Gamma_\beta$. Hence we may pick $1 \leqq s \in G_\beta$ so that $(\alpha r)s = \alpha$. Now $hrs \in (G_\alpha)_\beta$, and $\gamma hrs = \delta rs = \delta s \geqq \delta$, so that $\gamma$ and $\delta$ lie in the same (convex) orbit of $(G_\alpha)_\beta$. Therefore, $(G_\alpha)_\beta$ has only one positive orbit in $\Delta$, so that $(G_\alpha, \Delta)$ is $o$-2-transitive.

We can pick $g \in G$ such that $\alpha g < \alpha$ and $\beta g > \beta$. Then $1 \neq g \vee 1$ fixes each point in some segment $\Lambda$ of $\Omega$ which meets both $\Delta$ and $\Gamma$, so that $1 \neq (g \vee 1) \mid \Delta$ has support which is certainly bounded below, and by periodicity, is also bounded above. Therefore, $(G_\alpha, \Delta)$ is not pathological.

The author showed in [4, Theorem 7] that for a transitive $l$-subgroup $G$ of $A(\Omega)$, the following are equivalent:

(1) $G_\alpha$ is *closed* subgroup of $G$ for one (hence every) $\alpha \in \Omega$, i.e., if $g = \bigvee_{i \in I} g_i$ with each $g_i \in G_\alpha$, then $g \in G_\alpha$.

(2) $G$ is a *complete* subgroup of $A(\Omega)$.

(3) Sups *in $G$ are pointwise*, i.e., if $g = \bigvee_{i \in I} g_i$ with each $g_i \in G$, then for each $\beta \in \Omega$, $\beta g$ is the sup in $\Omega$ of $\{\beta g_i \mid i \in I\}$.

Moreover, it was shown in [4, Corollary 15] that in the presence of these conditions, we have

(4) $G$ is a *completely distributive* $l$-group, i.e., $\bigwedge_{i \in I} \bigvee_{k \in K} g_{ik} = \bigvee_{f \in K^I} \bigwedge_{i \in I} g_{if(i)}$ for any collection $\{g_{ik} \mid i \in I, k \in K\}$ of $G$ for which the indicated sups and infs exist.

THEOREM 6. *Let $(G, \Omega)$ be an o-primitive l-permutation group. Then Conditions* (1), (2), (3), *and* (4) *are all equivalent, and they fail if and only if G is pathologically o-2-transitive.*

*Proof.* The $o$-2-transitive case is precisely the content of [8, Theorem 1], and the conditions hold automatically in the regular case. In the periodically $o$-primitive case, the proof parallels the proof in [8] for the nonpathologically $o$-2-transitive case: Suppose $g = \bigvee_{i \in I} g_i$, with $g \in G$ and each $g_i \in G_\alpha$, but with $\alpha < \alpha g$. By Lemma 5, we can pick $1 > h \in G_\alpha$ such that $h \mid \Delta_0$ ($\Delta_0$ the last negative orbit of $G_\alpha$) has support contained in $(\alpha g^{-1}, \alpha)$. Then for each $i \in I$, $g_i \leqq hg < g$ on $\Delta_0$, and hence on all of $\Omega$ by periodicity, giving a contradiction. Therefore $G_\alpha$ is closed, and the other conditions follow.

Now we prove Theorem 2 for complete representations. Theorem 6 states that every $(G, \bar{\omega} G)$ is complete. For the rest, it suffices to show that every proper closed prime subgroup of $G$ is $G_{\bar{\omega}}$ for some $\bar{\omega} \in \bar{\Omega}$; and the proof of Theorem 11 of [5] shows that if this were not the case, $G$ would have a proper $o$-block, violating $o$-primitivity.

LEMMA 7. *Let $(G, \Omega)$ be periodically o-primitive, and let $\Gamma \neq \Omega$ be a (not necessarily convex) block of G. Then $\Gamma \subseteqq FxG_\alpha$ for every $\alpha \in \Gamma$.*

*Proof.* If $\delta \in \Gamma \backslash FxG_\alpha$, then $\Gamma$ would contain the segment $\delta G_\alpha$, so that $\Gamma = \Omega$ by periodicity.

THEOREM 8. *Let G be a transitive l-permutation group. Then G is primitive if and only if G is o-2-transitive or G is periodically* *o-primitive and has* Config $(\infty)$.

*Proof.* If $G$ is primitive, then *a fortiori*, $G$ is $o$-primitive. If $G$ is $o$-2-transitive, it is clearly primitive. If $G$ is periodically $o$-primitive and has Config $(\infty)$, $FxG_\alpha = \{\alpha\}$, so $G$ is primitive by the lemma. However, if $G$ has Config $(n)$ for some finite $n$, the block $FxG_\alpha$ violates primitivity; and if $(G, \Omega)$ is regular, it is the regular representation of some subgroup of the reals ([6, Proposition 24]) and thus is not primitive.

We now borrow some terminology from [1, pp. 142–4] and [7], assuming for convenience that $\Omega$ is homogeneous and dense in itself (which will necessarily be the case if $(G, \Omega)$ is $o$-primitive, unless $\Omega$ is the integers). The point or hole (i.e., proper Dedekind cut) $\bar{\omega}$ of $\Omega$ is said to have *character* $c_{\beta\gamma}$ if $\omega_\beta$ is the unique regular ordinal number which is $o$-isomorphic to a cofinal subset of $\{\sigma \in \Omega \mid \sigma < \bar{\omega}\}$ and dually for $\omega_\gamma$. All elements of any one orbit $\bar{\omega}G$ have the same character. $\Omega$ is said to have *final character* $c_\beta$ if $\omega_\beta$ is the unique regular ordinal $o$-isomorphic to a cofinal subset of $\Omega$; and dually for *initial character*. Alternately, any of these characters can be determined by using subsets not of $\Omega$, but of any dense subset of $\bar{\Omega}$.

LEMMA 9. *Let $(G, \Omega)$ be periodically o-primitive, and suppose that the points of $\Omega$ have character $c_{\beta\gamma}$. Then if $\Delta$ is any long orbit of $G_\alpha$, $\alpha \in \Omega$, the initial character of $\Delta$ is $c_\gamma$ and the final character is $c_\beta$.*

*Proof.* $\Delta = \Delta_j$ for some $j$. $\Delta$ has the same initial character as $\Delta z^{1-j}$; since the latter is dense in $\bar{\Delta}_1$, its initial character is that of $\Delta_1$, which is $c_\gamma$. A similar argument works for final characters.

PROPOSITION 10. *Suppose that $(G, \Omega)$ is periodically o-primitive, and that in its order topology, $\Omega$ satisfies the first countability axiom* (i.e., *the points of $\Omega$ have character $c_{00}$*). *Then all long orbits of $G_\alpha$, $\alpha \in \Omega$, are o-isomorphic.*

*Proof.* Let $\Gamma$ and $\Delta$ be long orbits of $G_\alpha$. Picking $g \in G$ such that $\Gamma g$ meets $\Delta$, we obtain an $o$-isomorphism between some interval of $\Gamma$ and some interval of $\Delta$. Since $G_\alpha \mid \Gamma$ and $G_\alpha \mid \Delta$ are $o$-2-transitive, all nondegenerate closed intervals of $\Gamma$ and of $\Delta$ are $o$-isomorphic to each other. Since the points of $\Omega$ have character $c_{00}$, the lemma guarantees that $\Gamma$ and $\Delta$ both have $c_0$ as initial and final characters. The proposition follows.

**4. Extracts of periodically $o$-primitive groups.** The results about $(G_\alpha, \Delta)$ mentioned in the introduction will be needed also for $G_{\bar{\beta}}$, $\bar{\beta} \in \bar{\Omega}$.

PROPOSITION 11. *Let $(G, \Omega)$ be periodically o-primitive, and let*

$\bar{\beta} \in \bar{\Omega}$. *Then the long orbits in* $\Omega$ *of* $G_{\bar{\beta}}$ *are the sets* $\Delta_j = \{\omega \in \Omega \mid \bar{\beta} z^{j-1} < \omega < \bar{\beta} z^j\}$, *j an integer; and* $G_{\bar{\beta}}$ *is faithful on each* $\Delta_j$.

*Proof.* The statement about the long orbit structure of $G_{\bar{\beta}}$ is equivalent to the statement that the fixed points in $\bar{\Omega}$ of $G_{\bar{\beta}}$ are precisely those of the form $\bar{\beta} z^j$. Periodicity guarantees that $G_{\bar{\beta}}$ fixes these points and that to show it fixes no others, it suffices to consider $\bar{\beta} < \bar{\gamma} < \bar{\beta} z$. Now pick $\alpha \in \Omega$ such that $\alpha < \bar{\beta} < \bar{\gamma} < \alpha z$. Lemma 5 guarantees that $G_\alpha$ is $o$-2-transitive on its first positive orbit, so there exists $h \in G_\alpha$ with $\bar{\beta} h \leqq \bar{\beta}$ and $\bar{\gamma} h > \bar{\gamma}$. Now $h \vee 1$ fixes $\bar{\beta}$ and moves $\bar{\gamma}$, as desired. Therefore, the long orbits of $G_{\bar{\beta}}$ are as described, and by periodicity, $G$ is faithful on each of them.

We now define the $\bar{\beta}$-*extract* of $(G, \Omega)$, where $G$ is periodically $o$-primitive and $\bar{\beta} \in \bar{\Omega}$, to be $(G_{\bar{\beta}}, \Delta_1)$. (Warning: $\Delta_1 \not\subseteq \bar{\beta} G$ unless $\bar{\beta} \in \Omega$.) Of course if $\bar{\beta}$ and $\bar{\gamma}$ lie in the same orbit of $G$, the $\bar{\beta}$- and $\bar{\gamma}$-extracts are isomorphic as $o$-permutation groups.

LEMMA 12. *Let* $\mathscr{P}$ *be an o-permutation group property which carries over from* $(H, \Sigma)$ *to* $(H, \bar{\sigma} H)$ *when* $\bar{\sigma} H$ *is dense in* $\bar{\Sigma}$ (cf. *Theorem* 2). *Suppose that* $\mathscr{P}$ *holds for every* $\alpha$-*extract*, $\alpha \in \Omega$, *of every periodically o-primitive group* $(G, \Omega)$. *Then for every periodically o-primitive* $(G, \Omega)$, $\mathscr{P}$ *holds for every* $\bar{\beta}$-*extract*, $\bar{\beta} \in \bar{\Omega}$, *and hence for* $(G_{\bar{\beta}}, \Delta)$, *where* $\Delta$ *is any long orbit of* $G_{\bar{\beta}}$.

*Proof.* $\mathscr{P}$ holds for the $\bar{\beta}$-extract of $(G, \Omega)$ because it carries over from the $\bar{\beta}$-extract of $(G, \bar{\beta} G)$. ($(G, \bar{\beta} G)$ is periodically $o$-primitive by Theorem 3.) $(G_{\bar{\beta}}, \Delta)$ is merely the $\bar{\gamma}$-extract of $G$, where $\bar{\gamma} = \inf \Delta$.

THEOREM 13. *Every extract of a periodically o-primitive l-permutation group is a nonpathologically o-2-transitive l-permutation group.*

*Proof.* Use Lemmas 5 and 12.

LEMMA 14. *Let* $(G, \Omega)$ *be periodically o-primitive, let* $\bar{\beta} \in \bar{\Omega}$, *and let* $g \in G$ *be such that* $\bar{\beta} g \notin \bar{F}x G_{\bar{\beta}}$ (i.e., $\{\bar{\omega} \in \bar{\Omega} \mid \bar{\omega} G_{\bar{\beta}} = \bar{\omega}\}$). *Then G is generated as a group by* $G_{\bar{\beta}}$ *and g.*

*Proof.* We may assume that $\bar{\beta} \in \Omega$. (If not, replace $(G, \Omega)$ by $(G, \bar{\beta} G)$.) Now let $C$ be the subgroup of $G$ generated by $G_{\bar{\beta}}$ and $g$. Then $\bar{\beta} C$ is a block of $G$ (by [9, Theorem 7.5]), contradicting Lemma 7.

LEMMA 15. *Let* $(G, \Omega)$ *be full and let H and K be periodically o-primitive l-subgroups of G having the same period z as G. If there exists* $\bar{\beta} \in \bar{\Omega}$ *such that* $H_{\bar{\beta}} = K_{\bar{\beta}}$ *and* $\bar{\beta} g \notin \bar{F}x G_{\bar{\beta}}$ *for some* $g \in H \cap K$, *then* $H = K$.

*Proof.* $\bar{F}xG_{\bar{\beta}} = \bar{F}xH_{\bar{\beta}} = \bar{F}xK_{\bar{\beta}}$, so that this lemma follows from the previous one.

LEMMA 16. *Let* $(G, \Omega)$ *be full and let* $H \neq G$ *be a periodically o-primitive l-subgroup of* $G$. *Then* $H_{\bar{\beta}} \neq G_{\bar{\beta}}$ *for every* $\bar{\beta} \in \bar{\Omega}$.

*Proof.* If $H$ has the same period $z$ as $G$, the previous lemma suffices. But if $z$ is not the period of $H$, $\bar{F}xH_{\bar{\beta}} \neq \bar{F}xG_{\bar{\beta}}$, so $H_{\bar{\beta}} \neq G_{\bar{\beta}}$.

THEOREM 17. *Let* $(G_{\bar{\beta}}, \Delta)$ *be any extract of a periodically o-primitive group* $(G, \Omega)$. *Then* $G_{\bar{\beta}}$ *preserves the subsets* $\bar{\Delta} \cap \Omega_k (k \in I_n)$ *of* $\bar{\Delta}$. *Moreover,* $G$ *is full if and only if* $G_{\bar{\beta}}$ *consists of all o-permutations of* $\Delta$ *which preserve these sets.*

*Proof.* $G_{\bar{\beta}}$ preserves $\bar{\Delta}$ and $\Omega_k$, and thus preserves $\bar{\Delta} \cap \Omega_k$. Suppose $G$ is full. If $h \in A(\Delta)$ preserves the sets $\bar{\Delta} \cap \Omega_k$, the unique o-permutation of $\bar{\Omega}$ which extends $h$ and commutes with $z$ will preserve $\Omega$, so that $G_{\bar{\beta}}$ will be as described. Conversely, if $G_{\bar{\beta}}$ fits the description, and if $K$ is the full periodically o-primitive group containing $G$ and having the same period $z$ as $G$ ([6, Proposition 53]), then $G_{\bar{\beta}} = K_{\bar{\beta}}$, so that $G = K$ (Lemma 16) and $G$ is full.

THEOREM 18. *Let* $(G, \Omega)$ *be periodically o-primitive. If one of its extracts is entire, so are they all; and similarly for the splice property. If the extracts are entire,* $G$ *is full (and conversely if* $G$ *has* Config (1)). *If* $G$ *is full, the extracts have the splice property.*

*Proof.* Suppose that the extract $(G_{\bar{\beta}}, \Pi)$ has the splice property. In the extract $(G_{\bar{\gamma}}, \Lambda)$, let $\mathscr{D}_j = \{\Delta_{i,j} \mid i \in I\}$, $f$, and $\{g_i \mid i \in I\}$ satisfy the conditions of the splice property. With no loss of generality, we may suppose first (since $\bar{\beta}G$ is dense in $\bar{\Omega}$) that $\bar{\beta} \in \bar{\Lambda}$, and next (by multiplying by an appropriate element of $G_{\bar{\gamma}}$) that $\bar{\beta}$ is fixed by the permutation of $\bar{\Lambda}$ obtained by splicing the $g_i$'s. Now since $(G_{\bar{\beta}}, \Pi)$ has the splice property, when we splice together the $g_i$'s for which $\Delta_{i,1} \subseteqq \Pi \cap \Lambda$ and $\hat{g}$ (the identity on $\Pi \backslash \Lambda$), we obtain an element $h_1$ of $G_{\bar{\beta}}$ which acts as desired on $\Pi \cap \Lambda$ and (by periodicity) is the identity on $\Lambda \backslash \Pi$. Similarly, for an appropriate $\bar{\eta}$ $(\bar{\beta}z^{-1} < \bar{\eta} < \bar{\gamma})$, there exists $h_2 \in G_{\bar{\eta}}$ which acts as desired on $\Lambda \backslash \Pi$ and is the identity on $\Pi \cap \Lambda$. The product $h_1 h_2$ satisfies the conclusion of the splice property.

For entirety of extracts, we proceed similarly, letting $g \in A(\Lambda)$. We may not assume that $\bar{\beta}$ is fixed, but we do obtain an $h_1$ which agrees with $g$ on $\Pi \cap \Lambda$ and an $h_2$ which agrees with $g$ on $\Lambda \backslash \Pi$. Splicing these together, we find that $g \in G_{\bar{\gamma}}$. (Since one extract is entire, it, and hence every extract, satisfies the splice property.)

If the extracts are entire, then by Theorem 17, $G$ is certainly full; and if $G$ has Config (1), there is only one $\bar{\Delta} \cap \Omega_k$, namely $\Delta$, so the converse holds. If $G$ is full, Theorem 17 guarantees that its extracts have the splice property.

THEOREM 19. *Let* $(G, \Omega)$ *be periodically o-primitive, and let* $\bar{\beta} \in \bar{\Omega}$. *Then* $G_{\bar{\beta}}$ *is laterally complete if and only if* $G$ *is laterally complete; and these conditions obtain for all full groups.*

*Proof.* For the "iff" statement, we can assume that $\bar{\beta} = \beta \in \Omega$. (If not, consider $(G, \bar{\beta}G)$.) Certainly if $G$ is laterally complete, so is $G_\beta$, for $G_\beta$ is closed under arbitrary sups in $G$ (Theorem 6). Now suppose that $G_\beta$ is laterally complete, and let $\{h_i \mid i \in I\}$ be a set of pairwise disjoint elements of $G$. At most one of the $h_i$'s moves $\beta$, so we may suppose with no loss of generality that $\{h_i \mid i \in I\} \subseteqq G_\beta$. Let $h$ be the sup in $G_\beta$ of $\{h_i\}$. If this sup is pointwise, $h$ will also be the sup in $G$ of $\{h_i\}$, and we shall be finished. Thus let $\Delta$ be a long orbit of $G_\beta$. $G_\beta$ is faithful on $\Delta$, so in $G_\beta \mid \Delta$, $h \mid \Delta = \sup\{h_i \mid \Delta\}$. Since $G_\beta \mid \Delta$ is nonpathologically $o$-2-transitive, this sup is pointwise on $\Delta$ (Theorem 6); by periodicity, the sup is then pointwise on $\Omega$, as required. If $G$ is full, the $\bar{\beta}$-extract has the splice property by Theorem 18, so $G_{\bar{\beta}}$ is laterally complete.

5. **Periodically $o$-primitive groups constructed from $\alpha$-sets.** Let $\omega_\alpha$ be a regular ordinal number. An *$\alpha$-set* is a chain $\Omega$ of cardinality $\aleph_\alpha$ in which for any two (possibly empty) subsets $\Gamma < \Delta$ of cardinality less than $\aleph_\alpha$, there exists $\omega \in \Omega$ such that $\Gamma < \omega < \Delta$. If we consider only nonempty $\Gamma$ and $\Delta$ (though still requiring that $\Omega$ has neither a first nor a last point), so that the terminal characters need not be $c_\alpha$, we obtain a generalization which we shall call a *truncated $\alpha$-set*. We shall need some information from [7] about truncated $\alpha$-sets $\Omega$. For any regular $\omega_\alpha$, and any regular $\omega_\beta$, $\omega_\gamma$ less than or equal to $\omega_\alpha$, there exists (assuming the generalized continuum hypothesis) a truncated $\alpha$-set $\Omega$ having initial character $\omega_\beta$ and final character $\omega_\gamma$; and it is unique up to $o$-isomorphism. The points of $\Omega$ have character $c_{\alpha\alpha}$, and the holes have character $c_{\alpha\alpha}$, $c_{\alpha\beta}$, or $c_{\beta\alpha}$ (with $\omega_\beta$ regular and $\omega_\beta < \omega_\alpha$), with all of these characters actually occurring. Conversely, these conditions on characters (including the terminal characters), together with the requirement that card $(\Omega) \leqq \aleph_\alpha$, force $\Omega$ to be the truncated $\alpha$-set above. Hence any segment (without end points) of an $\alpha$-set is a truncated $\alpha$-set, and every truncated $\alpha$-set arises in this way. If both terminal characters are $c_0$, the set will be called *countably truncated*. The set of all holes in a truncated $\alpha$-set $\Omega$ of a given character $c_{\gamma\delta}$ form an orbit $\Omega_{\gamma\delta}$ of $(A(\Omega), \bar{\Omega})$, and these orbits are dense

in $\bar{\Omega}$. All have cardinality $\aleph_\alpha$ except for $\Omega_{\alpha\alpha}$, which has cardinality $2^{\aleph_\alpha}$.

LEMMA 20. *Suppose that $(G, \Omega)$ is a periodically o-primitive group having an $\alpha$-set $\Delta$ as the first positive orbit of a stabilizer subgroup $G_\xi$. Then all long orbits of $G_\xi$ are o-isomorphic to the $\alpha$-set $\Delta$, and $\Omega$ is o-isomorphic to the countably truncated $\alpha$-set.*

*Proof.* Use characters. The terminal characters of any long orbit are $c_\alpha$ by Lemma 9; and those of $\Omega$ are $c_0$ because of the configuration of $G$.

LEMMA 21. *Let $\Phi_j$, $j = 1, 2$, be truncated $\alpha$-sets having the same initial character and same final character. Let $\{\Psi_{j,i} \mid i \in I\}$, $I$ the positive integers, be a collection of dense pairwise disjoint subsets of $\bar{\Phi}_j$ for which $\Psi_{j,1} = \Phi_j$ and each $\Psi_{j,i}$ is o-isomorphic to $\Phi_j$. Then there exists an o-isomorphism $g$ from $\Phi_1$ onto $\Phi_2$ such that $\Psi_{1,i}g = \Psi_{2,i}$ for each $i$.*

*Proof.* First, the lemma holds for nontruncated $\alpha$-sets; for we may apply the standard proof of the uniqueness of $\alpha$-sets [1, p. 182], noting that if $\Gamma < \Delta$ in an $\alpha$-set $\Omega$, and if both sets have cardinality less than $\aleph_\alpha$, then $\{\omega \in \Omega \mid \Gamma < \omega < \Delta\}$ contains more than one point of $\Omega$. Now for truncated $\alpha$-sets, we may proceed exactly as in the proof of [7, Theorem 5].

THEOREM 22. *Let $n = 1, 2, \cdots$, or $\infty$, let $\omega_\alpha$ be a regular ordinal number, and let $\Delta$ be an $\alpha$-set. Then there exists a unique (up to o-isomorphism) full periodically o-primitive group $(G, \Omega)$ having $\Delta$ as the first positive orbit of a stabilizer subgroup $G_\xi$ and having* Config $(n)$. *Its extracts are entire if and only if $n = 1$. Let $\hat{\Omega}_{\alpha\alpha} = \Omega_{\alpha\alpha} \backslash \bigcup \{\Omega_k \mid k \in I_n\}$, which is o-isomorphic to $\Omega_{\alpha\alpha}$. $(G, \Omega)$ itself, $(G, \hat{\Omega}_{\alpha\alpha})$, and the $(G, \Omega_{\alpha\beta})$'s and $(G, \Omega_{\beta\alpha})$'s constitute (up to o-isomorphism) all weakly o-primitive (alternately, all complete transitive) representations of the l-group $G$, and distinct representations in the list are non-o-isomorphic. All except $(G, \Omega)$ have* Config (1). *The $(G, \Omega_{\alpha\beta})$'s and $(G, \Omega_{\beta\alpha})$'s are never full; and $(G, \hat{\Omega}_{\alpha\alpha})$ is full if and only if $n = 1$.*

*Proof.* First we show that $\Delta$ satisfies Conditions (a), (b), and (c) of [6, Theorem 54]. Let $\Sigma_1 = \Delta$. $\Delta_{\alpha\alpha}$ is dense in $\bar{\Delta}$. From each open interval of $\Delta$, pick an element of $\Delta_{\alpha\alpha}$; and let $\Sigma_2$ be the ordered set of holes thus obtained. Now $\Delta$ is a dense subset of $\bar{\Sigma}_2 = \bar{\Delta}$, so we may use subsets of $\Delta$ to determine characters for $\Sigma_2$. Hence each point in $\Sigma_2$ has character $c_{\alpha\alpha}$; each hole in $\Sigma_2$ has character $c_{\alpha\alpha}$, $c_{\alpha\beta}$, or

$c_{\beta\alpha}$; the initial and final characters of $\Sigma_2$ are $c_\alpha$; and card $(\Sigma_2) \leqq \aleph_\alpha$. Therefore $\Sigma_2$ is an $\alpha$-set, and of course $\Sigma_2 \subseteqq \bar{\Delta}$ and $\Sigma_2 \cap \Sigma_1 = \Box$. $\Sigma_1 \cup \Sigma_2$ is also an $\alpha$-set, so we may continue this process, obtaining a collection $\{\Sigma_i \mid i \in I_n\}$ of dense pairwise disjoint subsets of $\bar{\Omega}$, all of them $\alpha$-sets. Thus Condition (a) is satisfied. If $\bar{\eta} \in \bar{\Delta}$ has character $c_{\alpha\alpha}$, then for any $\Sigma_i$, $\{\lambda \in \Omega \mid \lambda < \bar{\eta}\}$ and $\{\lambda \in \Omega \mid \lambda > \bar{\eta}\}$ are $\alpha$-sets, so by applying Lemma 21 we get Conditions (b) and (c). This proves the existence of $(G, \Omega)$; and by the proof of [6, Theorem 54], $\Sigma_k = \bar{\Delta} \cap \Omega_k$ for each $k \in I_n$. But Lemmas 20 and 21 show that for a given $n$, there is (up to $o$-isomorphism) only one signature with $\Sigma_1$ an $\alpha$-set, so that the uniqueness of $(G, \Omega)$ follows from Theorem 4. That the extracts of $G$ are entire if and only if $n = 1$ follows from Theorem 17 and that fact that $\Delta_{\alpha\alpha}$ is an orbit of $A(\Delta)$.

By Theorem 2, every weakly $o$-primitive or complete transitive representation of $G$ is $o$-isomorphic to some $(G, \bar{\omega}G)$, $\bar{\omega} \in \bar{\Omega}$, and hence to some $(G, \bar{\delta}G)$, $\bar{\delta} \in \bar{\Delta} \cup \{\xi\}$. These representations (all periodically $o$-primitive by Theorem 3) are of three kinds:

(1) $\bar{\delta} \in \bigcup \Sigma_k$, so that $G_{\bar{\delta}} = G_\omega$ for some $\omega \in \Omega$ and thus $(G, \bar{\delta}G)$ is $o$-isomorphic to $(G, \Omega)$.

(2) $\bar{\delta}$ has character $c_{\alpha\alpha}$, but $\bar{\delta} \notin \bigcup \Sigma_k$. $G_\xi \mid \Delta$ is the set of all $o$-permutations of $\Delta$ which preserve the $\Sigma_k$'s (by Theorem 17), so by Lemma 21, $\bar{\delta}G_\xi = \Delta_{\alpha\alpha} \backslash \bigcup \Sigma_k = (\Omega_{\alpha\alpha} \backslash \Pi) \cap \bar{\Delta}$, where $\Pi = \bigcup \Omega_k$. Hence $\bar{\delta}G \supseteqq (\Delta_{\alpha\alpha}G) \backslash (\Pi G) = \Omega_{\alpha\alpha} \backslash \Pi$. Since $\bar{\delta} \notin \Pi$ and $\Pi G = \Pi$, $\bar{\delta}G = \Omega_{\alpha\alpha} \backslash \Pi = \hat{\Omega}_{\alpha\alpha}$. $\Pi$ is a countably truncated $\alpha$-set, and $\hat{\Omega}_{\alpha\alpha} = \Pi_{\alpha\alpha}$, which is $o$-isomorphic to $\Omega_{\alpha\alpha}$. Now card $(\hat{\Omega}_{\alpha\alpha}) = 2^{\aleph_\alpha}$, whereas $\hat{\Omega}_{\alpha\alpha} = \Pi_{\alpha\alpha}$ and card $(\bar{\Pi}_{\alpha\alpha} \backslash \Pi_{\alpha\alpha}) = \aleph_\alpha$, forcing $(G, \hat{\Omega}_{\alpha\alpha})$ to have Config (1), for otherwise $\tilde{\Omega}_{\alpha\alpha}z$ would be disjoint from $\hat{\Omega}_{\alpha\alpha}$. Next, $(G, \hat{\Omega}_{\alpha\alpha})$ is full if and only if $G_\xi \mid (\bar{\Delta} \cap \hat{\Omega}_{\alpha\alpha})$ is entire (by Theorem 17); and since $\bar{\Delta} \cap \hat{\Omega}_{\alpha\alpha} = (\bar{\Delta} \cap \Pi)_{\alpha\alpha}$, entirety holds if and only if $G_\xi \mid (\bar{\Delta} \cap \Pi)$ is entire (by [7, Theorem 15]); which is the case if and only if $n = 1$.

(3) $\bar{\delta}$ has character $c_{\alpha\beta}$, $\beta < \alpha$. The argument used in (2) shows that $\bar{\delta}G = \Omega_{\alpha\beta}$. $(G, \Omega_{\alpha\beta})$ has Config (1), for $\Omega_{\alpha\beta}$ has no holes of character $c_{\alpha\beta}$; and the argument in (2) then shows that $(G, \Omega_{\alpha\beta})$ is not full.

(4) $\bar{\delta}$ has character $c_{\beta\alpha}$, $\beta < \alpha$. This case is dual to (3).

Finally, the chains for any two distinct representations in our list differ either in cardinality or in point character, and hence are not $o$-isomorphic.

PROPOSITION 23. *Let $\Delta$ be an $\alpha$-set and let $\Lambda = \Delta_{\alpha\alpha}$ or $\Lambda = \Delta \cup \Delta_{\alpha\alpha}$. (When $\alpha = 0$, $\Lambda$ is the irrationals or the reals.) Then there is a full periodically o-primitive group of* Config (1) *having $\Lambda$ as the first positive orbit of a stabilizer subgroup.*

*Proof.* $\Lambda$ is homogeneous ([7, Corollary 16]) and for any $\lambda \in \Lambda$,

$\{\beta \in \Lambda \mid \beta < \lambda\}$ and $\{\beta \in \Lambda \mid \beta > \lambda\}$ are both $o$-isomorphic to $\Lambda$. [6, Theorem 54] guarantees the existence of the desired group of Config (1). The proof of (2) in the previous theorem shows that any group having $\Lambda$ as the first positive orbit of a stabilizer subgroup must necessarily have Config (1).

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