

# Pacific Journal of Mathematics

**RADICAL PROPERTIES INVOLVING ONE-SIDED IDEALS**

R. F. ROSSA

## RADICAL PROPERTIES INVOLVING ONE-SIDED IDEALS

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A radical  $P$  is called **strongly right hereditary (srh)** if  $P(I) = I \cap P(R)$  for every right ideal  $I$  of each (not necessarily associative) ring  $R$  in a suitable universal class  $W$ . This is a one-sided version of the concept of a strongly hereditary radical class investigated by W.G. Leavitt and R.L. Tangeman. A discussion parallel to theirs is obtained including a construction of the minimal srh radical class in  $W$  containing a given class. Srh radicals are related to a new radical construction obtained by modifying the lower radical construction of Tangeman and D. Kreiling.

1. Introduction. A class  $M$  of not necessarily associative rings is called right hereditary if every right ideal of each ring in  $M$  is also in  $M$ . Subring hereditary classes are defined in a corresponding way. A universal class is a homomorphically closed, subring hereditary class of rings. A radical  $P$  of some universal class  $W$  is strongly hereditary if for all  $R \in W$  we have  $P(I) = I \cap P(R)$  for all ideals  $I$  of  $R$ , and strongly right hereditary (srh) if we have the same property for all right ideals  $I$  of  $R$ . Strongly hereditary radicals have been studied by W. G. Leavitt [4] and R. L. Tangeman [6] using the following property (a) which may be satisfied by a class  $M$  of rings in a universal class  $W$ :

(a) If  $J \in M$  is an ideal of an ideal  $I$  of some  $R \in W$ , then the ideal  $J'$  of  $R$  generated by  $J$  is also in  $M$ . In § 2, we obtain a parallel discussion of srh radicals using the following modification of (a):

( $\rho$ ) If  $J \in M$  is an ideal of a right ideal  $I$  of  $R \in W$ , then the ideal  $J'$  of  $R$  generated by  $J$  is also in  $M$ .

In a universal class  $W$ , the lower radical determined by a class  $M$  will be denoted by  $LM$ . In § 3, we introduce a new radical construction obtained by altering the construction of  $LM$  given by Tangeman and Kreiling [3] at the limit ordinal step. A brief summary of their construction may be found in [5], whose notation we will continue to use. Our construction is related to property ( $\rho$ ) by Theorem 3.2.

For a class  $M \subseteq W$ , the minimal right hereditary subclass of  $W$  containing  $M$  will be denoted by  $GM$ . Write  $G_1M = M$  and, for  $n \geq 2$ ,  $G_nM = \{R \in W : R \text{ is a right ideal of some ring in } G_{n-1}M\}$ . Then  $GM = \bigcup G_nM$ , as in [5]. If  $M = \{R\}$  consists of a single ring, we will omit braces and write, for example,  $G_nM = G_nR$ .

2. Srh radicals. The results of [4] and [6] all have one-sided

versions. In particular, following [4, Theorem 1], we have.

**THEOREM 2.1.** *A right hereditary radical class  $P \subseteq W$  is srh if and only if it has property  $(\rho)$ .*

Next we show that property  $(\rho)$  is inherited by the lower radical. Our proof is an adaptation of an unpublished proof by Tangeman of [6, Theorem 2.4].

**THEOREM 2.2.** *Suppose  $M \subseteq W$  is homomorphically closed and has property  $(\rho)$ . Then  $LM$  also satisfies  $(\rho)$ .*

*Proof.* We will use the construction of  $LM$  due to Tangeman and Kreiling and the notation of [5]. By hypothesis  $M_1 = M$  has property  $(\rho)$ . Let  $\beta > 1$  be an ordinal number and let  $J$  be an ideal of a right ideal  $I$  of a ring  $R \in W$  such that  $J \in M_\beta$ . Let  $J'$  denote the ideal of  $R$  generated by  $J$ . Suppose the classes  $M_\alpha$  satisfy  $(\rho)$  for all  $\alpha < \beta$ .

First suppose  $\beta$  is a limit ordinal. Then  $J = \bigcup J_\gamma$ , where  $\{J_\gamma\}$  is a chain of ideals of  $J$  contained in  $\bigcup_{\alpha < \beta} M_\alpha$ . For each index  $\gamma$ , let  $K_\gamma$  be the ideal of  $I$  generated by  $J_\gamma$ . Then  $J = \bigcup K_\gamma$ . By property  $(\rho)$ , each  $K_\gamma \in \bigcup_{\alpha < \beta} M_\alpha$ . Now let  $K'_\gamma$  be the ideal of  $R$  generated by  $K_\gamma$ . By  $(\rho)$  again, each  $K'_\gamma \in \bigcup_{\alpha < \beta} M_\alpha$ . Since  $J' \supseteq K_\gamma$  for each  $\gamma$  and  $J'$  is an ideal of  $R$ ,  $J' \supseteq \bigcup K'_\gamma$ . On the other hand, since  $\bigcup K'_\gamma$  is an ideal of  $R$  containing  $\bigcup K_\gamma = J$ , we have  $\bigcup K'_\gamma \supseteq J'$ . Hence  $J' = \bigcup K'_\gamma \in M_\beta$ .

If  $\beta$  is not a limit ordinal, then  $J$  has an ideal  $K$  with  $K, J/K \in M_{\beta-1}$ . Now if  $P \subseteq J$  is the ideal of  $I$  generated by  $K$ , then  $P \in M_{\beta-1}$  by property  $(\rho)$ . Moreover,  $J/P \in M_{\beta-1}$  because  $J/P$  is a homomorphic image of  $J/K$  and  $M_{\beta-1}$  is homomorphically closed [3, Lemma 2]. Now  $P$  generates an ideal  $Q$  of  $R$  with  $Q \in M_{\beta-1}$  by the inductive hypothesis. The ideal of  $R/Q$  generated by  $J + Q/Q$  is  $J'/Q$ . Since  $P \subseteq J \cap Q$ ,  $J + Q/Q \simeq J/J \cap Q$  is a homomorphic image of  $J/P$ . Hence  $J + Q/Q \in M_\beta$  and so, using  $(\rho)$  again,  $J'/Q \in M_{\beta-1}$ . Since  $Q, J'/Q \in M_{\beta-1}$ , we have  $J' \in M_\beta$ . The theorem follows by transfinite induction.

Let  $EM = \{J': J \text{ is an ideal of a right ideal of a ring } R \in W, J \in M, \text{ and } J' \text{ is the ideal of } R \text{ generated by } J\}$ . The homomorphic closure of  $M$  will be denoted by  $HM$ . We have the following one-sided version of [6, Corollary to Theorem 2.5].

**THEOREM 2.3.** *If  $W$  is a universal class and  $M \subseteq W$ , then there exists a unique minimal radical class in  $W$  containing  $M$  and satisfying property  $(\rho)$ .*

*Proof.* Let  $M_1^* = EHM$  and, inductively,  $M_n^* = EHM_{n-1}^*$  for all integers  $n > 1$ . Then  $M^* = \bigcup M_n^*$  is a homomorphically closed class satis-

fying  $(\rho)$ , so that  $LM^*$  satisfies  $(\rho)$  by Theorem 2.2. On the other hand, any radical class which satisfies property  $(\rho)$  and contains  $M$  may be seen by induction to contain  $M^*$  and hence  $LM^*$ .

EXAMPLE 2.1. The class  $LM^*$  need not be hereditary when  $M$  is hereditary. For let  $K = GF(2)$  and let  $R = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} : a, b \in K \right\}$ . We identify isomorphic rings; thus  $K \simeq \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in K \right\}$  is a right ideal of  $R$  with  $K' = R$ .  $R$  has the ideal  $I = \left\{ \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} : b \in K \right\}$ . Let  $W = \{R, K, I, 0\}$ ,  $M = \{K, 0\}$ . Then  $M$  is hereditary, while  $LM^* = M^* = \{R, K, 0\}$  is not.

As in [6, Corollary (c)] and [8, Corollary 2.7], we also have

THEOREM 2.4. *If  $W$  is a universal class and  $M \subseteq W$ , then there is a unique minimal srh radical class in  $W$  containing  $M$ .*

*Proof.* Define  $\bar{M}_1 = EGHM$  and, for all  $n > 1$ ,  $\bar{M}_n = EGH\bar{M}_{n-1}$ . Then  $\bar{M} = \bigcup \bar{M}_n$  is the minimal subclass of  $W$  which contains  $M$ , satisfies property  $(\rho)$  and is right hereditary and homomorphically closed. Then  $L\bar{M}$  is the desired srh radical class, for by Theorem 2.2  $L\bar{M}$  has property  $(\rho)$  and by [5, Theorem 2],  $L\bar{M}$  is right hereditary. Thus by Theorem 2.1,  $L\bar{M}$  is srh; it is again easy to see that  $L\bar{M}$  is minimal.

We turn to a consideration of two properties similar to property  $(\rho)$ .

THEOREM 2.5. *Let  $M$  be a class of rings satisfying property  $(\rho)$ . For all  $R \in W$ , if  $I \in M$  is in  $GR$ , then the ideal  $I'$  of  $R$  generated by  $I$  is also in  $M$ .*

*Proof.* The theorem is trivially satisfied when  $I \in M \cap G_1R$ . Thus for induction assume for all  $R \in W$  and all  $I \in M \cap G_nR$  that  $I' \in M$ , where  $I'$  is the ideal of  $R$  generated by  $I$ . Let  $K \in M \cap G_{n+1}R$  so that  $K \in M \cap G_nJ$  for some right ideal  $J$  of  $R$ . By induction  $K^* \in M$  where  $K^*$  is the ideal of  $J$  generated by  $K$ . But then  $(\rho)$  implies  $K' = K^* \in M$ .

COROLLARY. *Property  $(\rho)$  is equivalent to the following property  $(\rho')$ :*

$(\rho')$  *If  $J \in M$  is a right ideal of a right ideal  $I$  of  $R \in W$ , then the ideal  $J'$  of  $R$  generated by  $J$  is also in  $M$ .*

Consider the following property  $(\sigma)$ : If  $J \in M$  is a right ideal of  $R$ , then the ideal of  $J'$  of  $R$  generated by  $J$  is also in  $M$ . In general this property is not inherited by  $LM$  as may be seen from the following example (for which we thank the referee).

EXAMPLE 2.2. Let  $K$  be generated over  $GF(2)$  by  $x, y, z$  where  $x^2 = y^2 = 0, xy = yx = x, yz = xz = zy = y$ , and  $zx = z^2 = z$ . Then  $I = \{0, x\}$  is an ideal of  $R = \{0, x, y, x + y\}$  and  $R$  is the only proper right ideal of  $K$ . Also  $K$  is simple so that  $R' = K$ . For the universal class  $\mathcal{W}$  consisting of  $K$  and all its subrings, the class  $\mathcal{M} = \{0, I\}$  has property  $(\sigma)$ . However,  $LM$  does not have the property since  $R \in LM$  whereas  $R' = K \notin LM$ .

For semisimple classes, we have the following one-sided version of [1, Theorem 4.1] and [6, Theorem 3.1], which we state without proof.

THEOREM 2.6.  $\mathcal{Q}$  is a semisimple class for a radical class  $\mathcal{P}$  with property  $(\rho)$  if and only if  $\mathcal{Q}$  has properties (b), (c), and (d) of [1, Theorem 4.1] and is right hereditary.

In general it cannot be expected that semisimple subideals will generate semisimple ideals, as in property (a). Indeed, if the radical class is not hereditary, a semisimple subideal may even generate a radical ideal. We give two examples using well-known radicals in the universal class of associative rings.

EXAMPLE 2.3. Let  $A$  be a ring isomorphic to the ring of even integers with generator  $a$ . Let  $B = \{0, x\}, C = \{0, y\}$  be zero rings of order two. Let  $I = A \oplus B$ , and form  $R$  by adjoining  $C$  to  $I$  in such a way that the additive group of  $R$  is  $I + C$  (direct sum),  $(na)y = y(na) = nx$  for all integers  $n$ , and  $xy = yx = 0$ . Then  $I$  is an ideal of  $R$  and  $A$  is a nil-semisimple ideal of  $I$ , but  $A' = I$  has the nil ideal  $B$ .

EXAMPLE 2.4. Let  $A$  be the zero ring whose additive group is  $Z_p(\infty)$  and let  $B$  be the ring of polynomials of degree  $\geq 1$  over  $GF(2)$ . Define the commutative ring  $R$  as follows. The additive group of  $R$  is the direct sum  $A + B$ ; the multiplication within  $A$  and  $B$  is as usual, and we define  $(a/p^n)x^i = a/p^{n+i}$ , extending this multiplication to  $R$  in the natural way.

Let  $I$  be the subring of  $A$  of order  $p$ . Thus  $I$  is an ideal of  $A$ , and the ideal  $I$  generates in  $R$  is  $A$ . In the upper radical of the class of all simple rings (see [2, page 14]),  $I$  is semisimple and  $A$  is radical.

3. Radical constructions involving one-sided ideals. Let  $\mathcal{M}$  be any class contained in a universal class  $\mathcal{W}$ . We will construct a class  $\Delta\mathcal{M}$  (depending of course on the universal class  $\mathcal{W}$ ) by modifying the radical construction of [3]. Briefly, let  $\Delta_1\mathcal{M}$  be the homomorphic closure of  $\mathcal{M}$ . We proceed inductively to define a class  $\Delta_\beta\mathcal{M}$  for each ordinal number  $\beta$ . If  $\beta - 1$  exists, let  $\Delta_\beta\mathcal{M} = \{R \in \mathcal{W}: R \text{ has an ideal } J \text{ such that } J, R/J \in \Delta_{\beta-1}\mathcal{M}\}$ . If  $\beta$  is a limit ordinal, define  $R \in$

$\Delta_\beta M$  if and only if  $R$  is the union of a chain  $\{I_\gamma\}$  of right ideals of  $R$  such that each  $I_\gamma \in \bigcup_{\alpha < \beta} \Delta_\alpha M$ . Finally, let  $\Delta M = \bigcup_\beta \Delta_\beta M$ .

By modifying suitably the proof of [3, Theorem 2] we have

**THEOREM 3.1.**  *$\Delta M$  is a radical class.*

The corresponding construction using left ideals yields a radical class we will call  $LM$ .

**THEOREM 3.2.** *If  $M$  is homomorphically closed and has property  $(\rho)$ , then  $LM = \Delta M$ .*

*Proof.* Since  $M \subseteq \Delta M$  and  $\Delta M$  is radical,  $LM \subseteq \Delta M$ . Thus assume for induction that, for  $\beta$  a given ordinal,  $\Delta_\alpha M \subseteq LM$  for all  $\alpha < \beta$ . If  $R \in M_\beta$  is a nonlimit ordinal then  $I, R/I \in \Delta_{\beta-1} M \subseteq LM$ , so that  $R \in LM$ . If  $\beta$  is a limit ordinal then  $R = I_\gamma$  for some chain  $\{I_\gamma\}$  of right ideals contained in  $\bigcup_{\alpha < \beta} \Delta_\alpha M \subseteq LM$ . But by Theorem 2.2,  $LM$  has property  $(\rho)$ . Thus if  $I'_\gamma$  is the ideal of  $R$  generated by  $I_\gamma$ , then  $I'_\gamma \in LM$  and so  $R = \bigcup I'_\gamma \in LM$ . Thus  $\Delta_\beta M \subseteq LM$  and so  $\Delta M \subseteq LM$ .

This is not a necessary condition, for let  $M$  be the nil radical class in the universal class of associative rings. Then  $M = LM = \Delta M$ , but  $M$  does not have property  $(\rho)$  by Theorem 2.6 because the nil-semisimple rings do not form a right hereditary class.

Even in the associative case,  $\Delta M$  and  $LM$  may be unequal.

**EXAMPLE 3.1.** Let  $R$  be the associative algebra over the field  $GF(2)$  generated by a countable number of symbols  $\{x_i: i = 1, 2, \dots\}$  subject to the relations  $x_i x_j = x_j$  for all  $i, j$ . For each  $n$ , let  $I_n$  be the left ideal generated by  $\{x_1, \dots, x_n\}$ . Then  $M = \{I_n: n = 1, 2, \dots\}$  is a chain of left ideals of  $R$  and  $R = I_n$ , so that  $R \in LM$ . Since  $R$  has no proper right ideals and  $R \notin M_1 = \Delta_1 M$ ,  $R$  cannot be in  $\Delta M$ .

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