RADICAL PROPERTIES INVOLVING ONE-SIDED IDEALS

R. F. Rossa
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A radical $P$ is called strongly right hereditary (srh) if $P(I) = I \cap P(R)$ for every right ideal $I$ of each (not necessarily associative) ring $R$ in a suitable universal class $W$. This is a one-sided version of the concept of a strongly hereditary radical class investigated by W.G. Leavitt and R.L. Tangeman. A discussion parallel to theirs is obtained including a construction of the minimal srh radical class in $W$ containing a given class. Srh radicals are related to a new radical construction obtained by modifying the lower radical construction of Tangeman and D. Kreiling.

1. Introduction. A class $M$ of not necessarily associative rings is called right hereditary if every right ideal of each ring in $M$ is also in $M$. Subring hereditary classes are defined in a corresponding way. A universal class is a homomorphically closed, subring hereditary class of rings. A radical $P$ of some universal class $W$ is strongly hereditary if for all $R \in W$ we have $P(I) = I \cap P(R)$ for all ideals $I$ of $R$, and strongly right hereditary (srh) if we have the same property for all right ideals $I$ of $R$. Strongly hereditary radicals have been studied by W. G. Leavitt [4] and R. L. Tangeman [6] using the following property (a) which may be satisfied by a class $M$ of rings in a universal class $W$:

(a) If $J \in M$ is an ideal of an ideal $I$ of some $R \in W$, then the ideal $J'$ of $R$ generated by $J$ is also in $M$. In § 2, we obtain a parallel discussion of srh radicals using the following modification of (a):

(ρ) If $J \in M$ is an ideal of a right ideal $I$ of $R \in W$, then the ideal $J'$ of $R$ generated by $J$ is also in $M$.

In a universal class $W$, the lower radical determined by a class $M$ will be denoted by $LM$. In § 3, we introduce a new radical construction obtained by altering the construction of $LM$ given by Tangeman and Kreiling [3] at the limit ordinal step. A brief summary of their construction may be found in [5], whose notation we will continue to use. Our construction is related to property (ρ) by Theorem 3.2.

For a class $M \subseteq W$, the minimal right hereditary subclass of $W$ containing $M$ will be denoted by $GM$. Write $G_0M = M$ and, for $n \geq 2$, $G_nM = \{R \in W: R \text{ is a right ideal of some ring in } G_{n-1}M\}$. Then $GM = \bigcup G_nM$, as in [5]. If $M = \{R\}$ consists of a single ring, we will omit braces and write, for example, $G_nM = G_nR$.

2. Srh radicals. The results of [4] and [6] all have one-sided
versions. In particular, following [4, Theorem 1], we have.

**THEOREM 2.1.** A right hereditary radical class $P \subseteq W$ is srh if and only if it has property $(\rho)$.

Next we show that property $(\rho)$ is inherited by the lower radical. Our proof is an adaptation of an unpublished proof by Tangeman of [6, Theorem 2.4].

**THEOREM 2.2.** Suppose $M \subseteq W$ is homomorphically closed and has property $(\rho)$. Then $LM$ also satisfies $(\rho)$.

*Proof.* We will use the construction of $LM$ due to Tangeman and Kreiling and the notation of [5]. By hypothesis $M_\beta = M$ has property $(\rho)$. Let $\beta > 1$ be an ordinal number and let $J$ be an ideal of a right ideal $I$ of a ring $R \in W$ such that $J \in M_\beta$. Let $J'$ denote the ideal of $R$ generated by $J$. Suppose the classes $M_\alpha$ satisfy $(\rho)$ for all $\alpha < \beta$.

First suppose $\beta$ is a limit ordinal. Then $J = \bigcup J_\gamma$, where $\{J_\gamma\}$ is a chain of ideals of $J$ contained in $\bigcup_{\alpha < \beta} M_\alpha$. For each index $\gamma$, let $K_\gamma$ be the ideal of $R$ generated by $J_\gamma$. Then $J = \bigcup K_\gamma$. By property $(\rho)$, each $K_\gamma \in \bigcup_{\alpha < \beta} M_\alpha$. Now let $K' \in \bigcup_{\alpha < \beta} M_\alpha$. Since $J' \supseteq K$ for each $\gamma$ and $J'$ is an ideal of $R$, $J' \supseteq \bigcup K'$. On the other hand, since $\bigcup K'$ is an ideal of $R$ containing $\bigcup K_\gamma = J$, we have $\bigcup K' \supseteq J'$. Hence $J' = \bigcup K'_\gamma \in M_\beta$.

If $\beta$ is not a limit ordinal, then $J$ has an ideal $K$ with $J/K \in M_{\beta-1}$. Now if $P \subseteq J$ is the ideal of $I$ generated by $K$, then $P \in M_{\beta-1}$ by property $(\rho)$. Moreover, $J/P \in M_{\beta-1}$ because $J/P$ is a homomorphic image of $J/K$ and $M_{\beta-1}$ is homomorphically closed [3, Lemma 2]. Now $P$ generates an ideal $Q$ of $R$ with $Q \in M_{\beta-1}$ by the inductive hypothesis. The ideal of $R/Q$ generated by $J + Q/Q$ is $J'/Q$. Since $P \subseteq J \cap Q$, $J + Q/Q \cong J/J \cap Q$ is a homomorphic image of $J/P$. Hence $J + Q/Q \in M_\beta$ and so, using $(\rho)$ again, $J'/Q \in M_{\beta-1}$. Since $Q$, $J'/Q \in M_{\beta-1}$, we have $J' \in M_\beta$. The theorem follows by transfinite induction.

Let $EM = \{J': J$ is an ideal of a right ideal of a ring $R \in W$, $J \in M$, and $J'$ is the ideal of $R$ generated by $J\}$. The homomorphic closure of $M$ will be denoted by $HM$. We have the following one-sided version of [6, Corollary to Theorem 2.5].

**THEOREM 2.3.** If $W$ is a universal class and $M \subseteq W$, then there exists a unique minimal radical class in $W$ containing $M$ and satisfying property $(\rho)$.

*Proof.* Let $M^*_\alpha = EHM$ and, inductively, $M^*_n = EHM^*_n$ for all integers $n > 1$. Then $M* = UM^*_n$ is a homomorphically closed class satis-
fying (\(\rho\)), so that \(LM^*\) satisfies (\(\rho\)) by Theorem 2.2. On the other hand, any radical class which satisfies property (\(\rho\)) and contains \(M\) may be seen by induction to contain \(M^*\) and hence \(LM^*\).

**Example 2.1.** The class \(LM^*\) need not be hereditary when \(M\) is hereditary. For let \(K = GF(2)\) and let \(R = \{ (\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}) : a, b \in K \}\). We identify isomorphic rings; thus \(K \simeq \{ (\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}) : a \in K \}\) is a right ideal of \(R\) with \(K' = R\). \(R\) has the ideal \(I = \{ (\begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}) : b \in K \}\). Let \(W = \{R, K, I, 0\}\), \(M = (K, 0)\). Then \(M\) is hereditary, while \(LM^* = M^* = \{R, K, 0\}\) is not.

As in [6, Corollary (c)] and [8, Corollary 2.7], we also have

**Theorem 2.4.** If \(W\) is a universal class and \(M \subseteq W\), then there is a unique minimal srh radical class in \(W\) containing \(M\).

*Proof.* Define \(\bar{M}_1 = EGHM\) and, for all \(n > 1\), \(\bar{M}_n = EGH\bar{M}_{n-1}\). Then \(\bar{M} = \bigcup \bar{M}_n\) is the minimal subclass of \(W\) which contains \(M\), satisfies property (\(\rho\)) and is right hereditary and homomorphically closed. Then \(LM\) is the desired srh radical class, for by Theorem 2.2 \(LM\) has property (\(\rho\)) and by [5, Theorem 2], \(LM\) is right hereditary. Thus by Theorem 2.1, \(LM\) is srh; it is again easy to see that \(LM\) is minimal.

We turn to a consideration of two properties similar to property(\(\rho\)).

**Theorem 2.5.** Let \(M\) be a class of rings satisfying property (\(\rho\)). For all \(R \in W\), if \(I \in M\) is in \(GR\), then the ideal \(I'\) of \(R\) generated by \(I\) is also in \(M\).

*Proof.* The theorem is trivially satisfied when \(I \in M \cap G, R\). Thus for induction assume for all \(R \in W\) and all \(I \in M \cap G, R\) that \(I' \in M\), where \(I'\) is the ideal of \(R\) generated by \(I\). Let \(K \in M \cap G_{a+1}, R\) so that \(K \in M \cap G, J\) for some right ideal \(J\) of \(R\). By induction \(K^* \in M\) where \(K^*\) is the ideal of \(J\) generated by \(K\). But then (\(\rho\)) implies \(K' = K^{**} \in M\).

**Corollary.** Property (\(\rho\)) is equivalent to the following property (\(\rho'\)):

(\(\rho'\)) If \(J \in M\) is a right ideal of a right ideal \(I\) of \(R \in W\), then the ideal \(J'\) of \(R\) generated by \(J\) is also in \(M\).

Consider the following property (\(\sigma\)): If \(J \in M\) is a right ideal of \(R\), then the ideal of \(J'\) of \(R\) generated by \(J\) is also in \(M\). In general this property is not inherited by \(LM\) as may be seen from the following example (for which we thank the referee).
EXAMPLE 2.2. Let $K$ be generated over $GF(2)$ by $x, y, z$ where $x^2 = y^2 = 0, xy = yx = x, yz = zx = zy = y$, and $zx = z^2 = z$. Then $I = \{0, x\}$ is an ideal of $R = \{0, x, y, x + y\}$ and $R$ is the only proper right ideal of $K$. Also $K$ is simple so that $R' = K$. For the universal class $W$ consisting of $K$ and all its subrings, the class $M = \{0, I\}$ has property (a). However, $LM$ does not have the property since $R \in LM$ whereas $R' = K \in LM$.

For semisimple classes, we have the following one-sided version of [1, Theorem 4.1] and [6, Theorem 3.1], which we state without proof.

**Theorem 2.6.** $Q$ is a semisimple class for a radical class $P$ with property (a) if and only if $Q$ has properties (b), (c), and (d) of [1, Theorem 4.1] and is right hereditary.

In general it cannot be expected that semisimple subideals will generate semisimple ideals, as in property (a). Indeed, if the radical class is not hereditary, a semisimple subideal may even generate a radical ideal. We give two examples using well-known radicals in the universal class of associative rings.

**Example 2.3.** Let $A$ be a ring isomorphic to the ring of even integers with generator $a$. Let $B = \{0, x\}, C = \{0, y\}$ be zero rings of order two. Let $I = A \oplus B$, and form $R$ by adjoining $C$ to $I$ in such a way that the additive group of $R$ is $I + C$ (direct sum), $(na)y = y(na) = nx$ for all integers $n$, and $xy = yx = 0$. Then $I$ is an ideal of $R$ and $A$ is a nil-semisimple ideal of $I$, but $A' = I$ has the nil ideal $B$.

**Example 2.4.** Let $A$ be the zero ring whose additive group is $Z_{p(q)}$ and let $B$ be the ring of polynomials of degree $> 1$ over $GF(2)$. Define the commutative ring $R$ as follows. The additive group of $R$ is the direct sum $A + B$; the multiplication within $A$ and $B$ is as usual, and we define $(a/p^n)x^i = a/p^{n+i}$, extending this multiplication to $R$ in the natural way.

Let $I$ be the subring of $A$ of order $p$. Thus $I$ is an ideal of $A$, and the ideal $I$ generates in $R$ is $A$. In the upper radical of the class of all simple rings (see [2, page 14]), $I$ is semisimple and $A$ is radical.

3. Radical constructions involving one-sided ideals. Let $M$ be any class contained in a universal class $W$. We will construct a class $\Delta M$ (depending of course on the universal class $W$) by modifying the radical construction of [3]. Briefly, let $\Delta_\beta M$ be the homomorphic closure of $M$. We proceed inductively to define a class $\Delta_\beta M$ for each ordinal number $\beta$. If $\beta = 1$ exists, let $\Delta_\beta M = \{R \in W: R$ has an ideal $J$ such that $J, R/J \in \Delta_{\beta-1} M\}$. If $\beta$ is a limit ordinal, define $R \in \Delta_\beta M$. For semisimple classes, we have the following one-sided version of [1, Theorem 4.1] and [6, Theorem 3.1], which we state without proof.

**Theorem 2.6.** $Q$ is a semisimple class for a radical class $P$ with property (a) if and only if $Q$ has properties (b), (c), and (d) of [1, Theorem 4.1] and is right hereditary.

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$\Delta M$ if and only if $R$ is the union of a chain $\{I_\gamma\}$ of right ideals of $R$ such that each $I_\gamma \in \bigcup_{\alpha < \beta} \Delta_\alpha M$. Finally, let $\Delta M = \bigcup_\beta \Delta_\beta M$.

By modifying suitably the proof of [3, Theorem 2] we have

**Theorem 3.1.** $\Delta M$ is a radical class.

The corresponding construction using left ideals yields a radical class we will call $\Lambda M$.

**Theorem 3.2.** If $M$ is homomorphically closed and has property $(\rho)$, then $LM = \Delta M$.

*Proof.* Since $M \subseteq \Delta M$ and $\Delta M$ is radical, $LM \subseteq \Delta M$. Thus assume for induction that, for $\beta$ a given ordinal, $\Delta_\alpha M \subseteq LM$ for all $\alpha < \beta$. If $R \in M_\beta$ is a nonlimit ordinal then $I, R/I \in \Delta_\alpha M \subseteq LM$, so that $R \in LM$. If $\beta$ is a limit ordinal then $R = I_\gamma$ for some chain $\{I_\gamma\}$ of right ideals contained in $\bigcup_{\alpha < \beta} \Delta_\alpha M \subseteq LM$. But by Theorem 2.2, $LM$ has property $(\rho)$. Thus if $I_\gamma$ is the ideal of $R$ generated by $I_\gamma$, then $I_\gamma \in LM$ and so $R = \bigcup I_\gamma \in LM$. Thus $\Delta_\beta M \subseteq LM$ and so $\Delta M \subseteq LM$.

This is not a necessary condition, for let $M$ be the nil radical class in the universal class of associative rings. Then $M = LM = \Delta M$, but $M$ does not have property $(\rho)$ by Theorem 2.6 because the nilsemisimple rings do not form a right hereditary class.

Even in the associative case, $\Delta M$ and $\Lambda M$ may be unequal.

**Example 3.1.** Let $R$ be the associative algebra over the field $GF(2)$ generated by a countable number of symbols $\{x_i; i = 1, 2, \ldots\}$ subject to the relations $x_i x_j = x_j$ for all $i, j$. For each $n$, let $I_n$ be the left ideal generated by $\{x_1, \ldots, x_n\}$. Then $M = \{I_n; n = 1, 2, \ldots\}$ is a chain of left ideals of $R$ and $R = I_n$, so that $R \in \Delta M$. Since $R$ has no proper right ideals and $R \notin M_1 = \Delta_1 M$, $R$ cannot be in $\Delta M$.

**References**


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