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# ON THE KONHAUSER SETS OF BIORTHOGONAL POLYNOMIALS SUGGESTED BY THE LAGUERRE POLYNOMIALS

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**Recently, Joseph D. E. Konhauser discussed two polynomial sets $\{Y_n^\alpha(x; k)\}$ and $\{Z_n^\alpha(x; k)\}$, which are biorthogonal with respect to the weight function $x^\alpha e^{-x}$ over the interval $(0, \infty)$, where $\alpha > -1$ and $k$ is a positive integer. For the polynomials $Y_n^\alpha(x; k)$, the following bilateral generating function is derived in this paper:**

$$\sum_{n=0}^{\infty} Y_n^\alpha(x; k)\zeta_n(y)t^n = (1-t)^{-(\alpha+1)/k} \exp\{x[1-(1-t)^{-1/k}]\} \cdot G[x(1-t)^{-1/k}, \quad yt/(1-t)] , \tag{1}$$

**where**

$$G[x, t] = \sum_{n=0}^{\infty} \lambda_n Y_n^\alpha(x; k)t^n , \tag{2}$$

**the $\lambda_n \neq 0$ are arbitrary constants, and $\zeta_n(y)$ is a polynomial of degree $n$ in $y$ given by**

$$\zeta_n(y) = \sum_{r=0}^{n} \binom{n}{r} \lambda_r y^r . \tag{3}$$

**It is also shown that the polynomials $Z_n^\alpha(x; k)$ can be expressed as a finite sum of $Z_n^\alpha(y; k)$ in the form**

$$Z_n^\alpha(x; k) = \left(\frac{x}{y}\right)^{kn} \sum_{r=0}^{n} \binom{\alpha+kn}{kr} \frac{(kr)!}{r!} [(y/x)^k - 1]^r Z_{n-r}^\alpha(y; k) . \tag{4}$$

For $k = 2$, formulas (1) and (4) yield corresponding properties for the polynomials introduced earlier by Preiser [4]. Moreover, when $k=1$, both (1) and (4) would reduce to similar results involving the generalized Laguerre polynomials $L_n^\alpha(x)$. For results analogous to (1) and (4), involving certain classes of functions, the reader may be referred to our papers [5] and [6], respectively.

2. The following results will be required in our analysis.

(i) The generating function [3, p. 803]:

$$\sum_{n=0}^{\infty} \binom{m+n}{n} Y_{m+n}^\alpha(x; k)t^n = (1-t)^{-(\alpha+mk+1)/k} \exp\{x[1-(1-t)^{-1/k}]\} Y_m^\alpha(x(1-t)^{-1/k}; k) , \tag{5}$$

where $m$ is any integer $\geqq 0$.

(ii) The explicit expression for $Z_n^\alpha(x; k)$:

$$Z_n^\alpha(x;k) = \frac{\Gamma(\alpha+kn+1)}{n!}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{x^{kj}}{\Gamma(\alpha+kj+1)}, \tag{6}$$

which is equation (5), p. 304 of Konhauser [2].

From (6) it follows fairly easily that

$$\begin{aligned}&\sum_{n=0}^{\infty} Z_n^\alpha(x;k)\frac{t^n}{(\alpha+1)_{kn}}\\ &= e^t{}_0F_k[-;(\alpha+1)/k,\cdots,(\alpha+k)/k;-(x/k)^k t],\end{aligned} \tag{7}$$

since $k$ is a positive integer.

**3. Proof of the bilateral generating function (1).** Substituting for the coefficients $\zeta_n(y)$ from (3) on the left-hand side of (1), we find that

$$\begin{aligned}\sum_{n=0}^{\infty} Y_n^\alpha(x;k)\zeta_n(y)t^n &= \sum_{n=0}^{\infty} Y_n^\alpha(x;k)t^n\sum_{r=0}^{n}\binom{n}{r}\lambda_r y^r\\ &= \sum_{r=0}^{\infty}\lambda_r(yt)^r\sum_{n=0}^{\infty}\binom{n+r}{r}Y_{n+r}^\alpha(x;k)t^n\\ &= (1-t)^{-(\alpha+1)/k}\exp\{x[1-(1-t)^{-1/k}]\}\\ &\quad\cdot\sum_{r=0}^{\infty}\lambda_r Y_r^\alpha(x(1-t)^{-1/k};k)(yt/(1-t))^r,\end{aligned}$$

by applying (5), and formula (1) would follow if we interpret this last expression by means of (2).

**4. Proof of the summation formula (4).** In the generating function (7), if we set $t=(y/k)^k z$, we shall get

$$\sum_{n=0}^{\infty} Z_n^\alpha(x;k)\frac{(y/k)^{kn}z^n}{(\alpha+1)_{kn}} = \exp\{(y/k)^k z\}{}_0F_k[-(xy/k^2)^k z], \tag{8}$$

which, on interchanging $x$ and $y$, gives us

$$\sum_{n=0}^{\infty} Z_n^\alpha(y;k)\frac{(x/k)^{kn}z^n}{(\alpha+1)_{kn}} = \exp\{(x/k)^k z\}{}_0F_k[-(xy/k^2)^k z], \tag{9}$$

where, for convenience,

$${}_0F_k[\xi] \equiv {}_0F_k[-;(\alpha+1)/k,\cdots,(\alpha+k)/k;\xi]. \tag{10}$$

From (8) and (9) it follows at once that

$$\begin{aligned}&\sum_{n=0}^{\infty} Z_n^\alpha(x;k)\frac{(y/k)^{kn}z^n}{(\alpha+1)_{kn}}\\ &= \exp\{z[(y/k)^k-(x/k)^k]\}\sum_{n=0}^{\infty} Z_n^\alpha(y;k)\frac{(x/k)^{kn}z^n}{(\alpha+1)_{kn}},\end{aligned} \tag{11}$$

and on equating coefficients of $z^n$ in (11), we shall be led to our summation formula (4).

**5. Applications.** First of all we notice that formula (4) may be rewritten as

$$Z_n^\alpha(\mu x; k) = \sum_{r=0}^{n} \binom{\alpha + kn}{kr} \frac{(kr)!}{r!} \mu^{k(n-r)}(1 - \mu^k)^r Z_{n-r}^\alpha(x; k) , \tag{12}$$

which provides us with a multiplication formula for the polynomials $Z_n^\alpha(x; k)$.

On the other hand, by assigning suitable values to the arbitrary coefficients $\lambda_n$ it is fairly straightforward to obtain, from our formula (1), a large variety of bilateral generating functions for the polynomials $Y_n^\alpha(x; k)$. For instance, if we let

$$\lambda_n = \frac{(-1)^n}{\Gamma(\beta + ln + 1)} , \quad n = 0, 1, 2, \cdots; l = 1, 2, 3, \cdots; \tag{13}$$

and make use of the definition (6), we shall readily arrive at the bilateral generating function

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{n!}{\Gamma(\beta + ln + 1)} Y_n^\alpha(x; k) Z_n^\beta(y; l) t^n \\ &= (1 - t)^{-(\alpha+1)/k} \exp\{x[1 - (1 - t)^{-1/k}]\} H[x(1 - t)^{-1/k}, -y^l t/(1 - t)] , \end{aligned} \tag{14}$$

where, for convenience,

$$H[x, t] = \sum_{n=0}^{\infty} Y_n^\alpha(x; k) \frac{t^n}{\Gamma(\beta + ln + 1)} . \tag{15}$$

For $k = l = 1$ and $\alpha = \beta$, the generating relation (14) would evidently reduce to the well-known Hille-Hardy formula for the Laguerre polynomials.

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