

# Pacific Journal of Mathematics

**ON THE KONHAUSER SETS OF BIORTHOGONAL  
POLYNOMIALS SUGGESTED BY THE LAGUERRE  
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# ON THE KONHAUSER SETS OF BIORTHOGONAL POLYNOMIALS SUGGESTED BY THE LAGUERRE POLYNOMIALS

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Recently, Joseph D. E. Konhauser discussed two polynomial sets  $\{Y_n^\alpha(x; k)\}$  and  $\{Z_n^\alpha(x; k)\}$ , which are biorthogonal with respect to the weight function  $x^\alpha e^{-x}$  over the interval  $(0, \infty)$ , where  $\alpha > -1$  and  $k$  is a positive integer. For the polynomials  $Y_n^\alpha(x; k)$ , the following bilateral generating function is derived in this paper:

$$(1) \quad \sum_{n=0}^{\infty} Y_n^\alpha(x; k) \zeta_n(y) t^n = (1-t)^{-(\alpha+1)/k} \exp \{x[1 - (1-t)^{-1/k}]\} \\ \cdot G[x(1-t)^{-1/k}, y t/(1-t)],$$

where

$$(2) \quad G[x, t] = \sum_{n=0}^{\infty} \lambda_n Y_n^\alpha(x; k) t^n,$$

the  $\lambda_n \neq 0$  are arbitrary constants, and  $\zeta_n(y)$  is a polynomial of degree  $n$  in  $y$  given by

$$(3) \quad \zeta_n(y) = \sum_{r=0}^n \binom{n}{r} \lambda_r y^r.$$

It is also shown that the polynomials  $Z_n^\alpha(x; k)$  can be expressed as a finite sum of  $Z_n^\alpha(y; k)$  in the form

$$(4) \quad Z_n^\alpha(x; k) = \left(\frac{x}{y}\right)^{kn} \sum_{r=0}^n \binom{\alpha + kn}{kr} \frac{(kr)!}{r!} [(y/x)^k - 1]^r Z_{n-r}^\alpha(y; k).$$

For  $k = 2$ , formulas (1) and (4) yield corresponding properties for the polynomials introduced earlier by Preiser [4]. Moreover, when  $k=1$ , both (1) and (4) would reduce to similar results involving the generalized Laguerre polynomials  $L_n^\alpha(x)$ . For results analogous to (1) and (4), involving certain classes of functions, the reader may be referred to our papers [5] and [6], respectively.

2. The following results will be required in our analysis.

(i) The generating function [3, p. 803]:

$$(5) \quad \sum_{n=0}^{\infty} \binom{m+n}{n} Y_{m+n}^\alpha(x; k) t^n \\ = (1-t)^{-(\alpha+m k+1)/k} \exp \{x[1 - (1-t)^{-1/k}]\} Y_m^\alpha(x(1-t)^{-1/k}; k),$$

where  $m$  is any integer  $\geq 0$ .

(ii) The explicit expression for  $Z_n^\alpha(x; k)$ :

$$(6) \quad Z_n^\alpha(x; k) = \frac{\Gamma(\alpha + kn + 1)}{n!} \sum_{j=0}^n (-1)^j \binom{n}{j} \frac{x^{kj}}{\Gamma(\alpha + kj + 1)},$$

which is equation (5), p. 304 of Konhauser [2].

From (6) it follows fairly easily that

$$(7) \quad \sum_{n=0}^{\infty} Z_n^\alpha(x; k) \frac{t^n}{(\alpha + 1)_{kn}} = e {}_0F_k[-; (\alpha + 1)/k, \dots, (\alpha + k)/k; - (x/k)^k t],$$

since  $k$  is a positive integer.

3. **Proof of the bilateral generating function (1).** Substituting for the coefficients  $\zeta_n(y)$  from (3) on the left-hand side of (1), we find that

$$\begin{aligned} \sum_{n=0}^{\infty} Y_n^\alpha(x; k) \zeta_n(y) t^n &= \sum_{n=0}^{\infty} Y_n^\alpha(x; k) t^n \sum_{r=0}^n \binom{n}{r} \lambda_r y^r \\ &= \sum_{r=0}^{\infty} \lambda_r (yt)^r \sum_{n=0}^{\infty} \binom{n+r}{r} Y_{n+r}^\alpha(x; k) t^n \\ &= (1-t)^{-(\alpha+1)/k} \exp \{x[1 - (1-t)^{-1/k}]\} \\ &\quad \cdot \sum_{r=0}^{\infty} \lambda_r Y_r^\alpha(x(1-t)^{-1/k}; k) (yt/(1-t))^r, \end{aligned}$$

by applying (5), and formula (1) would follow if we interpret this last expression by means of (2).

4. **Proof of the summation formula (4).** In the generating function (7), if we set  $t = (y/k)^k z$ , we shall get

$$(8) \quad \sum_{n=0}^{\infty} Z_n^\alpha(x; k) \frac{(y/k)^{kn} z^n}{(\alpha + 1)_{kn}} = \exp \{(y/k)^k z\} {}_0F_k[-(xy/k^3)^k z],$$

which, on interchanging  $x$  and  $y$ , gives us

$$(9) \quad \sum_{n=0}^{\infty} Z_n^\alpha(y; k) \frac{(x/k)^{kn} z^n}{(\alpha + 1)_{kn}} = \exp \{(x/k)^k z\} {}_0F_k[-(xy/k^3)^k z],$$

where, for convenience,

$$(10) \quad {}_0F_k[\xi] \equiv {}_0F_k[-; (\alpha + 1)/k, \dots, (\alpha + k)/k; \xi].$$

From (8) and (9) it follows at once that

$$(11) \quad \begin{aligned} &\sum_{n=0}^{\infty} Z_n^\alpha(x; k) \frac{(y/k)^{kn} z^n}{(\alpha + 1)_{kn}} \\ &= \exp \{z[(y/k)^k - (x/k)^k]\} \sum_{n=0}^{\infty} Z_n^\alpha(y; k) \frac{(x/k)^{kn} z^n}{(\alpha + 1)_{kn}}, \end{aligned}$$

and on equating coefficients of  $z^n$  in (11), we shall be led to our summation formula (4).

5. **Applications.** First of all we notice that formula (4) may be rewritten as

$$(12) \quad Z_n^\alpha(\mu x; k) = \sum_{r=0}^n \binom{\alpha + kn}{kr} \frac{(kr)!}{r!} \mu^{k(n-r)} (1 - \mu^k)^r Z_{n-r}^\alpha(x; k),$$

which provides us with a multiplication formula for the polynomials  $Z_n^\alpha(x; k)$ .

On the other hand, by assigning suitable values to the arbitrary coefficients  $\lambda_n$  it is fairly straightforward to obtain, from our formula (1), a large variety of bilateral generating functions for the polynomials  $Y_n^\alpha(x; k)$ . For instance, if we let

$$(13) \quad \lambda_n = \frac{(-1)^n}{\Gamma(\beta + ln + 1)}, \quad n = 0, 1, 2, \dots; l = 1, 2, 3, \dots;$$

and make use of the definition (6), we shall readily arrive at the bilateral generating function

$$(14) \quad \sum_{n=0}^{\infty} \frac{n!}{\Gamma(\beta + ln + 1)} Y_n^\alpha(x; k) Z_n^\beta(y; l) t^n \\ = (1 - t)^{-(\alpha+1)/k} \exp \{x[1 - (1 - t)^{-1/k}]\} H[x(1 - t)^{-1/k}, -y^l t / (1 - t)],$$

where, for convenience,

$$(15) \quad H[x, t] = \sum_{n=0}^{\infty} Y_n^\alpha(x; k) \frac{t^n}{\Gamma(\beta + ln + 1)}.$$

For  $k = l = 1$  and  $\alpha = \beta$ , the generating relation (14) would evidently reduce to the well-known Hille-Hardy formula for the Laguerre polynomials.

## REFERENCES

1. Joseph D. E. Konhauser, *Some properties of biorthogonal polynomials*, J. Math. Anal. Appl., **11** (1965), 242-260.
2. ———, *Biorthogonal polynomials suggested by the Laguerre polynomials*, Pacific J. Math., **21** (1967), 303-314.
3. T. R. Prabhakar, *On the other set of the biorthogonal polynomials suggested by the Laguerre polynomials*, Pacific J. Math., **37** (1971), 801-804.
4. S. Preiser, *An investigation of biorthogonal polynomials derivable from ordinary differential equations of the third order*, J. Math. Anal. Appl., **4** (1962), 38-64.
5. J. P. Singhal and H. M. Srivastava, *A class of bilateral generating functions for certain classical polynomials*, Pacific J. Math., **42** (1972), 755-762.

6. H. M. Srivastava, *On  $q$ -generating functions and certain formulas of David Zeitlin*, Illinois J. Math., **15** (1971), 64-72.

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# Pacific Journal of Mathematics

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Wm. R. Allaway, <i>On finding the distribution function for an orthogonal polynomial set</i> .....	305
Eric Amar, <i>Sur un théorème de Mooney relatif aux fonctions analytiques bornées</i> ....	311
Robert Morgan Brooks, <i>Analytic structure in the spectrum of a natural system</i> .....	315
Bahattin Cengiz, <i>On extremely regular function spaces</i> .....	335
Kwang-nan Chow and Moses Glasner, <i>Atoms on the Royden boundary</i> .....	339
Paul Frazier Duvall, Jr. and Jim Maxwell, <i>Tame <math>Z^2</math>-actions on <math>E^n</math></i> .....	349
Allen Roy Freedman, <i>On the additivity theorem for <math>n</math>-dimensional asymptotic density</i> .....	357
John Griffin and Kelly Denis McKennon, <i>Multipliers and the group <math>L_p</math>-algebras</i> ....	365
Charles Lemuel Hagopian, <i>Characterizations of <math>\lambda</math> connected plane continua</i> .....	371
Jon Craig Helton, <i>Bounds for products of interval functions</i> .....	377
Ikuko Kayashima, <i>On relations between Nörlund and Riesz means</i> .....	391
Everett Lee Lady, <i>Slender rings and modules</i> .....	397
Shozo Matsuura, <i>On the Lu Qi-Keng conjecture and the Bergman representative domains</i> .....	407
Stephen H. McCleary, <i>The lattice-ordered group of automorphisms of an <math>\alpha</math>-set</i> .....	417
Stephen H. McCleary, <i><math>\alpha</math> – 2-transitive ordered permutation groups</i> .....	425
Stephen H. McCleary, <i><math>\alpha</math>-primitive ordered permutation groups. II</i> .....	431
Richard Rochberg, <i>Almost isometries of Banach spaces and moduli of planar domains</i> .....	445
R. F. Rossa, <i>Radical properties involving one-sided ideals</i> .....	467
Robert A. Rubin, <i>On exact localization</i> .....	473
S. Sribala, <i>On <math>\Sigma</math>-inverse semigroups</i> .....	483
H. M. (Hari Mohan) Srivastava, <i>On the Konhauser sets of biorthogonal polynomials suggested by the Laguerre polynomials</i> .....	489
Stuart A. Steinberg, <i>Rings of quotients of rings without nilpotent elements</i> .....	493
Daniel Mullane Sunday, <i>The self-equivalences of an <math>H</math>-space</i> .....	507
W. J. Thron and Richard Hawks Warren, <i>On the lattice of proximities of Čech compatible with a given closure space</i> .....	519
Frank Uhlig, <i>The number of vectors jointly annihilated by two real quadratic forms determines the inertia of matrices in the associated pencil</i> .....	537
Frank Uhlig, <i>On the maximal number of linearly independent real vectors annihilated simultaneously by two real quadratic forms</i> .....	543
Frank Uhlig, <i>Definite and semidefinite matrices in a real symmetric matrix pencil</i> ...	561
Arnold Lewis Villone, <i>Self-adjoint extensions of symmetric differential operators</i> ....	569
Cary Webb, <i>Tensor and direct products</i> .....	579
James Victor Whittaker, <i>On normal subgroups of differentiable homeomorphisms</i> .....	595
Jerome L. Paul, <i>Addendum to: "Sequences of homeomorphisms which converge to homeomorphisms"</i> .....	615
David E. Fields, <i>Correction to: "Dimension theory in power series rings"</i> .....	616
Peter Michael Curran, <i>Correction to: "Cohomology of finitely presented groups"</i> ....	617
Billy E. Rhoades, <i>Correction to: "Commutants of some Hausdorff matrices"</i> .....	617
Charles W. Trigg, <i>Corrections to: "Versum sequences in the binary system"</i> .....	619