

# Pacific Journal of Mathematics

**ON THE KONHAUSER SETS OF BIORTHOGONAL  
POLYNOMIALS SUGGESTED BY THE LAGUERRE  
POLYNOMIALS**

H. M. (HARI MOHAN) SRIVASTAVA

## ON THE KONHAUSER SETS OF BIORTHOGONAL POLYNOMIALS SUGGESTED BY THE LAGUERRE POLYNOMIALS

H. M. SRIVASTAVA

Recently, Joseph D. E. Konhauser discussed two polynomial sets  $\{Y_n^\alpha(x; k)\}$  and  $\{Z_n^\alpha(x; k)\}$ , which are biorthogonal with respect to the weight function  $x^\alpha e^{-x}$  over the interval  $(0, \infty)$ , where  $\alpha > -1$  and  $k$  is a positive integer. For the polynomials  $Y_n^\alpha(x; k)$ , the following bilateral generating function is derived in this paper:

$$(1) \quad \sum_{n=0}^{\infty} Y_n^\alpha(x; k) \zeta_n(y) t^n = (1-t)^{-(\alpha+1)/k} \exp \{x[1 - (1-t)^{-1/k}]\} \\ \cdot G[x(1-t)^{-1/k}, y t/(1-t)],$$

where

$$(2) \quad G[x, t] = \sum_{n=0}^{\infty} \lambda_n Y_n^\alpha(x; k) t^n,$$

the  $\lambda_n \neq 0$  are arbitrary constants, and  $\zeta_n(y)$  is a polynomial of degree  $n$  in  $y$  given by

$$(3) \quad \zeta_n(y) = \sum_{r=0}^n \binom{n}{r} \lambda_r y^r.$$

It is also shown that the polynomials  $Z_n^\alpha(x; k)$  can be expressed as a finite sum of  $Z_n^\alpha(y; k)$  in the form

$$(4) \quad Z_n^\alpha(x; k) = \left(\frac{x}{y}\right)^{kn} \sum_{r=0}^n \binom{\alpha + kn}{kr} \frac{(kr)!}{r!} [(y/x)^k - 1]^r Z_{n-r}^\alpha(y; k).$$

For  $k = 2$ , formulas (1) and (4) yield corresponding properties for the polynomials introduced earlier by Preiser [4]. Moreover, when  $k = 1$ , both (1) and (4) would reduce to similar results involving the generalized Laguerre polynomials  $L_n^\alpha(x)$ . For results analogous to (1) and (4), involving certain classes of functions, the reader may be referred to our papers [5] and [6], respectively.

2. The following results will be required in our analysis.

(i) The generating function [3, p. 803]:

$$(5) \quad \sum_{n=0}^{\infty} \binom{m+n}{n} Y_{m+n}^\alpha(x; k) t^n \\ = (1-t)^{-(\alpha+m k+1)/k} \exp \{x[1 - (1-t)^{-1/k}]\} Y_m^\alpha(x(1-t)^{-1/k}; k),$$

where  $m$  is any integer  $\geq 0$ .

(ii) The explicit expression for  $Z_n^\alpha(x; k)$ :

$$(6) \quad Z_n^\alpha(x; k) = \frac{\Gamma(\alpha + kn + 1)}{n!} \sum_{j=0}^n (-1)^j \binom{n}{j} \frac{x^{kj}}{\Gamma(\alpha + kj + 1)},$$

which is equation (5), p. 304 of Konhauser [2].

From (6) it follows fairly easily that

$$(7) \quad \sum_{n=0}^{\infty} Z_n^\alpha(x; k) \frac{t^n}{(\alpha + 1)_{kn}} \\ = e^t {}_0F_k[-; (\alpha + 1)/k, \dots, (\alpha + k)/k; - (x/k)^k t],$$

since  $k$  is a positive integer.

3. **Proof of the bilateral generating function (1).** Substituting for the coefficients  $\zeta_n(y)$  from (3) on the left-hand side of (1), we find that

$$\sum_{n=0}^{\infty} Y_n^\alpha(x; k) \zeta_n(y) t^n = \sum_{n=0}^{\infty} Y_n^\alpha(x; k) t^n \sum_{r=0}^n \binom{n}{r} \lambda_r y^r \\ = \sum_{r=0}^{\infty} \lambda_r (yt)^r \sum_{n=0}^{\infty} \binom{n+r}{r} Y_{n+r}^\alpha(x; k) t^n \\ = (1-t)^{-(\alpha+1)/k} \exp\{x[1 - (1-t)^{-1/k}]\} \\ \cdot \sum_{r=0}^{\infty} \lambda_r Y_r^\alpha(x(1-t)^{-1/k}; k) (yt/(1-t))^r,$$

by applying (5), and formula (1) would follow if we interpret this last expression by means of (2).

4. **Proof of the summation formula (4).** In the generating function (7), if we set  $t = (y/k)^k z$ , we shall get

$$(8) \quad \sum_{n=0}^{\infty} Z_n^\alpha(x; k) \frac{(y/k)^{kn} z^n}{(\alpha + 1)_{kn}} = \exp\{(y/k)^k z\} {}_0F_k[-(xy/k^2)^k z],$$

which, on interchanging  $x$  and  $y$ , gives us

$$(9) \quad \sum_{n=0}^{\infty} Z_n^\alpha(y; k) \frac{(x/k)^{kn} z^n}{(\alpha + 1)_{kn}} = \exp\{(x/k)^k z\} {}_0F_k[-(xy/k^2)^k z],$$

where, for convenience,

$$(10) \quad {}_0F_k[\xi] \equiv {}_0F_k[-; (\alpha + 1)/k, \dots, (\alpha + k)/k; \xi].$$

From (8) and (9) it follows at once that

$$(11) \quad \sum_{n=0}^{\infty} Z_n^\alpha(x; k) \frac{(y/k)^{kn} z^n}{(\alpha + 1)_{kn}} \\ = \exp\{z[(y/k)^k - (x/k)^k]\} \sum_{n=0}^{\infty} Z_n^\alpha(y; k) \frac{(x/k)^{kn} z^n}{(\alpha + 1)_{kn}},$$

and on equating coefficients of  $z^n$  in (11), we shall be led to our summation formula (4).

5. Applications. First of all we notice that formula (4) may be rewritten as

$$(12) \quad Z_n^\alpha(\mu x; k) = \sum_{r=0}^n \binom{\alpha + kn}{kr} \frac{(kr)!}{r!} \mu^{k(n-r)} (1 - \mu^k)^r Z_{n-r}^\alpha(x; k),$$

which provides us with a multiplication formula for the polynomials  $Z_n^\alpha(x; k)$ .

On the other hand, by assigning suitable values to the arbitrary coefficients  $\lambda_n$  it is fairly straightforward to obtain, from our formula (1), a large variety of bilateral generating functions for the polynomials  $Y_n^\alpha(x; k)$ . For instance, if we let

$$(13) \quad \lambda_n = \frac{(-1)^n}{\Gamma(\beta + ln + 1)}, \quad n = 0, 1, 2, \dots; l = 1, 2, 3, \dots;$$

and make use of the definition (6), we shall readily arrive at the bilateral generating function

$$(14) \quad \sum_{n=0}^{\infty} \frac{n!}{\Gamma(\beta + ln + 1)} Y_n^\alpha(x; k) Z_n^\beta(y; l) t^n \\ = (1-t)^{-(\alpha+1)/k} \exp \{x[1 - (1-t)^{-1/k}]\} H[x(1-t)^{-1/k}, -y^l t/(1-t)],$$

where, for convenience,

$$(15) \quad H[x, t] = \sum_{n=0}^{\infty} Y_n^\alpha(x; k) \frac{t^n}{\Gamma(\beta + ln + 1)}.$$

For  $k = l = 1$  and  $\alpha = \beta$ , the generating relation (14) would evidently reduce to the well-known Hille-Hardy formula for the Laguerre polynomials.

#### REFERENCES

1. Joseph D. E. Konhauser, *Some properties of biorthogonal polynomials*, J. Math. Anal. Appl., **11** (1965), 242-260.
2. ———, *Biorthogonal polynomials suggested by the Laguerre polynomials*, Pacific J. Math., **21** (1967), 303-314.
3. T. R. Prabhakar, *On the other set of the biorthogonal polynomials suggested by the Laguerre polynomials*, Pacific J. Math., **37** (1971), 801-804.
4. S. Preiser, *An investigation of biorthogonal polynomials derivable from ordinary differential equations of the third order*, J. Math. Anal. Appl., **4** (1962), 38-64.
5. J. P. Singhal and H. M. Srivastava, *A class of bilateral generating functions for certain classical polynomials*, Pacific J. Math., **42** (1972), 755-762.

6. H. M. Srivastava, *On  $q$ -generating functions and certain formulas of David Zeitlin*, Illinois J. Math., **15** (1971), 64-72.

Received September 7, 1972 and in revised form March 8, 1973. Supported in part by NRC grant A-7353. See Abstract 72T-B97 in Notices Amer. Math. Soc., **19** (1972), p. A-437.

UNIVERSITY OF VICTORIA

# PACIFIC JOURNAL OF MATHEMATICS

## EDITORS

RICHARD ARENS (Managing Editor)  
University of California  
Los Angeles, California 90024

J. DUGUNDJI\*  
Department of Mathematics  
University of Southern California  
Los Angeles, California 90007

R. A. BEAUMONT  
University of Washington  
Seattle, Washington 98105

D. GILBARG AND J. MILGRAM  
Stanford University  
Stanford, California 94305

## ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

## SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA  
CALIFORNIA INSTITUTE OF TECHNOLOGY  
UNIVERSITY OF CALIFORNIA  
MONTANA STATE UNIVERSITY  
UNIVERSITY OF NEVADA  
NEW MEXICO STATE UNIVERSITY  
OREGON STATE UNIVERSITY  
UNIVERSITY OF OREGON  
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA  
STANFORD UNIVERSITY  
UNIVERSITY OF TOKYO  
UNIVERSITY OF UTAH  
WASHINGTON STATE UNIVERSITY  
UNIVERSITY OF WASHINGTON  
\* \* \*  
AMERICAN MATHEMATICAL SOCIETY  
NAVAL WEAPONS CENTER

---

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

---

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. 39. All other communications to the editors should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

---

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$48.00 a year (6 Vols., 12 issues). Special rate: \$24.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

\* C. R. DePrima California Institute of Technology, Pasadena, CA 91109, will replace J. Dugundji until August 1974.

Copyright © 1973 by  
Pacific Journal of Mathematics  
All Rights Reserved

|   |     |
|---|-----|
| Wm. R. Allaway, <i>On finding the distribution function for an orthogonal polynomial set</i> .....  | 305 |
| Eric Amar, <i>Sur un théorème de Mooney relatif aux fonctions analytiques bornées</i> .....   | 311 |
| Robert Morgan Brooks, <i>Analytic structure in the spectrum of a natural system</i> .....   | 315 |
| Bahattin Cengiz, <i>On extremely regular function spaces</i> .....  | 335 |
| Kwang-nan Chow and Moses Glasner, <i>Atoms on the Royden boundary</i> .....   | 339 |
| Paul Frazier Duvall, Jr. and Jim Maxwell, <i>Tame <math>Z^2</math>-actions on <math>E^n</math></i> .....  | 349 |
| Allen Roy Freedman, <i>On the additivity theorem for n-dimensional asymptotic density</i> .....   | 357 |
| John Griffin and Kelly Denis McKennon, <i>Multipliers and the group <math>L_p</math>-algebras</i> .....   | 365 |
| Charles Lemuel Hagopian, <i>Characterizations of <math>\lambda</math> connected plane continua</i> .....  | 371 |
| Jon Craig Helton, <i>Bounds for products of interval functions</i> .....  | 377 |
| Ikuko Kayashima, <i>On relations between Nörlund and Riesz means</i> .....  | 391 |
| Everett Lee Lady, <i>Slender rings and modules</i> .....  | 397 |
| Shozo Matsuura, <i>On the Lu Qi-Keng conjecture and the Bergman representative domains</i> .....  | 407 |
| Stephen H. McCleary, <i>The lattice-ordered group of automorphisms of an <math>\alpha</math>-set</i> .....  | 417 |
| Stephen H. McCleary, <i><math>o - 2</math>-transitive ordered permutation groups</i> .....  | 425 |
| Stephen H. McCleary, <i><math>o</math>-primitive ordered permutation groups. II</i> .....   | 431 |
| Richard Rochberg, <i>Almost isometries of Banach spaces and moduli of planar domains</i> .....  | 445 |
| R. F. Rossa, <i>Radical properties involving one-sided ideals</i> .....   | 467 |
| Robert A. Rubin, <i>On exact localization</i> .....   | 473 |
| S. Sribala, <i>On <math>\Sigma</math>-inverse semigroups</i> .....  | 483 |
| H. M. (Hari Mohan) Srivastava, <i>On the Konhauser sets of biorthogonal polynomials suggested by the Laguerre polynomials</i> .....                         | 489 |
| Stuart A. Steinberg, <i>Rings of quotients of rings without nilpotent elements</i> .....  | 493 |
| Daniel Mullane Sunday, <i>The self-equivalences of an H-space</i> .....   | 507 |
| W. J. Thron and Richard Hawks Warren, <i>On the lattice of proximities of Čech compatible with a given closure space</i> .....                              | 519 |
| Frank Uhlig, <i>The number of vectors jointly annihilated by two real quadratic forms determines the inertia of matrices in the associated pencil</i> ..... | 537 |
| Frank Uhlig, <i>On the maximal number of linearly independent real vectors annihilated simultaneously by two real quadratic forms</i> .....                 | 543 |
| Frank Uhlig, <i>Definite and semidefinite matrices in a real symmetric matrix pencil</i> .....  | 561 |
| Arnold Lewis Villone, <i>Self-adjoint extensions of symmetric differential operators</i> .....  | 569 |
| Cary Webb, <i>Tensor and direct products</i> .....  | 579 |
| James Victor Whittaker, <i>On normal subgroups of differentiable homeomorphisms</i> .....   | 595 |
| Jerome L. Paul, <i>Addendum to: "Sequences of homeomorphisms which converge to homeomorphisms"</i> .....  | 615 |
| David E. Fields, <i>Correction to: "Dimension theory in power series rings"</i> .....   | 616 |
| Peter Michael Curran, <i>Correction to: "Cohomology of finitely presented groups"</i> .....   | 617 |
| Billy E. Rhoades, <i>Correction to: "Commutants of some Hausdorff matrices"</i> .....   | 617 |
| Charles W. Trigg, <i>Corrections to: "Versum sequences in the binary system"</i> .....  | 619 |