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Daniel Mullane Sunday

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THE SELF-EQUIVALENCES OF AN H-SPACE

DANIEL M. SUNDAY, JR.

This paper studies the group $E(X)$ of self-homotopy-equivalences of a space X. Under mild (necessary) restrictions, it is shown that if X is an H-space then $E(X)$ is both finitely presented and Hopfian.

This paper studies the group of self-equivalences of a CW-complex X. This group, denoted by $E(X)$, is formed by taking the homotopy classes of homotopy equivalences from X to itself, and using composition as the group operation. Thus, categorically, $E(X)$ is the homotopy analog of an automorphism group. This group is important in topology because of its connection with the general problem of finding a complete set of homotopy invariants. It is known that a Postnikov system, in general, over-determines the homotopy type of a space. This happens because of the choices involved in picking the Postnikov invariants. $E(X)$ measures the indeterminacy that arises in this situation.

In addition, knowledge about $E(X)$ is related to the construction of classifying spaces. Let $LF(B)$ denote the fiber homotopy equivalence classes of Hurewicz fibrations over B with fibers the homotopy type of F; $H(F)$ denote the space of homotopy equivalences of F; and $B_{H(F)}$ denote the Dold-Lashof classifying space of $H(F)$. Then the space $B_{H(F)}$ represents the functor $LF(-)$. Since $E(F) = \Pi_1(B_{H(F)})$, knowledge about $E(F)$, such as whether or not it is finitely generated or presented, is of importance.

Previous investigations of the group $E(X)$ have been made from a general point of view in [2], [3], [10], [11], [16], and [17]. However, despite the extensive literature that exists, very little is known about this group and its properties. In particular, it is not known if $E(X)$ is finitely generated for finite complexes (in general, it is an infinite non-abelian group). W. Shih has claimed that, for finite complexes, $E(X)$ is finitely generated [2, p. 295]. However, no details of his work have appeared, and we have found objections to his results [18]. The finite generation question is regarded as open.

In studying $E(X)$, there is a natural restriction to place on the space X being considered. In this paper *it is always assumed that X is either finite-dimensional or has only finitely many nonzero homotopy groups*. Without one of these restrictions there are obvious counterexamples to the finite generation of $E(X)$. In addition, *it is always assumed that X is simply connected.* Modulo these restrictions, one hopes to show that:

1. $E(X)$ is finitely presented.
2. $E(X)$ is Hopfian.

In previous work along these lines, with the exception of [11], only the finite generation question has been investigated. The best partial results have been obtained by M. Arkowitz and C. R. Curjel. Their main theorem [3] is that if X is a homotopy associative H-space whose rational Pontryagin algebra is commutative, then $E(X)$ is finitely generated. This is sharpened considerably in:

THEOREM A. *If X is an H-space, then $E(X)$ is finitely presented.*

No associativity is assumed. Also, the conclusion of finite presentation is much stronger than that of finite generation. In fact, there are only a countable number of finitely presented groups, whereas there are an uncountable number of non-isomorphic groups with two generators [15]. The weaker conjecture of finite generation is open when X is not an H-space. (Note: dualization proves finite presentation for suspensions.)

As a secondary result, we prove a theorem about the 'size' of $E(X)$. Recall that a poly-finite-or-cyclic group is one which can be obtained from the trivial group by a finite number of finite or cyclic extensions.

THEOREM B. *If X is an H-space, then the following are true:*

(1) *$E(X)$ is a poly-finite-or-cyclic group if and only if* rank $(\Pi_i(X)) \leq 1$, *for all i.*

(2) *If* rank $(\Pi_i(X)) > 1$, *for some i, then $E(X)$ contains a non-abelian free subgroup.*

The second main concern of this paper is the question of whether $E(X)$ possesses the Hopfian property or not. This property provides another strong restriction on the class of groups in which $E(X)$ can lie, in that there are infinite families of finitely presented groups which are non-Hopfian [6].

THEOREM C. *If X is a space such that $E(X)$ is finitely generated, then $E(X)$ is Hopfian.*

COROLLARY. *If X is an H-space, then $E(X)$ is Hopfian.*

The organization of the paper is as follows. Section 1 contains preliminary material. Section 2 contains technical results needed to prove Theorems A and B, which are then proved in §3. Section 4 is the proof of Theorem C.

1. Preliminary material. Throughout this paper it is assumed that all spaces X are 1-connected CW-complexes with basepoint $*$, and with finitely generated homotopy groups. Where there is no ambiguity Π_n denotes $\Pi_n(X)$. All maps and homotopies are pointed, and the set of homotopy classes of maps from X to Y is denoted by $[X, Y]$. Usually we will not distinguish between a map and its homotopy class.

The reader is assumed to be familiar with the use of Postnikov systems [9]. For a space X, $\{X_j, p_j, k_j\}$ denotes such a system for X, where the projection $p_j: X_j \to X_{j-1}$ is induced by the jth k-invariant $k_j: X_{j-1} \to K_j = K(\Pi_j, j+1)$. If $E(-)$ denotes the group of self-equivalences, then the projection maps p_j induce homomorphisms $\bar{p}_j: E(X_j) \to E(X_{j-1})$. This and a simple obstruction theory argument [2] yield:

LEMMA 1.1. *If* $\dim(X) < \infty$, *then* $E(X_n) = E(X)$ *for* $n > \dim(X)$.

Because of this *it is always assumed that spaces have only finitely many nonzero homotopy groups.* In particular, all Postnikov systems are finite in length.

DEFINITION 1.2. Given a space X, put:

$$\operatorname{Aut}(X) = \bigoplus_i \operatorname{Aut}(\Pi_i)$$
$$\operatorname{Hom}(X) = \bigoplus_i \operatorname{Hom}(\Pi_i, \Pi_i)\ .$$

The sums involved in this definition are finite. Also $\operatorname{Aut}(X)$ is naturally embedded in $\operatorname{Hom}(X)$, and is precisely the group of units of the composition structure on $\operatorname{Hom}(X)$.

DEFINITION 1.3.
(1) $\sigma_X: [X, X] \to \operatorname{Hom}(X)$ is the representation by induced maps.
(2) $\psi_X: E(X) \to \operatorname{Aut}(X)$ is the restriction of σ_X to $E(X)$.
(3) $E_\#(X) = \ker(\psi_X)$.

LEMMA 1.4. (From [2]): *$E_\#(X)$ is a polycyclic group.*

In §§2 and 3 it is assumed that X is an H-space such that the basepoint $*$ is a two-sided identity. That is, there is a multiplication map $m: X \times X \to X$, such that $m(x, *) = x = m(*, x)$ for all $x \in X$. No generality is lost from the situation in which $*$ is a homotopy unit. Some properties of H-spaces needed in this paper are recorded in the following list:

1.5. If X is an H-space, then

(1) the Postnikov invariants, k_j, of X are primitive. That is, they are H-maps [9, Thm 3.2].

(2) for each j, k_j is of finite order in $H^{j+1}(X_{j-1}; \Pi_j)$, [1].

(3) ΩX is an H-space via two different natural multiplications; namely, Ωm (where m is the multiplication on X), and loop addition which we denote by '$+$'. By using the Moore path space for ΩX, and 'sliding' paths along each other, one can show that ΩM and $+$ are homotopic. This also shows that adding loops in either order gives homotopic H-structures. Hence, the additive structure induced on $[Y, \Omega X]$ is the same for both multiplications, and is abelian.

(4) since ΩX is homotopy abelian, the rational Pontryagin algebra of ΩX is commutative. In particular, all the results of [3] are valid for ΩX.

Finally, certain properties of finitely presented groups (that is, ones which can be defined by a finite set of generators and relations) are summarized (see: [13]).

1.6.

(1) An extension of finitely presented groups is finitely presented [7].

(2) A subgroup of finite index in a finitely presented group is finitely presented [13, p. 93].

(3) If $G_1, \cdots, G_k$ are finitely presented, so is $\bigoplus_{j=1}^{k} G_j$.

(4) If Π is a polycyclic group, then Aut (Π) is finitely presented [4].

2. Technical results. In this section, technical results needed for Theorems A and B are proved. Lemma 2.1 is the key lemma of the paper. A slightly modified form of this lemma holds for the general case of $[\Omega X, \Omega Y]$.

LEMMA 2.1. *Let X be a 1-connected H-space with a finite Postnikov system. Then there exists a positive integer $M(X)$, depending only on the homotopy type of X, with the following property: given any $f \in [\Omega X, \Omega X]$, there exists an $\bar{f} \in [X, X]$, such that $(\Omega\bar{f})$ and $M(X)f$ (addition via the loop structure) induce the same maps on the homotopy groups.*

Proof. Recall that X has finitely many nontrivial k-invariants, all of finite order. Define:

$$m_i = \begin{cases} 1, & \text{if } k_i \text{ is trivial} \\ \text{order } (k_i), & \text{otherwise}\,. \end{cases}$$

Put $M_j = \Pi_{i=2}^{j} m_i$, (note: $X_1 = *$, since $\Pi_1 = 0$), and define $M(X) = M_\infty$. This makes sense, since $m_i = 1$ for large i. $M(X)$ is a finite positive integer, and depends only on the homotopy type of X.

The assertion of the lemma is demonstrated by induction on Postnikov systems of X and ΩX. Fix a Postnikov system for X, and then loop it to obtain the system used for ΩX. Assume that at the $(j-1)$th stage of the argument it has been shown that there exists a map $\bar{f}_{j-1}: X_{j-1} \to X_{j-1}$, such that:

(1) $(\Omega \bar{f}_{j-1})$ and $M_j f_{j-1}$ induce the same maps on the homotopy groups,

(2) the following diagram homotopy commutes (in fact, both compositions are null-homotopic):

$$\begin{array}{ccc}
X_{j-1} & \xrightarrow{\bar{f}_{j-1}} & X_{j-1} \\
{\scriptstyle k_j}\downarrow & & \downarrow{\scriptstyle k_j} \\
K_j & \xrightarrow{\alpha_j} & K_j = K(\Pi_j, j+1)
\end{array}$$

where α_j is induced by $(M_j f_j)_\#$ on Π_j.

For $j = 2$, $X_1 = *$, and the above two conditions are trivial.

Now, because the above diagram homotopy commutes, $\bar{f}_{j-1}$ lifts to some map $\hat{f}_j: X_j \to X_j$, [9, p. 442]. Put $\bar{f}_j = m_{j+1}\hat{f}_j$. (Note: this means that $\hat{f}_j$ is added to itself m_{j+1}-times via the H-space structure on X_j. Since this structure may be nonassociative, insert parentheses so that the formula makes sense. This can be done arbitrarily since we are only interested in how $\Omega \bar{f}_j$ behaves, and looping recovers associativity.) Computation shows that $\bar{f}_j$ satisfies the induction hypothesis:

(1) On Π_j:

$$\begin{aligned}
(\Omega \bar{f}_j)_\# &= (\Omega(m_{j+1}\hat{f}_j))_\# = (\Omega(m_{j+1}\alpha_j))_\# = (m_{j+1}\alpha_j)_\# \\
&= (m_{j+1}M_j f_j)_\# = (M_{j+1} f_j)_\# .
\end{aligned}$$

On Π_i, for $i < j$:

$$\begin{aligned}
(\Omega \bar{f}_j)_\# &= (\Omega(m_{j+1} f_j))_\# = (m_{j+1}\Omega(\bar{f}_{j-1}))_\# \\
&= (m_{j+1}M_j f_{j-1})_\# = (M_{j+1} f_{j-1})_\# = (M_{j+1} f_j)_\# .
\end{aligned}$$

Hence, the first condition of the induction hypothesis holds.

(2)-(a)
$$\begin{aligned}
k_{j+1} \circ \bar{f}_j &= k_{j+1} \circ (m_{j+1}\hat{f}_j) \\
&= (m_{j+1}k_{j+1}) \circ \hat{f}_j ,
\end{aligned}$$

since k_{j+1} is primitive,

$$\cong * \circ \hat{f}_j = * .$$

$$\begin{aligned}
\text{-(b)}\quad \alpha_{j+1}\circ k_{j+1} &= (m_{j+1}\bar{\alpha}_{j+1})\circ k_{j+1},\ \bar{\alpha}_{j+1} \text{ induced by } (M_j f_{j+1})_\# ,\\
&= \bar{\alpha}_{j+1}\circ(m_{j+1}k_{j+1}),\ \text{since } \bar{\alpha}_{j+1} \text{ is primitive},\\
&\cong \bar{\alpha}_{j+1}\circ * = * .
\end{aligned}$$

Hence, the appropriate diagram homotopy commutes, and the second condition of the induction hypothesis is satisfied. This completes the proof of the lemma.

We now set up the framework to which the above lemma applies. Because of 1.5 (3), $[\Omega X, \Omega X]$ has a near-ring structure with an abelian addition (multiplication is composition of maps, and left distribution fails).

DEFINITION 2.2. $[\Omega X, \Omega X]_H$ denotes the subset of $[\Omega X, \Omega X]$ represented by H-maps, and $[\Omega X, \Omega X]_\Omega$ denotes the subset of classes represented by loop maps.

LEMMA 2.3. *$[\Omega X, \Omega X]_H$ and $[\Omega X, \Omega X]_\Omega$ are subrings of $[\Omega X, \Omega X]$.*

Proof. Since these two sets are closed under composition, and H-maps distribute on the left, we just need to show that they are closed under addition. This is an easy exercise.

Now, consider the following diagram of (near-)rings and homomorphisms, where $\operatorname{Hom}(\Omega X)$ is as in 1.2, and the maps involved are natural representations by induced maps.

$$\begin{tikzcd}
{[\Omega X, \Omega X]} \arrow[dr, "\sigma"] & \\
{[\Omega X, \Omega X]_H} \arrow[u, phantom, "\cup"] \arrow[r, "\sigma_H"] & \operatorname{Hom}(\Omega X) \\
{[\Omega X, \Omega X]_\Omega} \arrow[u, phantom, "\cup"] \arrow[ur, "\sigma_\Omega"'] &
\end{tikzcd}$$

LEMMA 2.4. *When we view the above diagram as consisting of additive abelian groups, then* (1) *σ, σ_H, and σ_Ω have finite cokernels,* (2) *σ_H and σ_Ω have finite kernels.*

Proof. According to [3, Lemma 5], σ_H has a finite kernel and cokernel. This implies that $\ker(\sigma_\Omega)$ and $\operatorname{coker}(\sigma)$ are finite. We just need to show that $\operatorname{coker}(\sigma_\Omega)$ is finite.

By Lemma 2.1, there exists a positive integer $M(X)$ such that $M(X)(\operatorname{im}\sigma) \subset \operatorname{im}(\sigma_\Omega)$. Since these are finitely generated abelian groups, $\operatorname{im}(\sigma_\Omega)$ has finite index in $\operatorname{im}(\sigma)$. Also, $\operatorname{im}(\sigma)$ has finite index in $\operatorname{Hom}(\Omega X)$. Hence, $\operatorname{coker}(\sigma_\Omega)$ is finite.

COROLLARY 2.5. *$[\Omega X, \Omega X]_\Omega$ has finite index in $[\Omega X, \Omega X]_H$. Further, their* rank $= \sum_i r_i^2$, *where* $r_i = \operatorname{rank}(\Pi_i)$.

Proof. The first statement is clear. The rank formula is obtained by noting that everything is equal to the rank of Hom (ΩX).

REMARK. Analogues of the above results hold for the more general case of $[\Omega X, \Omega Y]$. In this situation, it is assumed that Y is an H-space, and that X has Postnikov invariants of finite order. Further, these results can be generalized to spaces which have been looped n-times.

DEFINITION 2.6. Let $A(\Omega X)$ denote group of units of $[\Omega X, \Omega X]_H$ with the composition structure; and let $E_\Omega(\Omega X)$ denote the group of units of $[\Omega X, \Omega X]_\Omega$.

Consider the following diagram of groups and homomorphisms, where Aut (ΩX) is as in 1.2, and the maps are the obvious ones.

$$\begin{array}{ccc}
E(\Omega X) & \searrow^{\psi} & \\
\cup & & \\
A(\Omega X) & \xrightarrow{\psi_H} & \operatorname{Aut}(\Omega X) \\
\cup & & \\
E_\Omega(\Omega X) & \nearrow_{\psi_\Omega} &
\end{array}$$

LEMMA 2.7. *In the above diagram:*
(1) *ψ_H and ψ_Ω have finite kernels,*
(2) im (ψ), im (ψ_H) *and* im (ψ_Ω) *have finite index in* Aut (ΩX).

Proof. According to [3, pp. 144–146], the above is true for ψ_H. Hence, ψ_Ω has finite kernel, and im (ψ) has finite index in Aut (ΩX). We just need to show that im (ψ_Ω) has finite index in Aut (ΩX). The proof of this is exactly parallel to the one given in [3] for im (ψ_H). The only replacements needed are the fact that coker (σ_Ω) is finite (Lemma 2.4), and the fact that $[\Omega X, \Omega X]_\Omega$ is closed under addition (Lemma 2.3).

COROLLARY 2.8. *$E_\Omega(\Omega X)$ has finite index in $A(\Omega X)$.*

3. **The finite presentation of $E(X)$.** In this section, Theorems A and B as stated in the paper's introduction are proved. These results follow directly from the next proposition.

PROPOSITION 3.1. *If X is a 1-connected H-space with a finite Postnikov system, then $E_\Omega(\Omega X)$ is:*

(1) *finitely presented.*

(2) *finite if and only if* rank $(\Pi_i) \leqq 1$, *for all* i.

(3) *contains a nonabelian free subgroup if* rank $(\Pi_i) > 1$, *for some* i.

Proof. Aut (Π_i) is finitely presented for each i (by 1.6 (4)), implying that Aut (ΩX) is finitely presented (by 1.6(3)). Since a subgroup of finite index in a finitely presented group is itself finitely presented (by 1.6(2)), Lemma 2.7 implies that im (ψ_Ω) is finitely presented. Further, by Lemma 2.7, ker (ψ_Ω) is finite, and hence finitely presented. Statement (1) now follows from the fact that an extension of finitely presented groups is finitely presented (by 1.6 (1)).

For statement (2), the 'if' follows because Aut (ΩX) is finite when the condition holds. The 'only if' is implied by statement (3).

Finally, statement (3) follows from the observation that the proof of Lemma 2.7 also shows that, whenever rank $(\Pi_i) > 1$ for some i, im (ψ_Ω) contains a free nonabelian subgroup. Pull this subgroup back.

Proof of Theorem A. By Lemma 1.1, assume that X has a finite Postnikov system. Hence, Prop. 3.1 is valid. Consider the diagram:

$$\begin{array}{ccccc}
\ker(\hat{\Omega}) & = & \ker(\hat{\Omega}) & \longrightarrow & 0 \\
\cap & & \cap & & \cap \\
\ker(\psi_X) & \longrightarrow & E(X) & \xrightarrow{\psi_X} & \mathrm{Aut}\,(X) \\
\downarrow & & \downarrow{\scriptstyle \hat{\Omega}} & & \downarrow{\scriptstyle \theta} \\
\ker(\psi_\Omega) & \longrightarrow & E_\Omega(\Omega X) & \xrightarrow{\psi_\Omega} & \mathrm{Aut}\,(\Omega X)
\end{array}$$

where $\hat{\Omega}$ is the loop functor restricted to $E(X)$, ψ_X is the natural representation, and θ is the identity modulo a dimension shift. It is easy to see that the diagram commutes and that $\hat{\Omega}$ is onto. In particular, ker $(\hat{\Omega}) \subset$ ker (ψ_X). By Lemma 1.4 (taken from [2]), ker (ψ_X) is polycyclic. This implies that ker (ψ_X) and all of its subgroups are finitely presented. Hence, the same is true of ker $(\hat{\Omega})$. By Prop. 3.1, $E_\Omega(\Omega X)$ is finitely presented. Thus, $E(X)$ is an extension of finitely presented groups; and, by 1.6 (1), $E(X)$ is finitely presented.

Proof of Theorem B. The condition of statement (1) implies that Aut (X) is finite; hence, im (ψ_X) is finite. Also, ker (ψ_X) is polycyclic (Lemma 1.4). This proves the 'if' part. The 'only if' part follows from statement (2).

By Prop. 3.1, the condition of statement (2) implies that $E_\Omega(\Omega X)$ has a nonabelian free subgroup. Since $\hat{\Omega}: E(X) \to E_\Omega(\Omega X)$ is onto, pull the subgroup back.

REMARK. In Theorem B, the 'if' part of (1) holds in general for 1-connected spaces. The 'only if' of (1), and statement (2) fail in general, since $E(S^2 \times S^2)$ is finite [12, Prop. 2], whereas $\Pi_i(S^2 \times S^2)$ has rank 2 for $i = 2, 3$.

4. The Hopfian property.

In this section, Theorem C of the introduction is proved. Before doing so, however, some group theoretic results need proof.

DEFINITION 4.1.

(1) A group G is said to be an *H-group* (Hopfian group) if it is not isomorphic to a proper quotient of itself.

(2) A group G is said to be an *RF-group* (residually finite group) if given any $g \in G$ there exists a normal subgroup K of finite index in G such that $g \notin K$.

LEMMA 4.2. (Mal'cev: [14]). *A finitely generated RF-group is an H-group.*

LEMMA 4.3. (G. Baumslag: [5]). *If G is a finitely generated RF-group, then* Aut(G) *is an RF-group.*

DEFINITION 4.4.

(1) Given a group G, let $\{H_a\}_{a \in I}$ be the set of all normal subgroups of finite index in G. Define: $G_{RF} = \bigcap_{a \in I} H_a$.

(2) The RF-series of a group G is the sequence of subgroups: $G = G^{(0)} \supset G^{(1)} \supset \cdots \supset G^{(j)} \supset \cdots$ defined inductively by the rule: $G^{(j+1)} = G^{(j)}_{RF}$.

LEMMA 4.5.

(1) *G/G_{RF} is an RF-group.*

(2) *If $f: G \to G$ is an endomorphism, then $f(G_{RF}) \subset G_{RF}$. Further, if f is onto and G finitely generated, then $f(G_{RF}) = G_{RF}$.*

Proof. Statement (1) is obvious from the definition.

The first part of (2) follows from the fact that, for each $a \in I$, $f^{-1}(H_a)$ has finite index in G. Hence, $G_{RF} \subset f^{-1}(G_{RF})$.

Now, suppose that f is onto and that G is finitely generated. Let S_n be the set of all normal subgroups of index $\leqq n$. By [8, Thm. 5.2], S_n is a finite set. Hence, f^{-1} is a one-to-one correspondence between S_n and itself; and, given any $H_a \in S_n$, there exists an $H_b \in S_n$ such that $f^{-1}(H_b) = H_a$. This implies:

$$f^{-1}(G_{RF}) = f^{-1}\Big(\bigcap_b H_b\Big) = \bigcap_b f^{-1}(H_b) = \bigcap_a H_a = G_{RF} .$$

PROPOSITION 4.6. *If, in the RF-series of the group G, $G^{(j)}$ is finitely generated for all j, and $\bigcap_j G^{(j)} = 1$, then G is an H-group.*

Proof. Suppose G is not an H-group. Then there is a proper quotient of G, given by $f\colon G \to \bar{G}$, and an isomorphism $\phi\colon G \to \bar{G}$. Let $N = \ker(f)$. We prove by induction that $N \subset G^{(j)}$ for all j. Hence, by hypothesis, N is forced to be trivial.

For the induction hypothesis, assume that $N \subset G^{(j)}$, and that $f(G^{(j)}) = \phi(G^{(j)})$. This is trivially true for $j = 0$. Now, by assumption, $(\phi^{-1} \circ f)$ restricted to $G^{(j)}$ is an epimorphism. Since $G^{(j)}$ is finitely generated, and $G^{(j+1)} = G^{(j)}_{RF}$ by definition, Lemma 4.5 implies that $(\phi^{-1} \circ f)(G^{(j+1)}) = G^{(j+1)}$; that is $f(G^{(j+1)}) = \phi(G^{(j+1)})$.

This gives induced maps $\bar{f}\colon G^{(j)}/G^{(j+1)} \to \bar{G}^{(j)}/\bar{G}^{(j+1)}$, and $\bar{\phi}\colon G^{(j)}/G^{(j+1)} \to \bar{G}^{(j)}/\bar{G}^{(j+1)}$, where $\bar{G}^{(i)} = f(G^{(i)})$. Clearly $\bar{\phi}$ is an isomorphism. Also, $G^{(j)}/G^{(j+1)}$ is finitely generated, and (by Lemma 4.5) is an RF-group. Thus, by Lemma 4.2, it is an H-group and $\ker(\bar{f})$ is trivial. However, since $N \subset G^{(j)}$, $\ker(\bar{f}) = N/(N \cap G^{(j+1)})$, implying that $N \subset G^{(j+1)}$ as required.

COROLLARY 4.7. *If G is an extension of a poly-finite-or-cyclic group by a finitely generated RF-group, then G is an H-group.*

Proof. Let K be the group of which G is an extension. K and all of its subgroups are finitely generated. Also, $G^{(1)} = G_{RF} \subset K$. The corollary now follows from Prop. 4.6 (in fact, $G^{(2)} = 1$).

Proof of Theorem C. By Lemma 1.1, assume that $\Pi_i(X) = 0$ for $i > M$, M a finite integer. Since Π_i is a finitely generated RF-group (this is easy to check) for all i, Lemma 4.3 implies that $\mathrm{Aut}(\Pi_i)$ is an RF-group for all i. Furthermore, RF-groups are closed under direct products. Thus, $\mathrm{Aut}(X)$ is an RF-group. Let $\psi_X\colon E(X) \to \mathrm{Aut}(X)$ be the natural representation. Since subgroups of RF-groups are also RF-groups, $\mathrm{im}(\psi_X)$ is an RF-group. In addition, $\mathrm{im}(\psi_X)$ is finitely generated by hypothesis. Further, by Lemma 1.4, $\ker(\psi_X)$ is polycyclic. Thus, by Corollary 4.7, $E(X)$ is an H-group.

COROLLARY 4.8. *If X is an H-space, then $E(X)$ is an H-group.*

Proof. By Theorem A, $E(X)$ is finitely presented.

Question. Is $E(X)$ an RF-group?

The answer to this is complicated by the fact that there exists an example, shown to me by G. Baumslag, of an extension of Z_2 by a finitely generated RF-group which is not an RF-group.

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* C. R. DePrima California Institute of Technology, Pasadena, CA 91109, will replace J. Dugundji until August 1974.

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