Pacific Journal of Mathematics

# ON THE LATTICE OF PROXIMITIES OF ČECH COMPATIBLE WITH A GIVEN CLOSURE SPACE

W. J. THRON AND RICHARD HAWKS WARREN

Vol. 49, No. 2 June 1973

PACIFIC JOURNAL OF MATHEMATICS
Vol. 49, No. 2, 1973

# ON THE LATTICE OF PROXIMITIES OF ČECH COMPATIBLE WITH A GIVEN CLOSURE SPACE

W. J. THRON AND R. H. WARREN

**Let $(X, c)$ be a Čech closure space. By $\mathfrak{M}$ we denote the family of all proximities of Čech on $X$ which induce $c$. $\mathfrak{M}$ is known to be a complete lattice under set inclusion as ordering. The analogue of the $R_0$ separation axiom as defined for topological spaces is introduced into closure spaces. $R_0$-closure spaces are exactly those spaces for which $\mathfrak{M} \neq \phi$. Other characterizations for $R_0$-closure spaces are presented. The most interesting one is: every $R_0$-closure space is a subspace of a product of a certain number of copies of a fixed $R_0$-closure space. A number of techniques for constructing elements of $\mathfrak{M}$ are developed. By means of one of these constructions, all covers of any member of $\mathfrak{M}$ can be obtained. Using these constructions the following structural properties of $\mathfrak{M}$ are derived: $\mathfrak{M}$ is strongly atomic, $\mathfrak{M}$ is distributive, $\mathfrak{M}$ has no antiatoms, $|\mathfrak{M}| = 0, 1$ or $|\mathfrak{M}| \geqq 2^{2^{\aleph_0}}$.**

**1. Introduction.** E. Čech in [2] has studied a basic proximity structure (see Definition 1.3). The closure operator induced by such a structure is in general not a Kuratowski closure operator, since it may fail to satisfy the condition $c(c(A)) \subset c(A)$, however it satisfies the other three conditions and thus $(X, c)$ is a closure space (Definition 1.1). Since Čech called his basic proximity just a "proximity" and since this term is commonly used to denote a proximity of Efremovič, we shall refer to the basic proximities of Čech as $\check{C}$-proximities. We did not wish to use the name "Čech proximity" because this term already has another meaning in the literature [2, p. 447].

This paper is primarily concerned with a study of the order structure of the family $\mathfrak{M}$ of all $\check{C}$-proximities which induce the same closure operator on a given set. Čech [2] proved that $\mathfrak{M}$ is a complete lattice. He characterized least upper bounds in $\mathfrak{M}$, the least and greatest elements in $\mathfrak{M}$, and those closure spaces for which $\mathfrak{M} \neq \phi$.

The symbol $\mathscr{P}(X)$ denotes the power set of $X$, $|A|$ indicates the cardinal number of the set $A$, the triple bar $\equiv$ is reserved for definitions and $\square$ signals the end of a proof.

DEFINITION 1.1. [2, p. 237] Let $X$ be a set. A function $c: \mathscr{P}(X) \to \mathscr{P}(X)$ is called a *Čech closure operator* on $X$ iff it satisfies the following three axioms:

C1: $\bar{\phi} = \phi$;

C2: for every $A \subset X$, $A \subset \bar{A}$;

C3: for all $A, B \subset X$, $\bar{A} \cup \bar{B} = \overline{A \cup B}$.

In stating these axioms, we have denoted $c(A)$ by $\bar{A}$. We shall also use this notation in the following material, since from the context one can determine whether $\bar{A}$ denotes a topological closure or a Čech closure. The pair $(X, c)$ is called a *closure space*. We note that C1, C2, C3 are three of the four Kuratowski closure axioms. The fourth is: for every $A \subset X$, $\bar{\bar{A}} \subset \bar{A}$.

DEFINITION 1.2. [2, p. 270] Let $(X, c)$, $(Y, d)$ be closure spaces and let $f: X \to Y$. Then $f$ is *continuous* iff, given $A \subset X$, it follows that $f(\bar{A}) \subset \overline{f(A)}$.

DEFINITION 1.3. [2, p. 439] A relation $\mathscr{P}$ on $\mathscr{P}(X)$ is said to define a *Č-proximity* on a set $X$ iff it satisfies the conditions:

P1: $(A, B) \in \mathscr{P}$ implies $(B, A) \in \mathscr{P}$;

P2: $(A, B \cup C) \in \mathscr{P}$ iff $(A, B) \in \mathscr{P}$ or $(A, C) \in \mathscr{P}$;

P3: $(\phi, A) \notin \mathscr{P}$ for every $A \subset X$;

P4: $A \cap B \neq \phi$ implies $(A, B) \in \mathscr{P}$.

We now list a number of basic results about Č-proximities which were established by Čech. Let $\mathscr{P}$ be a Č-proximity on $X$. The function $c = c(\mathscr{P}): \mathscr{P}(X) \to \mathscr{P}(X)$ defined by $x \in \bar{A} = c(A)$ iff $([x], A) \in \mathscr{P}$ is a Čech closure operator which satisfies: $x \in [\bar{y}]$ implies $y \in [\bar{x}]$. We say that $\mathscr{P}$ induces $c$ or that $\mathscr{P}$ is compatible with $c$. More generrally, for a relation $\mathscr{S}$ on $\mathscr{P}(X)$, we say that $\mathscr{S}$ induces $c$ if for each $A \subset X$, $c(A) = [x: ([x], A) \in \mathscr{S}]$. If $(X, c)$ is a closure space satisfying $x \in [\bar{y}]$ implies $y \in [\bar{x}]$, then

$$\mathscr{R} \equiv [(A, B): (\bar{A} \cap B) \cup (A \cap \bar{B}) \neq \phi]$$

is a Č-proximity on $X$ compatible with $c$. Let $\mathfrak{M} = \mathfrak{M}(X, c)$ be the family of all Č-proximities on $X$ which induce $c$. If $[\mathscr{P}_i: i \in I] \subset \mathfrak{M}$, then $\bigcup [\mathscr{P}_i: i \in I] \in \mathfrak{M}$. Let $\mathfrak{M}$ be partially ordered by set inclusion. Then $\mathfrak{M}$ has a least element $\mathscr{R}$ (defined above) and a greatest element

$$\mathscr{W} \equiv \mathscr{R} \cup [(A, B): A \text{ and } B \text{ are infinite subsets of } X] .$$

It then follows [4, pp. 7-10] that $\mathfrak{M}$ is a complete lattice with the operator $\vee = \cup$.

The following definitions will be useful in describing some of our results in this paper.

DEFINITION 1.4. Let $(L, \leqq)$ be a partially ordered set. If $a, b \in L$, we say *a covers b* or *b is covered by a* when $a > b$ and $a > c > b$ is not satisfied for any $c \in L$. Moreover, $(L, \leqq)$ is said to be *covered* iff, given $x \in L$ such that there is $y \in L$ satisfying $y > x$, then there is $z \in L$ which covers $x$ and satisfies $z \leqq y$. Also $(L, \leqq)$ is said to be *anticovered* iff the dual of $(L, \leqq)$ is covered.

DEFINITION 1.5. Let $(L, \leqq)$ be a partially ordered set. If $(L, \leqq)$ has a least element $\ell$, then $a \in L$ is an *atom* iff $a$ covers $\ell$. Also $c \in L$ is an *antiatom* iff $c$ is an atom in the dual of $(L, \leqq)$. Furthermore, $(L, \leqq)$ is called *atomic* when each $x \in L$, $x$ not the least element, is the least upper bound of the atoms $\leqq x$. Moreover, $(L, \leqq)$ is called *strongly atomic* iff, given $a \in L$, the partially ordered set $[b\colon a \leqq b \in L]$ is atomic. Also $(L, \leqq)$ is *antiatomic* iff the dual of $(L, \leqq)$ is atomic.

We note that if $(L, \leqq)$ is strongly atomic and has a least element, then $(L, \leqq)$ is atomic. Also if $(L, \leqq)$ is strongly atomic, then $(L, \leqq)$ is covered. However, if $(L, \leqq)$ is covered, then $(L, \leqq)$ may not be atomic or strongly atomic. To verify the last statement, let $N$ be the set of natural numbers and define $a \leqq b$ iff $a$ divides $b$. To see that $(N, \leqq)$ is not atomic, we observe that the only atom $\leqq 4$ is 2. To see that $(N, \leqq)$ is covered, let $a$ properly divide $b$. Then there is prime $p$ such that $ap$ divides $b$. Thus $ap$ covers $a$.

DEFINITION 1.6. A lattice $(L, \vee, \wedge)$ is *infinitely meet distributive* iff, given nonempty $B \subset L$ and $a \in L$, then $a \wedge (\bigvee B) = \bigvee [a \wedge b\colon b \in B]$.

DEFINITION 1.7. A lattice $(L, \vee, \wedge)$ with least element $\ell$ and greatest element $\hslash$ is said to be *complemented* iff, for each $x \in L$, there is $y \in L$ such that $x \vee y = \hslash$ and $x \wedge y = \ell$.

2. **$R_0$-closure spaces.** Since a closure space $(X, c)$ has a compatible $\check{C}$-proximity iff $x \in [\bar{y}]$ implies $y \in [\bar{x}]$, it seems appropriate to give this condition a name. Moreover, a topological space is $R_0$ iff this condition is satisfied [3, p. 106].

DEFINITION 2.1. Let $(X, c)$ be a closure space. We say that $(X, c)$ is $R_0$ iff, given $x, y$ in $X$ such that $x \in [\bar{y}]$, then $y \in [\bar{x}]$.

Clearly, every $R_0$-topological space is an $R_0$-closure space. The following example of an $R_0$-closure space, which is not a topological space, will be useful in the sequel.

EXAMPLE 2.1. Let $S = [r, s, t]$ and let $d\colon \mathscr{P}(S) \rightarrow \mathscr{P}(S)$ be defined by:

$d(\phi) = (\phi)$,
$d([r]) = d(S) = d([r, s]) = d([r, t]) = d([s, t]) = S$,
$d([s]) = [r, s]$ and
$d([t]) = [r, t]$.

THEOREM 2.1. *Let* $(X, c)$ *be a closure space. Then the following are equivalent*:

(a) $(X, c)$ *is* $R_0$.

(b) *There is a* $\check{C}$*-proximity on* $X$ *which induces* $c$, i.e., $\mathfrak{M} \neq \phi$.

(c) *There is a semi-uniformity on* $X$ *which induces* $c$.

(d) *Given* $A \subset X$ *and* $x \notin \bar{A}$, *then* $[\bar{x}] \cap A = \phi$; i.e., *each subset of* $X$ *is separated from the points excluded from its closure.*

(e) *Given* $A \subset X$ *and* $x \in (X - \bar{A})$, *then* $[\bar{x}] \subset (X - A)$; i.e., *each subset of* $X$ *contains the closure of the points in its interior.*

(f) $(X, c)$ *is homeomorphic to a subspace of a product of spaces* $(S, d)$ *given in Example* 2.1.

*Proof*. In [2] it is shown that (a), (b), and (c) are equivalent, although the name $R_0$ is not used. The proof that (a), (d), and (e) are equivalent is straightforward and therefore is omitted.

(a) $\Rightarrow$ (f). By Theorem 17 C.17 in [2], it suffices to show that there is a family $[f_\alpha : X \rightarrow S]$ such that:

(1) Each $f_\alpha$ is continuous.

(2) The family distinguishes points.

(3) If $x \in X$ and $A \subset X$ such that $x \notin \bar{A}$, then there is an $\alpha$ such that $f_\alpha(x) \notin \overline{f_\alpha(A)}$.

To form such a family, if $A, B \subset X$ and $(\bar{A} \cap B) \cup (A \cap \bar{B}) = \phi$, then we define $g: X \rightarrow S$ by

$$g(x) = \begin{cases} r & \text{if } \; x \in X - (A \cup B) \\ s & \text{if } \; x \in A \\ t & \text{if } \; x \in B \,. \end{cases}$$

To verify that $g$ is continuous, it suffices to show that if $C \subset X$ and $\overline{g(C)} \neq S$, then $g(\bar{C}) \subset \overline{g(C)}$. If $g(C) = [s]$, then $C \subset A$ and $\bar{C} \subset \bar{A}$. Since $\bar{A} \cap B = \phi$, it follows that $g(\bar{C}) \subset [r, s] = \overline{g(C)}$. Similarly, if $g(C) = [t]$, then $g(\bar{C}) \subset \overline{g(C)}$.

If $y, z \in X$, $y \neq z$ and $y \in [\bar{z}]$, then we define $h: X \rightarrow S$ by

$$h(x) = \begin{cases} r & \text{if } \; x \neq z \\ s & \text{if } \; x = z \,. \end{cases}$$

To see that $h$ is continuous, it suffices to show that if $C \subset X$ and $\overline{h(C)} \neq S$, then $h(\bar{C}) \subset \overline{h(C)}$. So we consider $h(C) = [s]$. Then $C = [z]$ and $h(\bar{C}) = [r, s] = \overline{h(C)}$.

We define the family $[f_\alpha]$ to consist of all those maps $g, h$ which we have specified above.

To verify (2), let $y, z \in X$ and $y \neq z$. If $y \in [\bar{z}]$, then there is a map $h$ which distinguishes $y$ and $z$. If $y \notin [\bar{z}]$, then, since $(X, c)$ is $R_0$, $z \notin [\bar{y}]$ and there is a map $g$ which distinguishes $y$ and $z$.

To verify (3), let $x \notin \bar{A}$. Given $a \in A$, then $[\bar{a}] \subset \bar{A}$ and $x \notin [\bar{a}]$. Since $(X, c)$ is $R_0$, $a \notin [\bar{x}]$. Therefore, $[\bar{x}] \cap A = \phi$ and there is a map $g$ such that $g(x) \notin \overline{g(A)}$.

(f) $\Rightarrow$ (a). Since products (Theorem 23 D.11 in [2]), subspaces and homeomorphic images of $R_0$-closure spaces are $R_0$-closure spaces, the result follows. □

It is well known that in a topological space $(X, \mathscr{T})$ the $R_0$-axiom is equivalent to each of the following statements: given $x, y \in X$, then $[\bar{x}] = [\bar{y}]$ or $[\bar{x}] \cap [\bar{y}] = \phi$; or, given $x \in G \in \mathscr{T}$, then $[\bar{x}] \subset G$. However, these statements are not equivalent to the $R_0$-axiom for closure spaces.

THEOREM 2.2. *Let $(X, c)$ be a closure space. If $[\bar{x}] = [\bar{y}]$ or $[\bar{x}] \cap [\bar{y}] = \phi$ for all $x, y$ in $X$, then $(X, c)$ is $R_0$; but the converse is false.*

*Proof.* The proof of the positive assertion is straightforward and therefore is omitted. The converse fails in the $R_0$-closure space given in Example 2.1.

THEOREM 2.3. *Let $(X, c)$ be a closure space. If $(X, c)$ is $R_0$, then each open set contains the closure of each of its points; but the converse is false.*

*Proof.* The positive assertion is easily established. To see that the converse is false, consider the following example: Let $X = [a, b, c]$ and let $c: \mathscr{P}(X) \to \mathscr{P}(X)$ be defined by

$$c(A) = \begin{cases} \phi & \text{if} \quad A = \phi \\ [a, c] & \text{if} \quad A = [c] \\ X & \text{otherwise.} \end{cases} \quad \square$$

Similarly, one shows that if a closure space is $R_0$, then closed sets are separated from the points they exclude; but the converse is false.

## 3. Construction of proximites of Čech.

In this section we characterize the least member of $\mathfrak{M}$ in three ways, describe several techniques for constructing members of $\mathfrak{M}$ and derive some properties of these constructions.

THEOREM 3.1. *Let $(X, c)$ be an $R_0$-closure space, and let $\mathscr{S}$ be a relation on $\mathscr{P}(X)$. If $\mathscr{R} \subset \mathscr{S} \subset \mathscr{W}$, then $\mathscr{S}$ induces $c$ and satisfies* P3 *and* P4.

*Proof.* Obvious.

THEOREM 3.2. *Let $(X, c)$ be an $R_0$-closure space. Let $\mathscr{D} \equiv [([x], A)\colon x \in \bar{A}, A \subset X]$ and let $\mathscr{E} \equiv [(C, D)\colon \exists([x], A) \in \mathscr{D}$ such that $(x \in C$ and $A \subset D)$ or $(x \in D$ and $A \subset C)]$. Then $\mathscr{E} = \mathscr{R}$.*

*Proof.* The proof is an easy verification. □

In order to analyze Theorem 3.2, let $\mathscr{P} \in \mathfrak{M}(X, c)$. Since $c$ is compatible with $\mathscr{P}$, it is necessary that $\mathscr{D} \subset \mathscr{P}$. Also from that part of P2 which insures that $C \supset B$ and $A\delta B$ implies $A\delta C$ and from P1, it follows that $\mathscr{D} \subset \mathscr{E} \subset \mathscr{P}$. What is surprising is that no further alteration of $\mathscr{E}$, to accommodate the second part of P2 as well as P3 and P4, is necessary to obtain $\mathscr{R}$.

THEOREM 3.3. *Let $(X, c)$ be an $R_0$-closure space and let $(S, d)$ be the closure space in Example 2.1. Then the least $\check{C}$-proximity $\mathscr{R}$ in $\mathfrak{M}(X, c)$ is defined by $(A, B) \notin \mathscr{R}$ iff there is a continuous function $g\colon (X, c) \to (S, d)$ such that $g(A) \subset [s]$ and $g(B) \subset [t]$.*

*Proof.* Assume $(A, B) \notin \mathscr{R}$. Then $(\bar{A} \cap B) \cup (A \cup \bar{B}) = \phi$ and the existence of a suitable function $g$ was shown in the proof of Theorem 2.1.

Conversely, assume there is a continuous function $g$ such that $g(A) \subset [s]$ and $g(B) \subset [t]$. Then $g(\bar{A}) \subset \overline{g(A)} \subset [r, s]$, and thus $\bar{A} \cap B = \phi$. Similarly $A \cap \bar{B} = \phi$. Hence $(A, B) \notin \mathscr{R}$.

DEFINITION 3.1. [2, 25 A.7] A mapping $f$ from a $\check{C}$-proximity space $(X, \mathscr{P})$ to a $\check{C}$-proximity space $(Y, \mathscr{P}^*)$ is said to be *p-continuous* iff $(A, B) \in \mathscr{P}$ implies $(f(A), f(B)) \in \mathscr{P}^*$.

An equivalent formulation of this definition is: $f$ is $p$-continuous iff for all $(C, D) \notin \mathscr{P}^*$ with $C, D \subset Y$, it is true that $(f^{-1}(C), f^{-1}(D)) \notin \mathscr{P}$.

It is known [2, 25 A. 10] that every $p$-continuous function is a continuous function with respect to the induced closure operators. It is easily verified that there is only one-proximity $\mathscr{R}^d$ on $S$ compatible with $d$ (the space $(S, d)$ is defined in Example 2.1) and that

$$\sim\mathscr{R}^d = [(\phi, A)\colon A \subset S] \cup [(B, \phi)\colon B \subset S] \cup [([s], [t])] .$$

In this context the following theorem may be of interest.

THEOREM 3.4. *Let $\mathscr{P} \in \mathfrak{M}(X, c)$. Then $\mathscr{P} = \mathscr{R}$ iff all functions which are continuous from $(X, c)$ to $(S, d)$ are p-continuous from*

*$(X, \mathscr{P})$ to $(S, \mathscr{R}^d)$. Here $\mathscr{R}$ is the least Č-proximity in $\mathfrak{M}(X, c)$, $(S, d)$ is the space in Example 2.1 and $\mathscr{R}^d$ is the unique Č-proximity in $\mathfrak{M}(S, d)$.*

*Proof.* Assume $\mathscr{P} = \mathscr{R}$. Let $f: (X, c) \to (S, d)$ be continuous and let $(A, B) \in \mathscr{P}$. Then $\bar{A} \cap B \neq \phi$ or $A \cap \bar{B} \neq \phi$; say $A \cap \bar{B} \neq \phi$. Choose $a$ in $A \cup \bar{B}$. Since $f$ is continuous, $f(a) \in f(\bar{B}) \subset \overline{f(B)}$. Thus $f(A) \cap \overline{f(B)} \neq \phi$ and $(f(A), f(B)) \in \mathscr{R}^d$. Therefore, $f$ is $p$-continuous.

Conversely, assume $\mathscr{P} \neq \mathscr{R}$. Then there is $(A, B) \in \mathscr{P} - \mathscr{R}$. So $(\bar{A} \cap B) \cup (A \cap \bar{B}) = \phi$. We define $g: X \to S$ by

$$g(x) = \begin{cases} r & \text{if } \; x \in X - (A \cup B) \\ s & \text{if } \; x \in A \\ t & \text{if } \; x \in B\,. \end{cases}$$

In the proof of Theorem 2.1 we verified that $g$ is continuous. However, $g$ is not $p$-continuous since $(g(A), g(B)) \notin \mathscr{R}^d$.

THEOREM 3.5 *Let $(X, c)$ be an $R_0$-closure space and let $\mathscr{R}_1 \equiv [(A, B): \bar{A} \cap \bar{B} \neq \phi]$. Then $\mathscr{R}_1 \in \mathfrak{M}$ iff, given $A \subset X$ and $x \in X$ such that $\bar{A} \cap [\bar{x}] \neq \phi$, it follows that $x \in \bar{A}$.*

*Proof.* The proof is a straightforward verification. □

It is easily shown that if $(X, c)$ is a closure space, then $\mathscr{R}_1 = [(A, B): A, B \subset X$ and if $f: (X, c) \to (X, c)$ is continuous, then $\overline{f(A)} \cap \overline{f(B)} \neq \phi]$.

DEFINITION 3.2. Let $(X, c)$ be an $R_0$-closure space, let $E \subset X$, let $m$ be an infinite cardinal number and let $\mathscr{P} \in \mathfrak{M}$. We introduce the following notations.

(i) $\mathscr{P}(E, m) \equiv \mathscr{W} \cap (\mathscr{P} \cup [(A, B): |A \cap E| \geqq m$ or $|B \cap E| \geqq m])$.

(ii) $\mathscr{P}\{E, m\} \equiv \mathscr{P} \cup [(A, B): |A \cap E| \geqq m$ and $|B \cap E| \geqq m]$.

THEOREM 3.6. *The relation $\mathscr{P}\{E, m\}$ is in $\mathfrak{M}$ and has the following properties:*

(i) *If $E \subset F$, $m \leqq m_1$ and $\mathscr{P}' \subset \mathscr{P}$, then $\mathscr{P}'\{E, m_1\} \subset \mathscr{P}\{F, m\}$.*

(ii) *If $|F| < m$, then $\mathscr{P}\{E, m\} = \mathscr{P}\{E \cup F, m\} = \mathscr{P}\{E - F, m\}$.*

(iii) *$\mathscr{P}\{E, m\} \wedge \mathscr{P}\{F, m\} = \mathscr{P}\{E \cap F, m\}$.*

(iv) *$\mathscr{P}\{E, m\} \cup \mathscr{P}\{F, m\} \subset \mathscr{P}\{E \cup F, m\}$ and, in general, equality does not hold.*

(v) *If $m \leqq m_1$ and $|F - E| \leqq m_1$, then $(\mathscr{P}\{E, m\})\{F, m_1\} = \mathscr{P}\{E, m\}$.*

(vi) *If $m \geqq m_1$ and $|E - F| < m$, then $(\mathscr{P}\{E, m\})\{F, m_1\} = \mathscr{P}\{F, m_1\}$.*

(vii) *If* $\mathscr{P}, \mathscr{P}' \in \mathfrak{M}$ *and* $\mathscr{P} \cup \mathscr{P}' \subsetneqq \mathscr{P}\{E, m\}$, *then there exists* $\mathscr{P}^* \in \mathfrak{M}$ *such that* $\mathscr{P}' \subsetneqq \mathscr{P}^* \subsetneqq \mathscr{P}\{E, m\}$.

*Proof*. Clearly $\mathscr{P}\{E, m\} \subset \mathscr{W}$. Since $\mathscr{R} \subset \mathscr{P} \subset \mathscr{P}\{E, m\}$, it follows from Theorem 3.1 that $\mathscr{P}\{E, m\}$ induces $c$ and satisfies P3, P4. Clearly P1 holds in $\mathscr{P}\{E, m\}$. The verification of P2 is straightforward.

(i) Let $(A, B) \in \mathscr{P}'\{E, m_1\}$. If $(A, B) \in \mathscr{P}'$, then $(A, B) \in \mathscr{P} \subset \mathscr{P}\{F, m\}$. If $(A, B) \notin \mathscr{P}'$, then $|A \cap E| \geqq m_1$ and $|B \cap E| \geqq m_1$. Since $E \subset F$ and $m \leqq m_1$, it follows that $|A \cap F| \geqq m$ and $|B \cap F| \geqq m$. Thus $(A, B) \in \mathscr{P}\{F, m\}$.

(ii) Since $|F| < m$ and $m$ is an infinite cardinal number, it is known from set theory that $|A \cap E| \geqq m$ iff $|A \cap (E \cup F)| \geqq m$ iff $|A \cap (E - F)| \geqq m$.

(iii) Let $(A, B) \in \mathscr{P}\{E, m\} \wedge \mathscr{P}\{F, m\}$. If $(A, B) \in \mathscr{P}$, then $(A, B) \in \mathscr{P}\{E \cap F, m\}$. So we assume that $(A, B) \notin \mathscr{P}$. Suppose that $|B \cap E \cap F| < m$. Since $(A, B) \notin \mathscr{P}$, P2 implies that $(A, B \cap E \cap F) \notin \mathscr{P}$. Thus $(A, B \cap E \cap F) \notin \mathscr{P}\{E, m\} \wedge \mathscr{P}\{F, m\}$. Because $B = (B \cap E \cap F) \cup (B - (E \cap F))$, it follows from P2 that $(A, B - (E \cap F)) \in \mathscr{P}\{E, m\} \wedge \mathscr{P}\{F, m\}$. In as much as $B - (E \cap F) = (B - E) \cup (B - F)$, P2 implies that:

*Case* 1. $(A, B - E) \in \mathscr{P}\{E, m\} \wedge \mathscr{P}\{F, m\}$. Then $(A, B - E) \in \mathscr{P}\{E, m\}$. Again, since $(A, B) \notin \mathscr{P}$, P2 implies that $(A, B - E) \notin \mathscr{P}$. Therefore, $|(B - E) \cap E| \geqq m$ which is a contradiction.

*Case* 2. $(A, B - F) \in \mathscr{P}\{E, m\} \wedge \mathscr{P}\{F, m\}$. An argument similar to Case 1 leads to the contradiction $|(B - F) \cap F| \geqq m$.

As a result of the contradictions, $|B \cap E \cap F| \geqq m$. Similarly, $|A \cap E \cap F| \geqq m$. Therefore, $(A, B) \in \mathscr{P}\{E \cap F, m\}$, and $\mathscr{P}\{E, m\} \wedge \mathscr{P}\{F, m\} \subset \mathscr{P}\{E \cap F, m\}$.

By (i), $\mathscr{P}\{E \cap F, m\} \subset \mathscr{P}\{E, m\} \cap \mathscr{P}\{E, m\}$. Since $\mathscr{P}\{E, m\} \wedge \mathscr{P}\{F, m\}$ is the union of all members of $\mathfrak{M}$ contained in $\mathscr{P}\{E, m\} \cap \mathscr{P}\{F, m\}$, it follows that $\mathscr{P}\{E \cap F, m\} \subset \mathscr{P}\{E, m\} \wedge \mathscr{P}\{F, m\}$.

(iv) By (i), $\mathscr{P}\{E, m\} \subset \mathscr{P}\{E \cup F, m\}$ and $\mathscr{P}\{F, m\} \subset \mathscr{P}\{E \cup F, m\}$. Therefore, $\mathscr{P}\{E, m\} \cup \mathscr{P}\{F, m\} \subset \mathscr{P}\{E \cup F, m\}$.

Next we give an example where $\mathscr{P}\{E \cup F, m\} \not\subset \mathscr{P}\{E, m\} \cup \mathscr{P}\{F, m\}$. Let $X$ be the set of real numbers, $\mathscr{T}$ the usual topology on $X$, $m = |X|$, $\mathscr{P} = \mathscr{R}$, $E = [x \in X \colon 0 \leqq x \leqq 1]$ and $F = [x \in X \colon 2 \leqq x \leqq 3]$. Then $(E, F) \in \mathscr{P}\{E \cup F, m\}$ but $(E, F) \notin \mathscr{P}\{E, m\} \cup \mathscr{P}\{F, m\}$.

(v) Clearly $\mathscr{P}\{E, m\} \subset \mathscr{P}\{E, m\} \cup [(A, B) \colon |A \cap F| \geqq m_1$ and $|B \cap F| \geqq m_1] = (\mathscr{P}\{E, m\})\{F, m_1\}$.

Let $|A \cap F| \geqq m_1$. We write $A \cap F = (A \cap (F - E)) \cup (A \cap E \cap F)$. Since $|A \cap (F - E)| \leqq |F - E| < m_1$, it follows that $|A \cap E \cap F| \geqq m_1$.

So $|A \cap E| \geqq m_1 \geqq m$. Therefore, we have shown that $[(A, B); |A \cap F| \geqq m_1$ and $|B \cap F| \geqq m_1] \subset [(A, B): |A \cap E| \geqq m$ and $|B \cap E| \geqq m]$. Thus $[(A, B): |A \cap E| \geqq m$ and $|B \cap E| \geqq m] \subset \mathscr{P}\{E, m\}$ implies $(\mathscr{P}\{E, m\})\{F, m_1\} \subset \mathscr{P}\{E, m\}$.

(vi) The proof is similar to (v) and therefore is omitted.

(vii) Let $(C, D) \in \mathscr{P}\{E, m\} - (\mathscr{P} \cup \mathscr{P}')$. Thus $|C \cap E| \geqq m$ and $|D \cap E| \geqq m$. We partition $D$ into two disjoint sets $D_1, D_2$ such that $|D_1 \cap E| = |D_2 \cap E| = |D \cap E|$. Then $D_1 \cap E = D_1 \cap (E - D_2)$. So $|D_1 \cap (E - D_2)| \geqq m$.

We write $C \cap E = (C \cap E \cap D_1) \cup (C \cap E \cap D_2) \cup ((C \cap E) - (D_1 \cap D_2))$. Since $|C \cap E| \geqq m$, it follows that $|(C \cap E \cap D_1) \cup ((C \cap E) - (D_1 \cap D_2))| \geqq m$ or $|(C \cap E \cap D_2) \cup ((C \cap E) - (D_1 \cap D_2))| \geqq m$; say the former is true. Let $F = C - (E \cap D_2)$. Then $F \cap (E - D_2) = (C \cap E \cap D_1) \cup ((C \cap E) - (D_1 \cup D_2))$ and $|F \cap (E - D_2)| \geqq m$.

Let $\mathscr{P}^* = \mathscr{P}'\{E - D_2, m\}$. Then $(F, D_1) \in \mathscr{P}^*$ by the above work. Since $(C, D) \notin \mathscr{P}'$, $F \subset C$ and $D_1 \subset D$, P1 and P2 imply that $(F, D_1) \notin \mathscr{P}'$. Thus $\mathscr{P}' \subsetneqq \mathscr{P}^*$.

Clearly $(C, D_2) \in \mathscr{P}\{E, m\}$ and $\mathscr{P}^* \subset \mathscr{P}\{E, m\}$. Since $(C, D_2) \in \mathscr{P}'$ implies by P2 that $(C, D) \in \mathscr{P}'$, contrary to assumption, we must have $(C, D_2) \notin \mathscr{P}'$. Also $D_2 \cap (E - D_2) = \phi$ implies that $(C, D_2) \notin \mathscr{P}^*$. Thus $\mathscr{P}^* \subsetneqq \mathscr{P}\{E, m\}$.

THEOREM 3.7. *The relation $\mathscr{P}(E, m)$ is in $\mathfrak{M}$ and has the following properties*:

(i) *If $E \subset F$, $m \leqq m_1$ and $\mathscr{P}' \subset \mathscr{P}$, then $\mathscr{P}'(E, m_1) \subset \mathscr{P}(F, m)$.*

(ii) *If $|F| < m$, then $\mathscr{P}(E, m) = \mathscr{P}(E \cup F, m) = \mathscr{P}(E - F, m)$.*

(iii) *$\mathscr{P}(E, m) \wedge \mathscr{P}(F, m) \supset \mathscr{P}(E \cap F, m)$ and, in general, equality does not hold.*

(iv) *$\mathscr{P}(E, m) \cup \mathscr{P}(F, m) = \mathscr{P}(E \cup F, m)$.*

(v) *If $m \leqq m_1$ and $|F - E| < m_1$, then $(\mathscr{P}(E, m))(F, m_1) = \mathscr{P}(E, m)$.*

(vi) *If $m \geqq m_1$ and $|E - F| < m$, then $(\mathscr{P}(E, m))(F, m_1) = \mathscr{P}(F, m_1)$.*

(vii) *If $\mathscr{P}, \mathscr{P}' \in \mathfrak{M}$ and $\mathscr{P} \cup \mathscr{P}' \subsetneqq \mathscr{P}(E, m)$, then there exists $\mathscr{P}^* \in \mathfrak{M}$ such that $\mathscr{P}' \subsetneqq \mathscr{P}^* \subsetneqq \mathscr{P}(E, m)$.*

*Proof*. The proof is similar to Theorem 3.6 and therefore is omitted.

THEOREM 3.8.

(i) $\mathscr{P}\{E, m\} \subset \mathscr{P}(E, m)$.

(ii) *In general, $[\mathscr{R}(E, m): E \subset X, m$ an infinite cardinal$]$ neither contains nor is contained in $[\mathscr{R}\{E, m\}: E \subset X, m$ an infinite cardinal$]$.*

(iii) *In general, $\mathfrak{M} \neq [\mathscr{R}(E, m), \mathscr{R}\{E, m\}: E \subset X, m$ an infinite cardinal$]$.*

*Proof.* (i) The result is immediate by comparing the definitions of $\mathscr{P}\{E, m\}$ and $\mathscr{P}(E, m)$.

(ii) Let $X$ be the set of real numbers, $\mathscr{T}$ the usual topology on $X$, $F = [x \in X: 0 \leqq x \leqq 4]$ and $t = |X|$. Then $\mathscr{R}\{F, t\} \notin [\mathscr{R}(E, m): E \subset X, m$ an infinite cardinal] and $\mathscr{R}(F, t) \notin [\mathscr{R}\{E, m\}: E \subset X, m$ an infinite cardinal].

(iii) Let $X$ be the set of real numbers, let $\mathscr{T}$ be the usual topology on $X$ and let $\mathscr{R}_1 = [(A, B): \bar{A} \cap \bar{B} \neq \phi]$. Then $\mathscr{R}_1 \in \mathfrak{M}$ by Theorem 3.5. However, $\mathscr{R}_1 \notin [\mathscr{R}(E, m), \mathscr{R}\{E, m\}: E \subset X, m$ an infinite cardinal].

THEOREM 3.9. *If $\mathscr{R} \neq \mathscr{R}\{E, m\}$, then $\mathscr{R}\{E, m\}$ covers no element of $\mathfrak{M}$. If $\mathscr{R} \neq \mathscr{R}(E, m)$, then $\mathscr{R}(E, m)$ covers no element of $\mathfrak{M}$.*

*Proof.* Suppose $\mathscr{P} \in \mathfrak{M}$ and $\mathscr{P} \subsetneqq \mathscr{R}\{E, m\}$. Since $\mathscr{R} \cup \mathscr{P} = \mathscr{P}$, we appeal to Theorem 3.6 (vii) to see that $\mathscr{R}\{E, m\}$ does not cover $\mathscr{P}$. The second statement follows from Theorem 3.7 (vii).

THEOREM 3.10. *Let $(X, c)$ be a closure space such that $|\mathfrak{M}| > 1$. In addition, let $\mathscr{P} \in \mathfrak{M}$ and $\mathscr{P} \neq \mathscr{W}$. Let $(C, D) \in \mathscr{W} - \mathscr{P}$ and let $\mathscr{F}, \mathscr{G}$ be nonprincipal ultrafilters on $X$ containing $C, D$ respectively. Then $\mathscr{P}' = \mathscr{P} \cup (\mathscr{F} \times \mathscr{G}) \cup (\mathscr{G} \times \mathscr{F})$ is in $\mathfrak{M}$ and $\mathscr{P}'$ covers $\mathscr{P}$.*

*Proof.* Clearly $\mathscr{R} \subset \mathscr{P} \subset \mathscr{P}'$. Since each member of a nonprincipal ultrafilter is an infinite set, $\mathscr{F} \times \mathscr{G} \subset \mathscr{W}$. Hence $\mathscr{P}' \subset \mathscr{W}$. By Theorem 3.1, $\mathscr{P}'$ induces $c$ and satisfies P3, P4.

Clearly $\mathscr{P}'$ satisfies P1. To verify P2, let $(A, B \cup C) \in \mathscr{F} \times \mathscr{G}$. Hence $B \cup C \in \mathscr{G}$. Since $\mathscr{G}$ is an ultrafilter, it is known [4, p. 84] that $B \in \mathscr{G}$ or $C \in \mathscr{G}$. Thus $(A, B) \in \mathscr{F} \times \mathscr{G}$ or $(A, C) \in \mathscr{F} \times \mathscr{G}$.

Conversely, let $(A, B) \in \mathscr{F} \times \mathscr{G}$ and $C \subset X$. Since $\mathscr{G}$ is a filter and $B \in \mathscr{G}$, $B \cup C$ is in $\mathscr{G}$. Thus $(A, B \cup C) \in \mathscr{F} \times \mathscr{G}$.

We have shown that $\mathscr{P}' \in \mathfrak{M}$.

Suppose there exists $\mathscr{P}^* \in \mathfrak{M}$ and $\mathscr{P} \subsetneqq \mathscr{P}^* \subset \mathscr{P}'$. We will show that $\mathscr{P}^* = \mathscr{P}'$. Let $(F, G)$ belong to $\mathscr{P}^* - \mathscr{P}$. Then $(F, G)$ is in $\mathscr{P}' - \mathscr{P}$ and $(F, G)$ belongs to $\mathscr{F} \times \mathscr{G}$ or $\mathscr{G} \times \mathscr{F}$, say $\mathscr{F} \times \mathscr{G}$. To verify that $\mathscr{F} \times \mathscr{G} \subset \mathscr{P}^*$, we let $(A, B)$ be in $\mathscr{F} \times \mathscr{G}$. Then $G = (G \cap B) \cup (G - B)$. By P2, $(F, G \cap B) \in \mathscr{P}^*$ or $(F, G - B) \in \mathscr{P}^*$. Since the assumption that $(F, G - B) \in \mathscr{P}^*$ leads to a contradiction, we conclude that $(F, G \cap B) \in \mathscr{P}^*$.

Because $\mathscr{F}$ and $\mathscr{G}$ are filters, a similar argument shows $(F \cap A, G \cap B) \in \mathscr{P}^*$. P1 and P2 imply that $(A, B) \in \mathscr{P}^*$. Therefore, $\mathscr{F} \times \mathscr{G} \subset \mathscr{P}^*$. P1 implies that $\mathscr{G} \times \mathscr{F} \subset \mathscr{P}^*$. Hence $\mathscr{P}' \subset \mathscr{P}^*$ and $\mathscr{P}^* = \mathscr{P}'$.

THEOREM 3.11. *Let $(X, c)$ be a closure space such that $|\mathfrak{M}| > 1$. Let $\mathscr{P} \in \mathfrak{M}$, let $\mathscr{P} \neq \mathscr{R}$ and let $(C, D)$ belong to $\mathscr{P} - \mathscr{R}$. Then there exist nonprincipal ultrafilters $\mathscr{F}, \mathscr{G}$ on $X$ such that $(C, D) \in \mathscr{F} \times \mathscr{G} \subset \mathscr{P}$.*

*Proof.* Fix $(C, D)$ in $\mathscr{P} - \mathscr{R}$. Let $\mathscr{H} = [E: E \subset D$ and $(C, E) \in \mathscr{P} - \mathscr{R}]$. Let $\mathfrak{S}$ be the family of all subsets $\mathscr{H}^*$ of $\mathscr{H}$ having the property: $A, B \in \mathscr{H}^*$ implies $A \cap B \in \mathscr{H}^*$. We partially order $\mathfrak{S}$ by set inclusion. By Zorn's lemma, $\mathfrak{S}$ has a maximal element $\mathscr{G}_1$.

$\mathscr{G}_1$ is a filter base on $X$ due to the formation of $\mathscr{H}$ and $\mathfrak{S}$. Hence, there exists an ultrafilter $\mathscr{G}$ on $X$ containing $\mathscr{G}_1$ [4, pp. 78, 79, 83]. Furthermore, $D \in \mathscr{G}_1$ and $\mathscr{G}$ is a nonprincipal ultrafilter. Also, if $G \in \mathscr{G}$, then $(C, G \cap D) \in \mathscr{P} - \mathscr{R}$.

Let $\mathscr{L} = [L: L \subset C$ and $(L, G \cap D) \in \mathscr{P} - \mathscr{R}$ for all $G \in \mathscr{G}]$. Let $\mathfrak{J}$ be the family of all subsets $\mathscr{L}^*$ of $\mathscr{L}$ having the property: $S, T \in \mathscr{L}^*$ implies $S \cap T \in \mathscr{L}^*$. We partially order $\mathfrak{J}$ by set inclusion. By Zorn's lemma, $\mathfrak{J}$ has a maximal element $\mathscr{F}_1$ which is a filter base on $X$. Hence there exists an ultrafilter $\mathscr{F}$ on $X$ containing $\mathscr{F}_1$.

Moreover, $C \in \mathscr{F}_1$ and $\mathscr{F}$ is a nonprincipal ultrafilter. Finally, if $(F, G)$ is in $\mathscr{F} \times \mathscr{G}$, then $(F \cap C, G \cap D) \in \mathscr{P}$. Thus P1, P2 imply that $(F, G) \in \mathscr{P}$. So $\mathscr{F} \times \mathscr{G} \subset \mathscr{P}$.

THEOREM 3.12. *Let $(X, c)$ be a closure space such that $\mathscr{P}, \mathscr{P}' \in \mathfrak{M}$. Then $\mathscr{P}'$ covers $\mathscr{P}$ iff, given $(C, D)$ in $\mathscr{P}' - \mathscr{P}$, there exist nonprincipal ultrafilters $\mathscr{F}, \mathscr{G}$ on $X$ containing $C, D$ respectively such that $\mathscr{P}' = \mathscr{P} \cup (\mathscr{F} \times \mathscr{G}) \cup (\mathscr{G} \times \mathscr{F})$.*

*Proof.* Assume $\mathscr{P}'$ covers $\mathscr{P}$. Let $(C, D)$ belong to $\mathscr{P}' - \mathscr{P}$. Since $\mathscr{R} \subset \mathscr{P}$, $(C, D)$ is in $\mathscr{P}' - \mathscr{R}$. By Theorem 3.11, there are nonprincipal ultrafilters $\mathscr{F}, \mathscr{G}$ on $X$ such that $(C, D) \in \mathscr{F} \times \mathscr{G} \subset \mathscr{P}'$. P1 implies $\mathscr{G} \times \mathscr{F} \subset \mathscr{P}'$. Thus $\mathscr{P} \subsetneq \mathscr{P} \cup (\mathscr{F} \times \mathscr{G}) \cup (\mathscr{G} \times \mathscr{F}) \subset \mathscr{P}'$. By Theorem 3.10, $\mathscr{P} \cup (\mathscr{F} \times \mathscr{G}) \cup (\mathscr{G} \times \mathscr{F})$ is in $\mathfrak{M}$. Since $\mathscr{P}'$ covers $\mathscr{P}$, $\mathscr{P}' = \mathscr{P} \cup (\mathscr{F} \times \mathscr{G}) \cup (\mathscr{G} \times \mathscr{F})$.

The converse is a direct application of Theorem 3.10.

**4. The structure of the lattice of Č-proximities compatible with a given $R_0$-closure space.** In this section we first characterize greatest lower bound in $\mathfrak{M}$. Then it is shown that $\mathfrak{M}$ is strongly atomic and distributive. Finally, we prove that $\mathfrak{M}$ has no antiatoms and that if $|\mathfrak{M}| > 1$, then $|\mathfrak{M}| \geqq 2^{2^{\aleph_0}}$.

LEMMA 4.1. [2, p. 441] *Let $\mathscr{P}$ be a Č-proximity on $X$ and let*

*$(A, B) \in \mathscr{P}$. If $A = \bigcup [A_i: 1 \leqq i \leqq n]$ and $B = \bigcup [B_j: 1 \leqq j \leqq m]$ where $n$ and $m$ are integers, then there exists $i, j$ such that $(A_i, B_j) \in \mathscr{P}$.*

THEOREM 4.1. *Let $(X, c)$ be an $R_0$-closure space. Let $K$ be a nonempty index set and $\mathscr{P} = \bigwedge [\mathscr{P}_\alpha: \mathscr{P}_\alpha \in \mathfrak{M}$ and $\alpha \in K]$. Then $(A, B) \in \mathscr{P}$ iff, given $A = \bigcup [A_i: i \in I]$ and $B = \bigcup [B_j: j \in J]$ where $I$ and $J$ are finite sets, it follows that there exists $i, j$ such that $(A_i, B_j) \in \mathscr{P}_\alpha$ for each $\alpha \in K$.*

*Proof.* Let $(A, B) \in \mathscr{P}$, let $A = \bigcup [A_i: i \in I]$ and let $B = \bigcup [B_j: j \in J]$ where $I, J$ are finite sets. We appeal to Lemma 4.1 to obtain $i, j$ such that $(A_i, B_j) \in \mathscr{P}$. Since $\mathscr{P} \subset \mathscr{P}_\alpha$ for each $\alpha \in K$, $(A_i, B_j) \in \mathscr{P}_\alpha$.

Conversely, let $\mathscr{P}' = [(A, B)$: if $A = \bigcup [A_i: i \in I]$ and $B = \bigcup [B_j: j \in J]$ where $I, J$ are finite sets, then $\exists i, j$ such that $(A_i, B_j) \in \mathscr{P}_\alpha$ for each $\alpha \in K]$. Čech has proved [2, p. 470] that $\mathscr{P}'$ is a $\check{C}$-proximity on $X$, $\mathscr{P}' \subset \mathscr{P}_\alpha$ for each $\alpha \in K$ and if $\mathscr{P}^*$ is any $\check{C}$-proximity on $X$ such that $\mathscr{P}^* \subset \mathscr{P}_\alpha$ for each $\alpha \in K$, then $\mathscr{P}^* \subset \mathscr{P}'$. We shall prove that $\mathscr{P}'$ induces $c$. It then follows that $\mathscr{P} \in \mathfrak{M}$ and $\mathscr{P}' = \bigwedge [\mathscr{P}_\alpha: \mathscr{P}_\alpha \in \mathfrak{M}$ and $\alpha \in K]$.

Let $(C, D) \in \mathscr{R}$ and suppose $C = \bigcup [C_i: i \in I]$ and $D = \bigcup [D_j: j \in J]$ where $I, J$ are finite sets. By Lemma 4.1 there exist $i$ and $j$ such that $(C_i, D_j) \in \mathscr{R}$. Since $\mathscr{R} \subset \mathscr{P}_\alpha$ for each $\alpha \in K$, $(C_i, D_j) \in \mathscr{P}_\alpha$. Consequently $(C, D) \in \mathscr{P}'$. Thus $\mathscr{R} \subset \mathscr{P}'$. Since $\mathscr{P}' \subset \mathscr{P}_\alpha \subset \mathscr{W}$, we have $\mathscr{R} \subset \mathscr{P}' \subset \mathscr{W}$. By Theorem 3.1, $\mathscr{P}'$ induces $c$. □

We observe that the operation of meet in $\mathfrak{M}(X, c)$ is the restriction of the operation of meet in the family of all $\check{C}$-proximities on $X$ (no compatibility requirement). This follows from Theorem 4.1 and [2, p. 470]. Čech has established the analogous conclusion for the operation of join in these two lattices [2, p. 448]. Therefore, $\mathfrak{M}(X, c)$ is a sublattice of the lattice of all $\check{C}$-proximities on $X$.

THEOREM 4.2. *If $\mathscr{P} \in \mathfrak{M}$ and $\mathscr{P} \neq \mathscr{W}$, then there exists $\mathscr{P}^* \in \mathfrak{M}$ such that $\mathscr{P} \subsetneqq \mathscr{P}^* \subsetneqq \mathscr{W}$. Therefore, the lattice $\mathfrak{M}$ has no antiatoms. Also, $\mathfrak{M}$ is not antiatomic and is not anticovered iff $|\mathfrak{M}| > 1$.*

*Proof.* Since $\mathscr{W} = \mathscr{R}\{X, \aleph_0\}$ and $\mathscr{R} \cup \mathscr{P} = \mathscr{P} \subsetneqq \mathscr{W}$, we appeal to Theorem 3.6 (vii) to obtain $\mathscr{P}^*$ satisfying the theorem. The last two statements follow from the appropriate definitions.

COROLLARY 4.1. *If $|\mathfrak{M}| > 1$, then $\mathfrak{M}$ is not lattice isomorphic to a power set lattice.*

*Proof.* Every power set lattice with more than one element has antiatoms.

THEOREM 4.3. *The lattice $\mathfrak{M}$ is strongly atomic and consequently, atomic and covered.*

*Proof.* Let $\mathscr{P} \in \mathfrak{M}$ and $\mathscr{P} \subset \mathscr{P}^* \in \mathfrak{M}$. If $(C, D) \in \mathscr{P}^* - \mathscr{P}$, then, by Theorem 3.11, there exist nonprincipal ultrafilters $\mathscr{F}$, $\mathscr{G}$ on $X$ such that $(C, D) \in \mathscr{F} \times \mathscr{G} \subset \mathscr{P}^*$. P1 implies $\mathscr{G} \times \mathscr{F} \subset \mathscr{P}^*$. Thus $\mathscr{P}_{C,D} \equiv \mathscr{P} \cup (\mathscr{F} \times \mathscr{G}) \cup (\mathscr{G} \times \mathscr{F}) \subset \mathscr{P}^*$. By Theorem 3.10, $\mathscr{P}_{C,D}$ is an atom in the lattice $([\mathscr{P}' \in \mathfrak{M}: \mathscr{P}' \supset \mathscr{P}], \subset)$. Since $\bigcup [\mathscr{P}_{C,D}: (C, D)$ is in $\mathscr{P}^* - \mathscr{P}] = \mathscr{P}^*$, the lattice $([\mathscr{P}' \in \mathfrak{M}: \mathscr{P}' \supset \mathscr{P}], \subset)$ is atomic.

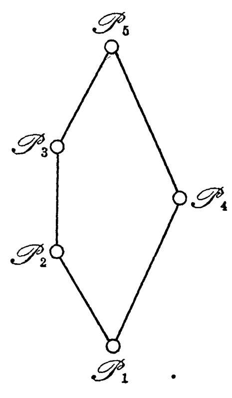
$\mathfrak{M}$ is atomic because every strongly atomic lattice with a least element is atomic. $\mathfrak{M}$ is covered since every strongly atomic lattice is covered.

COROLLARY 4.2. *If* $|\mathfrak{M}| > 1$, *then* $\mathfrak{M}$ *is not infinitely meet distributive.*

*Proof.* Suppose $\mathfrak{M}$ is infinitely meet distributive. Since it is well known that a complete, infinitely meet distributive lattice is a complete Boolean algebra, $\mathfrak{M}$ is a complete, atomic Boolean algebra. Consequently $\mathfrak{M}$ is isomorphic to a power set lattice which contradicts Corollary 4.1.

LEMMA 4.2. *The lattice $\mathfrak{M}$ is modular.*

*Proof.* It suffices [1, p. 13] to show that $\mathfrak{M}$ does not contain a sublattice of the form:




Suppose $\mathfrak{M}$ does contain such a sublattice. Let $\mathfrak{S}_1 =$ [atoms of $\mathfrak{M}$ which are contained in $\mathscr{P}_3$ but are not contained in $\mathscr{P}_2$] and $\mathfrak{S}_2 =$ [atoms of $\mathfrak{M}$ which are contained in $\mathscr{P}_4$ but are not contained in $\mathscr{P}_1$]. If $\mathfrak{S}_1 \cap \mathfrak{S}_2 \neq \phi$, then $\mathscr{P}_1 \subsetneqq \mathscr{P}_1 \cup (\bigcup (\mathfrak{S}_1 \cap \mathfrak{S}_2)) \subset \mathscr{P}_3 \wedge \mathscr{P}_4$. Since $\mathscr{P}_1 \cup$

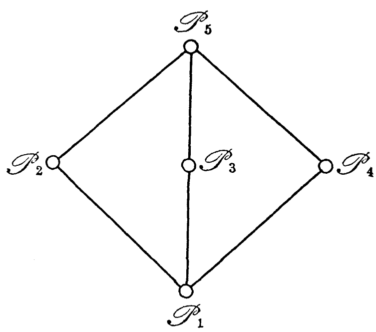
$(\bigcup(\mathfrak{S}_1 \cap \mathfrak{S}_2)) \in \mathfrak{M}$, we have contradicted $\mathscr{P}_1 = \mathscr{P}_3 \wedge \mathscr{P}_4$. Thus $\mathfrak{S}_1 \cap \mathfrak{S}_2 = \phi$.

Since $\mathfrak{M}$ is atomic, there exists $\mathscr{P}' \in \mathfrak{S}_1$. Because $\mathscr{P}' \not\subset \mathscr{P}_1$ and $\mathscr{P}' \notin \mathfrak{S}_2$, $\mathscr{P}' \not\subset \mathscr{P}_4$. Hence there exists $(A, B) \in \mathscr{P}' - \mathscr{P}_4$. Since $\mathscr{P}' \not\subset \mathscr{P}_2$, there exists $(C, D) \in \mathscr{P}' - \mathscr{P}_2$. Because $\mathscr{P}'$ is an atom in $\mathfrak{M}$, we appeal to Theorem 3.12 to obtain nonprincipal ultrafilters $\mathscr{F}, \mathscr{G}$ on $X$ containing $C, D$ respectively such that $\mathscr{P}' = \mathscr{R} \cup (\mathscr{F} \times \mathscr{G}) \cup (\mathscr{G} \times \mathscr{F})$. Now $(A, B)$ in $\mathscr{P}' - \mathscr{P}_4$ implies $(A, B)$ belongs to $\mathscr{F} \times \mathscr{G}$ or $\mathscr{G} \times \mathscr{F}$, say $\mathscr{F} \times \mathscr{G}$. Thus $(A \cap C, B \cap D) \in \mathscr{F} \times \mathscr{G} \subset \mathscr{P}' \subset \mathscr{P}_5$.

Since $(C, D) \notin \mathscr{P}_2$, P1 and P2 imply $(A \cap C, B \cap D) \notin \mathscr{P}_2$. Having seen $(A, B) \notin \mathscr{P}_4$, P1 and P2 imply $(A \cap C, B \cap D) \notin \mathscr{P}_4$. Thus $(A \cap C, B \cap D) \notin \mathscr{P}_2 \cup \mathscr{P}_4 = \mathscr{P}_5$, which is a contradiction.

THEOREM 4.4. *The lattice $\mathfrak{M}$ is distributive.*

*Proof.* In view of Lemma 4.2, it suffices [1, p. 39] to show that $\mathfrak{M}$ does not contain a sublattice of the form:




Suppose $\mathfrak{M}$ does contain such a sublattice. Let $\mathfrak{S}_i =$ [atoms of $\mathfrak{M}$ which are contained in $\mathscr{P}_i$ but are not contained in $\mathscr{P}_1$] $(i = 2, 3, 4)$. If $\mathfrak{S}_2 \cap \mathfrak{S}_3 \neq \phi$, then $\mathscr{P}_1 \subsetneqq \mathscr{P}_1 \cup (\bigcup (\mathfrak{S}_2 \cap \mathfrak{S}_3)) \subset \mathscr{P}_2 \wedge \mathscr{P}_3$. Since $\mathscr{P}_1 \cup (\bigcup (\mathfrak{S}_2 \cap \mathfrak{S}_3))$ is in $\mathfrak{M}$, we have contradicted $\mathscr{P}_1 = \mathscr{P}_2 \wedge \mathscr{P}_3$. Thus $\mathfrak{S}_2 \cap \mathfrak{S}_3 = \phi$. Similarly, $\mathfrak{S}_2 \cap \mathfrak{S}_4 = \phi$.

Since $\mathfrak{M}$ is atomic, there exists $\mathscr{P}' \in \mathfrak{S}_2$. Because $\mathscr{P}' \not\subset \mathscr{P}_1$ and $\mathscr{P}' \notin \mathfrak{S}_3$, $\mathscr{P}' \not\subset \mathscr{P}_3$. Hence there exists $(A, B) \in \mathscr{P}' - \mathscr{P}_3$. Similarly, there exists $(C, D) \in \mathscr{P}' - \mathscr{P}_4$. Because $\mathscr{P}'$ is an atom in $\mathfrak{M}$, we appeal to Theorem 3.12 to obtain nonprincipal ultrafilters $\mathscr{F}, \mathscr{G}$ on $X$ containing $C, D$ respectively such that $\mathscr{P}' = \mathscr{R} \cup (\mathscr{F} \times \mathscr{G}) \cup (\mathscr{G} \times \mathscr{F})$. $(A, B) \in \mathscr{P}' - \mathscr{P}_3$ implies $(A, B)$ is in $\mathscr{F} \times \mathscr{G}$ or $\mathscr{G} \times \mathscr{F}$, say $\mathscr{F} \times \mathscr{G}$. Therefore, $(A \cap C, B \cap D) \in \mathscr{F} \times \mathscr{G} \subset \mathscr{P}' \subset \mathscr{P}_5$.

Since $(C, D) \notin \mathscr{P}_4$, P1 and P2 imply $(A \cap C, B \cap D) \notin \mathscr{P}_4$. Having seen $(A, B) \notin \mathscr{P}_3$, P1 and P2 imply $(A \cap C, B \cap D) \notin \mathscr{P}_3$. Thus $(A \cap C, B \cap D) \notin \mathscr{P}_3 \cup \mathscr{P}_4 = \mathscr{P}_5$, which is a contradiction.

COROLLARY 4.3. *If* $|\mathfrak{M}| > 1$, *then* $\mathfrak{M}$ *is not complemented.*

*Proof.* If $\mathfrak{M}$ is complemented, then $\mathfrak{M}$ is a complete Boolean algebra, and thus is infinitely meet distributive [1, p. 118]. This contradicts Corollary 4.2.

THEOREM 4.5. *Let* $(X, c)$ *be an* $R_0$*-closure space. Then* $|\mathfrak{M}| = 1$ *iff, given two infinite subsets of* $X$, *at least one of them contains a point in the closure of the other.*

*Proof.* Assume $|\mathfrak{M}| = 1$. Let $A, B$ be two infinite subsets of $X$. Then $(A, B) \in \mathscr{W}$. $|\mathfrak{M}| = 1$ implies $\mathscr{R} = \mathscr{W}$. Thus $(\bar{A} \cap B) \cup (A \cap \bar{B}) \neq \phi$.

The converse is true because the assumption says $\mathscr{R} = \mathscr{W}$. Therefore, $|\mathfrak{M}| \leqq 1$. Since $(X, c)$ is $R_0$, $|\mathfrak{M}| \geqq 1$ by Theorem 2.1. □

We note that $|\mathfrak{M}| = 1$ for each of the following topological spaces: any $R_0$ topology on a finite set, any set with the indiscrete topology, any set with the minimum $T_1$ topology and any atom in the lattice of $T_1$ topologies on a fixed set. We also note that the characterization given in Theorem 4.5 can be expressed as: $|\mathfrak{M}| = 1$ iff any two infinite subsets of $X$ are not separated.

THEOREM 4.6. *Let* $(X, c)$ *be an* $R_0$*-closure space. Then* $|\mathfrak{M}| = 1$ *or* $2^{2^{\aleph_0}} \leqq |\mathfrak{M}| \leqq 2^{2^{|X|}}$. *Furthermore, if* $|X| \geqq \aleph_0$ *and* $m$ *is a cardinal number such that* $\aleph_0 \leqq m \leqq |X|$, *then there is a* $T_1$ *topology* $\mathscr{T}$ *on* $X$ *such that* $|\mathfrak{M}(X, \mathscr{T})| = 2^{2^m}$.

*Proof.* Theorem 2.1 implies $|\mathfrak{M}| \geqq 1$. If $|\mathfrak{M}| > 1$, then there exists $(C, D) \in \mathscr{W} - \mathscr{R}$. Thus $C, D$ are infinite and $C \cap D = \phi$. We appeal to Theorem 3.11 to obtain nonprincipal ultrafilters $\mathscr{U}, \mathscr{V}$ on $X$ containing $C, D$ respectively. Then $\mathscr{R} \cup (\mathscr{U} \times \mathscr{V}) \cup (\mathscr{V} \times \mathscr{U})$ is in $\mathfrak{M}$ by Theorem 3.10. We note that if $\mathscr{U}, \mathscr{F}$ are distinct nonprincipal ultrafilters on $X$ containing $C$, then $\mathscr{R} \cup (\mathscr{U} \times \mathscr{V}) \cup (\mathscr{V} \times \mathscr{U})$ and $\mathscr{R} \cup (\mathscr{F} \times \mathscr{V}) \cup (\mathscr{V} \times \mathscr{F})$ are distinct. From [2, p. 212] there are $2^{2^{|C|}}$ distinct ultrafilters on $X$ containining $C$. Since there are $|C|$ principal ultrafilters on $X$ containg $C$, there are $2^{2^{|C|}}$ distinct nonprincipal ultrafilters on $X$ containing $C$, and it follows that $2^{2^{\aleph_0}} \leqq 2^{2^{|C|}} \leqq |\mathfrak{M}|$.

On the other hand, since there are $2^{2^{|X|}}$ families of ordered pairs of subsets of $X$, and since each $\check{C}$-proximity is such a family, $|\mathfrak{M}| \leqq 2^{2^{|X|}}$.

To form $\mathscr{T}$, choose subsets $S, T$ of $X$ such that $S \cap T = \phi$ and $|S| = |T| = m$. Then $[\phi, X - S, X - T, X -$ any finite subset of $X]$ is a subbase for the desired topology $\mathscr{T}$. Since $(S, T) \in \mathscr{W} -$

$\mathscr{R}$, $|\mathfrak{M}(X, \mathscr{T})| > 1$. By the above argument, $|\mathfrak{M}(X, \mathscr{T})| \geqq 2^{2^{|S|}}$.

Let $\mathscr{T}'$ be the relative topology on $S \cup T$. Then $f\colon \mathfrak{M}(X, \mathscr{T}) \to \mathfrak{M}(S \cup T, \mathscr{T}')$ defined by $f(\mathscr{P}) = \mathscr{P} \cap (\mathscr{P}(S \cup T) \times \mathscr{P}(S \cup T))$ is a 1:1, onto map. Thus $|\mathfrak{M}(X, \mathscr{T})| = |\mathfrak{M}(S \cup T, \mathscr{T}')|$. By the argument above $|\mathfrak{M}(S \cup T, \mathscr{T}')| \leqq 2^{2^{|S \cup T|}}$, which establishes our result since $|S \cup T| = m$.

THEOREM 4.7. *Let $(X, c)$, $(X, d)$ be $R_0$-closure spaces. Let $\mathscr{R}_c$, $\mathscr{R}_d$ be the least members of $\mathfrak{M}(X, c)$, $\mathfrak{M}(X, d)$ respectively. If $c(A) \subset d(A)$ for each $A \subset X$, then $\mathscr{R}_c \subset \mathscr{R}_d$.*

*Proof.* The verification is straightforward.

THEOREM 4.8. *Let $(X, c)$, $(X, d)$ be $R_0$-closure spaces. Let $\mathscr{R}_c$, $\mathscr{R}_d$ be the least members of $\mathfrak{M}(X, c)$, $\mathfrak{M}(X, d)$ respectively. If $\mathscr{R}_c \subset \mathscr{R}_d$, then $|\mathfrak{M}(X, d)| \leqq |\mathfrak{M}(X, c)|$.*

*Proof.* Let $\mathfrak{A}$ be the family of atoms of $\mathfrak{M}(X, d)$. By Theorems 3.11 and 3.10, if $\mathscr{S} \in \mathfrak{A}$, then there are nonprincipal ultrafilters $\mathscr{U}$, $\mathscr{V}$ on $X$ such that $\mathscr{S} = \mathscr{R}_d \cup (\mathscr{U} \times \mathscr{V}) \cup (\mathscr{V} \times \mathscr{U})$. Since $\mathscr{R}_c \subset \mathscr{R}_d$, by Theorem 3.10 $\mathscr{R}_c \cup (\mathscr{U} \times \mathscr{V}) \cup (\mathscr{V} \times \mathscr{U})$ is an atom in $\mathfrak{M}(X, c)$.

Define $f\colon \mathfrak{A} \to \mathfrak{M}(X, c)$ by $f(\mathscr{R}_d \cup (\mathscr{U} \times \mathscr{V}) \cup (\mathscr{V} \times \mathscr{U})) = \mathscr{R}_c \cup (\mathscr{U} \times \mathscr{V}) \cup (\mathscr{V} \times \mathscr{U})$. Also, define $g\colon \mathfrak{M}(X, d) \to \mathfrak{M}(X, c)$ by

$$g(\mathscr{P}) = \begin{cases} \bigcup [f(\mathscr{S})\colon \mathscr{S} \in \mathfrak{A} \text{ and } \mathscr{S} \subset \mathscr{P}] & \text{if } \mathscr{P} \neq \mathscr{R}_d \\ \mathscr{R}_c & \text{if } \mathscr{P} = \mathscr{R}_d . \end{cases}$$

To verify that $g$ is 1:1, let $\mathscr{P}, \mathscr{P}' \in \mathfrak{M}(X, d)$ and $\mathscr{P} \neq \mathscr{P}'$. Since $\mathfrak{M}(X, d)$ is atomic, there exists $\mathscr{S}' \in \mathfrak{A}$ such that ($\mathscr{S}' \subset \mathscr{P}$ and $\mathscr{S}' \not\subset \mathscr{P}'$) or ($\mathscr{S}' \subset \mathscr{P}'$ and $\mathscr{S}' \not\subset \mathscr{P}$); say the former is true. Then $f(\mathscr{S}') \subset g(\mathscr{P})$ by the definition of $g$. Let $(A, B)$ belong to $\mathscr{S}' - \mathscr{P}'$. Since $\mathscr{R}_d \subset \mathscr{P}'$, $(A, B) \notin \mathscr{R}_d$. By Theorems 3.11 and 3.10 there are nonprincipal ultrafilters $\mathscr{U}$, $\mathscr{V}$ on $X$ such that $(A, B) \in \mathscr{U} \times \mathscr{V}$ and $\mathscr{S}' = \mathscr{R}_d \cup (\mathscr{U} \times \mathscr{V}) \cup (\mathscr{V} \times \mathscr{U})$. Hence $(A, B) \in f(\mathscr{S}') \subset g(\mathscr{P})$.

On the other hand, $(A, B) \notin \mathscr{P}'$ implies that $(A, B)$ is not a member of any atom contained in $\mathscr{P}'$. Therefore, $(A, B) \notin g(\mathscr{P}')$, and $g(\mathscr{P}) \neq g(\mathscr{P}')$.

## REFERENCES

1. G. Birkhoff, *Lattice Theory*, Amer. Math. Soc. Colloq. Pub., Vol. 25, Providence, Third Edition, 1967.

2. E. Čech, *Topological Spaces*, Interscience Publishers, John Wiley and Sons, New York, 1966.

3. S. A. Naimpally and B. D. Warrack, *Proximity Spaces*, Cambridge University Press, London, 1970.

4. W. J. Thron, *Topological Structures*, Holt, Rinehart and Winston, New York, 1966.

Received June 21, 1972.

UNIVERSITY OF COLORADO
AND
AEROSPACE RESEARCH LABORATORIES, WRIGHT-PATTERSON AFB, OHIO

# PACIFIC JOURNAL OF MATHEMATICS



Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. **39**. All other communications to the editors should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: $48.00 a year (6 Vols., 12 issues). Special rate: $24.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

# Pacific Journal of Mathematics

Vol. 49, No. 2 June, 1973