SELF-ADJOINT EXTENSIONS OF SYMMETRIC DIFFERENTIAL OPERATORS

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Let $\mathcal{H}$ denote the Hilbert space of square summable analytic function on the unit disk, and consider those formal differential operators

$$L = \sum_{i=0}^{n} p_i D^i$$

which give rise to symmetric operators in $\mathcal{H}$. This paper is devoted to a study of when these operators are actually self-adjoint or admit of self-adjoint extensions in $\mathcal{H}$. It is shown that in the first order case the operator is always self-adjoint. For $n > 1$ sufficient conditions on the $p_i$ are obtained for the existence of self-adjoint extensions. In particular a condition on the coefficients is obtained which insures that the operator has defect indices equal to the order of $L$.

Let $\mathcal{A}$ denote the space of functions analytic on the unit disk and $\mathcal{H}$ the subspace of square summable functions in $\mathcal{A}$ with inner product

$$(f, g) = \int_{|z|<1} \int f(z)\overline{g(z)} dxdy.$$

A complete orthonormal set for $\mathcal{H}$ is provided by the normalized powers of $z$,

$$e_n(z) = [(n + 1)/\pi]^{1/2} z^n, \quad n = 0, 1, \cdots.$$

From this it follows that $\mathcal{H}$ is identical with the space of power series $\sum_{n=0}^{\infty} a_n z^n$ which satisfy

$$\sum_{n=0}^{\infty} |a_n|^2/(n + 1) < \infty.$$

Consider the formal differential operator

$$L = p_n D^n + \cdots + p_1 D + p_0,$$

where $D = d/dz$ and the $p_i$ are in $\mathcal{H}$. We now associate two operators as follows. Let $\mathcal{D}_0$ denote the span of the $e_n$ and $\mathcal{D}$ the set of all $f$ in $\mathcal{H}$ for which $L f$ is in $\mathcal{H}$, and define $T_0$ and $T$ as

$$T_0 f = L f \quad f \in \mathcal{D}_0,$$

$$T f = L f \quad f \in \mathcal{D}.$$

It is shown in [2] that $T_0$ and $T$ are both densely defined operators.
in \( \mathcal{H} \), \( T_0 \subseteq T \) and \( T \) is closed. Moreover, \( T_0 \) is symmetric if and only if
\[
(Le_n, e_m) = (e_n, Le_m), \quad n, m = 0, 1, \cdots .
\]
Such a formal operator is said to be formally symmetric. Regarding symmetric \( T_0 \) we have the following result.

**Theorem 1.1.** If \( T_0 \) is symmetric, \( T_0^* = T \) and \( T^* \subseteq T \). The closure of \( T_0, S = T_0^{**} = T^* \), is self-adjoint if and only if \( S = T \).

**Proof.** See [2].

For \( f \) and \( g \) in \( \mathcal{D} \) consider the bilinear form
\[
\langle f, g \rangle = (Lf, g) - (f, Lg),
\]
and let \( \widetilde{\mathcal{D}} \) be the set of those \( f \) in \( \mathcal{D} \) for which \( \langle f, g \rangle = 0 \) for all \( g \) in \( \mathcal{D} \). Since \( S = T^* \) and \( \mathcal{D}(T^*) = \mathcal{D}, S \) has domain \( \mathcal{D} \).

Let \( \mathcal{D}^+ \) and \( \mathcal{D}^- \) denote the set of all solutions of the equation \( Lu = iu \) and \( Lu = -iu \) respectively, which are in \( \mathcal{H} \). It is known from the general theory of Hilbert space [1, p. 1227-1230] that \( \mathcal{D} = \widetilde{\mathcal{D}} + \mathcal{D}^+ + \mathcal{D}^- \), and every \( f \in \mathcal{D} \) has a unique such representation. Let the dimensions of \( \mathcal{D}^+ \) and \( \mathcal{D}^- \) be \( m^+ \) and \( m^- \) respectively. Clearly, \( m^+ \) and \( m^- \) cannot exceed the order of \( L \). These integers are referred to as the deficiency indices of \( S \), and \( S \) has self-adjoint extensions if and only if \( m^+ = m^- \). Moreover, \( S \) is self-adjoint if and only if \( m^+ = m^- = 0 \).

2. In [2] it is shown that the general formally symmetric first order operator is given by
\[
L = (cz^2 + az + c)D + (2cz + b)
\]
where \( a \) and \( b \) are real. In this case it is possible to compute the solutions of \( Lu = \pm iu \) explicitly and show that the solutions so obtained are not in \( \mathcal{H} \). Proceeding in this manner we obtain the following result.

**Theorem 2.1.** If \( L \) is a first order formally symmetric operator, the associated operator \( T \) is self-adjoint.

**Proof.** We shall show that \( m^+ \) and \( m^- \) are both zero. When \( c = 0 \) \( L \) is just the first order Euler operator, and hence \( T \) is self-adjoint by the corollary to Theorem 1.3 of [2]. When \( c \neq 0 \) we have
\[
(z^3 + (a/c)z + (c/b)u' + (2z + b/c - i/c)u = 0
\]
The coefficient of \( u' \) has zeros at
\[
\alpha = -a/2c + (a^2 - 4|c|^2)^{1/2}/2c, \\
\beta = -a/2c - (a^2 - 4|c|^2)^{1/2}/2c.
\]

There are three cases to consider:

1. \( a^2 < 4|c|^2 \)
2. \( a^2 = 4|c|^2 \)
3. \( a^2 > 4|c|^2 \).

In case 1 we have \( \alpha = -a/2c + iR/2c \), \( \beta = -a/2c - iR/2c \) where \( R = (4|c|^2 - \alpha^2)^{1/2} \), moreover \( |\alpha| = |\beta| = 1 \). Every solution of (2.2) is a multiple of the fundamental solution \( \varphi(z) = (z - a)^r(z - \beta)^s \) where \( r = (R - 1)/R + i(b - a)/R \) and \( s = (R + 1)/R + i(b - a)/R \). Hence every (nontrivial) solution of (2.2) is analytic in the open unit disc \( D \) with at least one singularity on the boundary at \( z = \beta \). We now show that \( \varphi \) is not in \( \mathcal{H} \), i.e., the integral \( \int_D \int_{|z|} \varphi(z)^2dxdy \) diverges.

Introduce polar coordinates at \( \beta \) so \( z - \beta = \rho e^{i\theta} \). Let \( \delta \) be less than \( |\beta - a| \), then there exist suitable \( \theta_1 \) and \( \theta_2 \) such that for \( 0 < \varepsilon < \delta \), the regions \( W_\varepsilon = \{z| \varepsilon < \rho < \delta, \theta_1 < \theta < \theta_2\} \) lie within \( D \) and \( \theta_1 < \beta < \theta_2 \).

Now
\[
\int_D \int_{|z|} \varphi(z)^2dxdy \geq \lim_{\varepsilon \to 0} \int_{W_\varepsilon} \int (z - \alpha)^{-r}(z - \beta)^{-s}dxdy.
\]

Since \( \alpha \in W_\varepsilon \), it follows from continuity that \( |(z - \alpha)^{-r}| \geq m > 0 \) for \( z \) in \( W_\varepsilon \), all \( 0 < \varepsilon < \delta \). Using this and the fact that \( |(z - \beta)^{-s}| = \rho^{-s}e^{s\theta} \), where \( s = u + iv \), the inequality of (2.4) becomes
\[
\int_D \int_{|z|} \varphi(z)^2dxdy \geq \lim_{\varepsilon \to 0} m \int_{\theta_1}^{\theta_2} \int_{\rho}^{\delta} \rho^{-2u+1}e^{2v}\rho d\theta d\rho,
\]
where \( k = \inf \) of \( e^{2v} \) on \( \theta_1 \leq \theta \leq \theta_2 \) which is greater than zero. But \( -2u + 1 = -2(R + 1)/R + 1 = -1 - 2/R < -1 \), hence the integral on the left diverges and \( \varphi \) is not square summable.

The fundamental solution for (2.3) is given by \( \varphi(z) = (z - \alpha)^{-r}(z - \beta)^{-s} \), where \( r = (R + 1)/R - i(b - a)/R \) and \( s = (R - 1)/R + i(b - a)/R \). Hence \( \varphi(z) \) is analytic in the open unit disc \( D \) with a singularity on the boundary at \( \alpha \). Let \( z - \alpha = \rho e^{i\theta} \), then there exist suitable \( \theta_1 \) and \( \theta_2 \) such that for \( 0 < \varepsilon < \delta < |\alpha - \beta| \), the regions \( W_\varepsilon = \{z| \varepsilon < \rho < \delta, \theta_1 \leq \theta \leq \theta_2\} \) lie within \( D \) and \( \beta \in W_\varepsilon \). As before, we obtain
where \(|z - \beta|^{-s}|^2 \geq m > 0\) for all \(z\) in \(W\), and \(0 < \varepsilon < \delta\), \(k\) is the infimum of \(e^{2\theta}\) on \(\theta_1 \leq \theta \leq \theta_2\) and \(r = u + iv\). But \(-2u + 1 = -(R + 2)/R < -1\), hence the integral on the left diverges and \(\phi\) is not square summable.

In case 2 the coefficient of \(w\) has a double zero at \(\alpha = -a/2c\) where \(|\alpha|^2 = a^2/4|c|^2 = 1\). The functions \(\phi_+(z) = (z - \alpha)^2 e^{(z-a)^{-1}}, r = (b - a - i)/c\) and \(\phi_-(z) = (z - \alpha)^-2 e^{(z-a)^{-1}}, r = (b - a + i)/c\) are fundamental solutions for (2.2) and (2.3) respectively. Let us introduce polar coordinates at \(z = \alpha\) so that \(2 - a: = \varepsilon e^{i\theta/2}\) and let us agree to set \(\theta = 0\) so that for \(|z| < 1\), the argument of \(z - \alpha\) is restricted to the intervals \(0 \leq \theta < \pi/2\) and \(3\pi/2 < \theta < 2\pi\). Let \(r = u + iv\), then

\[
|\phi_{\pm}(z)| = |\rho^{-2} e^{i\theta} e^{(u+iv)(\cos \theta - i \sin \theta)/\rho}|
\]

\[
= \rho^{-2} e^{(u \cos \theta + v \sin \theta)/\rho}.
\]

We note that \(u\) and \(v\) are not both zero, for then \(b - a \pm i = 0\) where \(a\) and \(b\) are real. Now consider the function \(F(\theta) = u \cos \theta + v \sin \theta\). If \(u > 0, F(0) = u > 0\) and by continuity there exist \(\theta_1\) and \(\theta_2\) such that \(F(\theta) \geq u/2 > 0\) for \(\theta_1 \leq \theta \leq \theta_2 < \pi/2\), similarly if \(v > 0, F(\pi/2) = v\) and there exist \(\theta_1\) and \(\theta_2\) such that \(F(\theta) \geq v/2 > 0\) for \(\theta_1 \leq \theta \leq \theta_2 \leq \pi/2\). If \(v < 0, F(3\pi/2) = -v > 0\) and there exist \(\theta_1\) and \(\theta_2\) such that \(F(\theta) \geq -v/2 > 0\) for \(3\pi/2 < \theta < \theta_2\). Hence for all \(r = u + iv\), except for the case \(u < 0, v = 0\), there exists a \(M > 0\) and suitable \(\theta_1\) and \(\theta_2\) for which \(F(\theta) \geq M, \theta_1 \leq \theta \leq \theta_2\). This case requires only a minor modification which will be provided shortly. It is easy to see that for given \(\theta_1\) and \(\theta_2\) we can find \(\delta > 0\) for which the regions \(W = \{z : |z| \leq \rho < \delta, \theta_1 \leq \theta \leq \theta_2\}\) lie entirely within the disc for \(0 < \varepsilon < \delta\).

Now consider \(||\phi_{\pm}||^2\):

\[
\int_0^\delta \int_{W_\varepsilon} |\phi_{\pm}(z)|^2 \, dx \, dy \geq \lim_{\varepsilon \to 0} \int_0^\delta \int_{W_\varepsilon} |\phi_{\pm}(z)|^2 \, dx \, dy
\]

\[
= \lim_{\varepsilon \to 0} \int_{\theta_2}^{\theta_2} \rho^{-3} e^{2F(\theta)\rho} \, d\rho \, d\theta
\]

\[
\geq \lim_{\varepsilon \to 0} (\theta_2 - \theta_1) \int_{\theta_1}^{\theta_2} e^{2M/\rho} \rho^{-3} \, d\rho.
\]

Since \(\int_{\theta_1}^{\theta_2} e^{2M/\rho} \rho^{-3} \, d\rho\) diverges it follows that the \(\phi_{\pm}\) are not square summable, provided \(r\) is not a negative number. When \(r = u + iv = u < 0\) we merely agree to set \(\theta = 0\) so that for \(|z| < 1\) the argument of \(z - \alpha\) is restricted to the interval \(\pi/2 < \theta < 3\pi/2\). Then \(F(\pi) = -u > 0\) and the argument is the same as before.
In case 3, $\alpha^2 > 4|c|^2$, the coefficient of $u'$ has distinct zeros at $\alpha = (-a + R)/2c$ and $\beta = (-a - R)/2c$ where $R = (\alpha^2 - 4|c|^2)^{1/2} > 0$. For $a > 0$,

$$\beta = \frac{a + R}{2|c|} > \frac{a}{2|c|} > 1,$$

and therefore $|\alpha| < 1$. For $a < 0$,

$$|\alpha| = \frac{R - a}{2|c|} > \frac{|a|}{2|c|} > 1,$$

and therefore $|\beta| < 1$. Without loss of generality we assume $|\alpha| < 1$, and $|\beta| > 1$. For $|z| < |\alpha| < 1$, the functions $\phi_+$ and $\phi_-$ given by

$$\phi_+(z) = (z - \alpha)^{-r}(z - \beta)^{-s},$$

$$\phi_-(z) = (z - \beta)^{-r}(z - \alpha)^{-s},$$

where $r = (R + b - a)/R - i/R$ and $s = (R + b - a)/R + i/R$, are fundamental solutions for $Lu = iu$ and $Lu = -iu$ respectively. Now suppose $\psi$ is any nontrivial element of $\mathcal{H}$ which satisfies $Lu = \pm iu$. In particular $\psi$ is analytic for $|z| < |\alpha| < 1$. From uniqueness results this implies that $\psi(z) = c\phi_\pm(z)$ for $|z| < |\alpha|$, where $c \neq 0$. By the identity theorem for analytic functions this implies $\psi(z) = c\phi_\pm(z)$ for $|z| < 1$, hence $\phi_\pm(z)$ is analytic in $|z| < 1$. But $\phi_\pm(z)$ has a singularity at $|\alpha| < 1$, therefore, the equations $Lu = \pm iu$ have no nontrivial solutions in $\mathcal{H}$.

3. In this section we obtain conditions on the coefficients of $L$ which insure that for all $\lambda$ every solution of $L\phi = \lambda\phi$ is in $\mathcal{H}$. If $L$ is a formally symmetric operator satisfying these conditions the defect indices of the operator $T_0$ are equal to the order of $L$ and $T_0$ has a self-adjoint extension in $\mathcal{H}$.

In [2] it was shown that if $L = \sum_{k=0}^n p_k D^k$ is formally symmetric then the $p_k$ are polynomials of degree at most $n + i$. Regarding such $L$ with polynomial coefficients we have

**Theorem 3.1.** Let $L = \sum_{k=0}^n p_k D^k$ where $n \geq 2$, $p_0(0) \neq 0$, and $p_k = \sum_{i=0}^{n+i} a_i(k)z^k$, and

$$A = |a_n(n)|^{-1} \sum_{i=1}^{2n} |a_i(n)|,$$

(3.1) \quad \hat{B} = n(n + 1)/2, \quad \text{and}\quad B = |a_n(n)|^{-1} \sum_{i=1}^{2n} |a_i(n)n[(n + 1)/2 - i] + a_{i-1}(n - 1)|.$$
If \( A < 1 \) or \( A = 1 \) and \( B < \hat{B} \) then every solution of \( L\phi = 0 \) is in \( \mathcal{H} \).

**Proof.** Since \( p_n(0) = a_0(n) \neq 0 \), every solution of \( Lu = 0 \) at the origin is analytic in some neighborhood of the origin. Let \( \phi(z) = \sum_{j=0}^{\infty} b_j z^j \) be any such solution, we will show that there exists a positive constant \( K \) and positive integer \( p \) such that \( |b_j| \leq K^{j-p} \) for \( j \) sufficiently large. Consequently the series \( \sum_{j=0}^{\infty} |b_j|/(j + 1) \) converges and \( \phi \) belongs to \( \mathcal{H} \).

We begin by obtaining a recursion formula for the \( b_j \). Substituting \( \phi(z) = \sum_{j=0}^{\infty} b_j z^j \) into the equation \( L\phi(z) = 0 \) we obtain

\[
L\phi(z) = \sum_{j=0}^{\infty} \sum_{k=0}^{n+k} a_i(k)\pi_k(j - i + k) b_{j-i+k} z^j,
\]

where

\[
\pi_k(\lambda) = \lambda(\lambda - 1) \cdots (\lambda - k + 1) \quad k \leq \lambda
\]

\[
= 0 \quad k > \lambda.
\]

Hence \( L\phi = 0 \) if and only if the following relationship holds for all \( j \).

(3.2) \[
\sum_{k=0}^{n} \sum_{i=0}^{n+k} a_i(k)\pi_k(j - i + k) b_{j-i+k} = 0.
\]

Hence,

\[
\sum_{k=0}^{n-i} \sum_{i=0}^{n+k} a_i(k)\pi_k(j - i + k) b_{j-i+k} + \sum_{i=1}^{2n} a_i(n)\pi_n(j - i + n) b_{j-i+n} + a_0(n)\pi_n(j + n) b_{j+n} = 0.
\]

Noting that the sums involve only the \( b_{j-n} \) thru \( b_{j+n-1} \) (where \( j > n \)) and \( \pi_n(j + n) \) never vanishes we may solve for \( b_{j+n} \) to obtain

(3.3) \[
b_{j+n} = -(S_1 + S_2)/a_0(n)\pi_n(j + n),
\]

where

\[
S_1 = \sum_{i=1}^{2n} a_i(n)\pi_n(j - i + n) b_{j-i+n},
\]

and

\[
S_2 = \sum_{k=0}^{n} \sum_{i=0}^{n+k} a_i(k)\pi_k(j - i + k) b_{j-i+k},
\]

for \( j > n \).

We now investigate the nature of \( S_1 \) and \( S_2 \) as polynomials in \( j \). It can be shown that \( \pi_n(j + n - 1) \) is a polynomial of degree \( n \) in \( j \),
\[
(3.4) \quad \pi_s(j + n - i) = j^n + \left[ \frac{n(n+1)}{2} - in \right] j^{n-1} + \cdots ,
\]

for \(i = 1, \ldots, 2n\). Using (3.4) in (3.3) we obtain
\[
S_1 = j^n \sum_{i=1}^{2n} a_i(n) b_{j-i+n}
+ j^{n-1} \sum_{i=1}^{2n} a_i(n) \left[ \frac{n(n+1)}{2} - in \right] + \text{lower powers of } j .
\]

Now consider \(S_2\). Since \(\pi_s(j - i + k)\) is a polynomial of degree \(k\) in \(j\), an examination of (3.3) shows that \(S_2\) is a polynomial of degree \(n - 1\) in \(j\), and that the only terms which contribute to the coefficient of \(j^{n-1}\) are those corresponding to \(k = n - 1\). Hence
\[
S_2 = j^{n-1} \sum_{i=1}^{2n} a_i(n - 1) b_{j-i+n-1}
+ \text{lower powers of } j .
\]

Combining (3.5) and (3.6) we obtain
\[
S_1 + S_2 = j^n \sum_{i=1}^{2n} a_i(n) b_{j-i+n}
+ j^{n-1} \sum_{i=1}^{2n} a_i(n) \left[ \frac{n(n+1)}{2} - in \right] + a_{i-1}(n - 1) b_{j-i+n}
+ \cdots , \quad (j > n) .
\]

Since \(\pi_s(j + n) = j^n + (n(n+1))/2j^{n-1} + \cdots\), is always positive (3.3) yields
\[
|b_{j+n}| = \frac{|S_1 + S_2|}{|a_o(n)| \left[ j^n + B_j^{n-1} + \cdots \right]} .
\]

We now estimate \(|S_1 + S_2|\). Let \(M(j) = \text{Max} (|b_{j-n}|, \ldots, |b_{j+n-1}|)\), then it follows from (3.1) and (3.7) that \(|S_1 + S_2| \leq |a_o(n)| [M(j) A_j^n + M(j) B_j^{n-1} + \cdots] .\) Hence
\[
|b_{j+n}| \leq \frac{A_j^n + B_j^{n-1}}{j^n + B_j^{n-1} + \cdots} M(j)
\]
for \(j > n\), where \(A, B,\) and \(\hat{B}\) are given by (3.1).

Consider the estimate (3.9) for \(|b_{j+n}|\),
\[
|b_{j+n}| \leq Q(j) M(j) \quad j > n ,
\]

where \(Q(j) = (A_j^n + B_j^{n-1} + \cdots)/(j^n + \hat{B}_j^{n-1} + \cdots)\). We note that for fixed \(\zeta\), \(Q(j) \leq 1 + \zeta j^{-1}\) for \(j\) sufficiently large if and only if \(A_j^n +
\[ B_j^{s-1} + \cdots \leq j^s + (\hat{B} + \zeta)j^{s-1} + \cdots. \]

Hence if \( A < 1 \) or \( A = 1 \) and \( B < \hat{B} + \zeta \) we have

\[ Q(j) \leq 1 + \zeta j^{-1} \]

for \( j \) sufficiently large. Now consider the expression

\[ (1 + \zeta(j + 1)^{-1}) (j - n + 1)^{-1/p}, \]

where \( \zeta < 0 \) and \( p \) a positive integer. It is not difficult to see that this is dominated by \( (j + n + 1)^{-1/p} \) for \( j \) sufficiently large if and only if

\[ j^{p+1} + (p + p\zeta + n + 1)j^p + \cdots \leq j^{p+1} + (p - n + 1)j^p + \cdots, \]

for \( j \) sufficiently large. Hence, we have

\[ (1 + \zeta(j + 1)^{-1}) (j - n + 1)^{-1/p} \leq (j + n + 1)^{-1/p} \]

for \( j \) sufficiently large if \( p \geq -2n\zeta^{-1}. \)

We now show that there exists a positive constant \( K \) and positive integer \( p \) for which \( |b_j| \leq K j^{-1/p}, j \) sufficiently large. By hypothesis either \( A < 1 \) or \( A = 1 \) and \( B < \hat{B} \). If \( A < 1 \) let \( \zeta = -1 \) and \( p = 2n \), if \( A = 1 \), select \( \zeta \) such that \( B - \hat{B} < \zeta < 0 \) and \( p > -2n\zeta^{-1}. \) For \( j \) sufficiently large, say \( j > j_1 \), (3.11) and (3.12) hold. Set

\[ K = \max_{j \leq j_1 + n} |b_j| j^{-1/p} \]

so that \( |b_j| \leq K j^{-1/p} \) for \( j \leq j_1 + n \). Using (3.10) and (3.11) it follows that

\[ |b_{j_1 + n+1}| \leq (1 + \zeta(j_1 + 1)^{-1})M(j_1 + 1), \]

where

\[ M(j_1 + 1) = \max (K(j_1 - n + 1)^{-1/p}, \cdots, K(j_1 + n)^{-1/p}) = K(j_1 - n + 1)^{-1/p}. \]

Hence \( |b_{j_1 + n+1}| \leq (1 + \zeta(j_1 + 1)^{-1})K(j_1 - n + 1)^{-1/p}, \) and using (3.12) this yields

\[ |b_{j_1 + n+1}| \leq K(j_1 + n + 1)^{-1/p}. \]

We now proceed inductively to establish

\[ |b_{j_1 + n+k}| \leq K(j_1 + n + k)^{-1/p} \quad k = 2, 3, \cdots. \]

Let \( K_i = \max_{j \leq j_1 + n+k} |b_j| j^{1/p} \), now \( K_i = \max (K, |b_{j_1 + n+1}| (j_1 + n + 1)^{1/p}) \) \( \leq K \), making use of (3.13). Using (3.11) yields

\[ |b_{j_1 + n+2}| \leq (1 + \zeta(j_1 + 2)^{-1})M(j_1 + 2) \]
where
\[ M(j_1 + 2) = \text{Max} \{ K(j_1 - n + 2)^{-1/p}, \ldots, K(j_1 + n + 1)^{-1/p} \} = K(j_1 - n + 2)^{-1/p}. \]
Using (3.12) it follows that
\[ |b_{j_1+n+2}| \leq K(j_1 + n + 2)^{-1/p}. \]
Continuing on in this manner we establish (3.14) and the theorem is proved.

We note that the conditions (3.1) of Theorem 3.1 involve only the coefficients of the polynomials \( p_n \) and \( p_{n-1} \), hence if \( L \) satisfies the conditions of (3.1) so do the operators \( L \pm i \). Hence we have established the following.

**Theorem 3.2.** Let \( L \) be a formally symmetric operator which satisfies (3.1), then the associated operator \( T_0 \) has defect indices \( n_+ = n_- = n \).

**Corollary 3.3.** The operator \( L = (c_2 z^4 + \bar{c}_2) d^2/dz^2 + (6c_1 z^2 + c_3 z^2 + a_2 z + \bar{c}_3) d/dz + (6c_3 z^3 + 2c_5 z + a_3) \), where \( a_2 \) and \( a_3 \) are real and \( |c_1| > |c_2| + |a_2|/2 \), has self-adjoint extensions.

**Proof.** Applying the algorithm given in Theorem 2.3 of [2] the general second order formally symmetric operator has coefficients
\[
\begin{align*}
p_2(z) &= c_1 z^4 + c_3 z^3 + a_2 z^2 + \bar{c}_3 z + c_1 \quad \text{and} \\
p_1(z) &= 6c_2 z^3 + (c_3 + 3c_1) z^2 + a_2 z + \bar{c}_3 \quad \text{and} \\
p_0(z) &= 6c_3 z^2 + 2c_5 z + a_3 \quad \text{.}
\end{align*}
\]
where \( a_1, a_2, \) and \( a_3 \) are real.
Now \( A = (|c_1| + 2|c_3| + |a_3|)/|c_1| \geq 1 \) and \( A = 1 \) if and only if \( c_3 = a_3 = 0 \). Now \( \tilde{B} = 3 \) and \( B = (|c_1| + |a_2| + 2|c_3|)/|c_1| < 3 \) if and only if \( |c_1| > |c_3| + |a_2|/2 \). Hence the result follows from the previous theorem.

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Received September 15, 1972 and in revised form January 5, 1973. This work was supported in part by NSF Grant GP-3594.

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