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SELF-ADJOINT EXTENSIONS OF SYMMETRIC DIFFERENTIAL OPERATORS

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Let \mathcal{H} denote the Hilbert space of square summable analytic function on the unit disk, and consider those formal differential operators

$$L = \sum_{i=0}^n p_i D^i$$

which give rise to symmetric operators in \mathcal{H} . This paper is devoted to a study of when these operators are actually self-adjoint or admit of self-adjoint extensions in \mathcal{H} . It is shown that in the first order case the operator is always self-adjoint. For $n > 1$ sufficient conditions on the p_i are obtained for the existence of self-adjoint extensions. In particular a condition on the coefficients is obtained which insures that the operator has defect indices equal to the order of L .

Let \mathcal{A} denote the space of functions analytic on the unit disk and \mathcal{H} the subspace of square summable functions in \mathcal{A} with inner product

$$(f, g) = \int_{|z|<1} \int f(z) \overline{g(\bar{z})} dx dy .$$

A complete orthonormal set for \mathcal{H} is provided by the normalized powers of z ,

$$e_n(z) = [(n + 1)/\pi]^{1/2} z^n , \quad n = 0, 1, \dots .$$

From this it follows that \mathcal{H} is identical with the space of power series $\sum_{n=0}^{\infty} a_n z^n$ which satisfy

$$(1.1) \quad \sum_{n=0}^{\infty} |a_n|^2 / (n + 1) < \infty .$$

Consider the formal differential operator

$$L = p_n D^n + \dots + p_1 D + p_0 ,$$

where $D = d/dz$ and the p_i are in \mathcal{H} . We now associate two operators as follows. Let \mathcal{D}_0 denote the span of the e_n and \mathcal{D} the set of all f in \mathcal{H} for which Lf is in \mathcal{H} , and define T_0 and T as

$$\begin{aligned} T_0 f &= Lf & f \in \mathcal{D}_0 \\ T f &= Lf & f \in \mathcal{D} . \end{aligned}$$

It is shown in [2] that T_0 and T are both densely defined operators

in \mathcal{H} , $T_0 \subseteq T$ and T is closed. Moreover, T_0 is symmetric if and only if

$$(1.2) \quad (Le_n, e_m) = (e_n, Le_m), \quad n, m = 0, 1, \dots$$

Such a formal operator is said to be formally symmetric. Regarding symmetric T_0 we have the following result.

THEOREM 1.1. *If T_0 is symmetric, $T_0^* = T$ and $T^* \subseteq T$. The closure of T_0 , $S = T_0^{**} = T^*$, is self-adjoint if and only if $S = T$.*

Proof. See [2].

For f and g in \mathcal{D} consider the bilinear form

$$(1.3) \quad \langle f, g \rangle = (Lf, g) - (f, Lg),$$

and let $\tilde{\mathcal{D}}$ be the set of those f in \mathcal{D} for which $\langle f, g \rangle = 0$ for all g in \mathcal{D} . Since $S = T^*$ and $\mathcal{D}(T^*) = \mathcal{D}$, S has domain $\tilde{\mathcal{D}}$.

Let \mathcal{D}^+ and \mathcal{D}^- denote the set of all solutions of the equation $Lu = iu$ and $Lu = -iu$ respectively, which are in \mathcal{H} . It is known from the general theory of Hilbert space [1, p. 1227-1230] that $\mathcal{D} = \tilde{\mathcal{D}} + \mathcal{D}^+ + \mathcal{D}^-$, and every $f \in \mathcal{D}$ has a unique such representation. Let the dimensions of \mathcal{D}^+ and \mathcal{D}^- be m^+ and m^- respectively. Clearly, m^+ and m^- cannot exceed the order of L . These integers are referred to as the deficiency indices of S , and S has self-adjoint extensions if and only if $m^+ = m^-$. Moreover, S is self-adjoint if and only if $m^+ = m^- = 0$.

2. In [2] it is shown that the general formally symmetric first order operator is given by

$$(2.1) \quad L = (cz^2 + az + \bar{c})D + (2cz + b)$$

where a and b are real. In this case it is possible to compute the solutions of $Lu = \pm iu$ explicitly and show that the solutions so obtained are not in \mathcal{H} . Proceeding in this manner we obtain the following result.

THEOREM 2.1. *If L is a first order formally symmetric operator, the associated operator T is self-adjoint.*

Proof. We shall show that m^+ and m^- are both zero. When $c = 0$ L is just the first order Euler operator, and hence T is self-adjoint by the corollary to Theorem 1.3 of [2]. When $c \neq 0$ we have

$$(2.2) \quad (z^2 + (a/c)z + \bar{c}/c)u' + (2z + b/c - i/c)u = 0$$

$$(2.3) \quad (z^2 + (a/c)z + \bar{c}/c)u' + (2z + b/c + i/c) = 0 .$$

The coefficient of u' has zeros at

$$\begin{aligned} \alpha &= -a/2c + (\alpha^2 - 4|c|^2)^{1/2}/2c . \\ \beta &= -a/2c - (\alpha^2 - 4|c|^2)^{1/2}/2c . \end{aligned}$$

There are three cases to consider:

1. $\alpha^2 < 4|c|^2$
2. $\alpha^2 = 4|c|^2$
3. $\alpha^2 > 4|c|^2$.

In case 1 we have $\alpha = -a/2c + iR/2c$, $\beta = -a/2c - iR/2c$ where $R = (4|c|^2 - \alpha^2)^{1/2}$, moreover $|\alpha| = |\beta| = 1$. Every solution of (2.2) is a multiple of the fundamental solution $\phi(z) = (z - \alpha)^{-r}(z - \beta)^{-s}$ where $r = (R - 1)/R - i(b - a)/R$ and $s = (R + 1)/R + i(b - a)/R$. Hence every (nontrivial) solution of (2.2) is analytic in the open unit disc D with at least one singularity on the boundary at $z = \beta$. We now show that ϕ is not in \mathcal{H} , i.e., the integral $\int_D \int |\phi(z)|^2 dx dy$ diverges. Introduce polar coordinates at β so $z - \beta = \rho e^{i\theta}$. Let δ be less than $|\beta - \alpha|$, then there exist suitable θ_1 and θ_2 such that for $0 < \varepsilon < \delta$, the regions $W_\varepsilon = \{z | \varepsilon \leq \rho \leq \delta, \theta_1 \leq \theta \leq \theta_2\}$ lie within D and $\alpha \notin W_\varepsilon$. Now

$$(2.4) \quad \int_D \int |\phi(z)|^2 dx dy \geq \lim_{\varepsilon \rightarrow 0} \int_{W_\varepsilon} \int |(z - \alpha)^{-r}|^2 |(z - \beta)^{-s}|^2 dx dy .$$

Since $\alpha \notin W_\varepsilon$ it follows from continuity that $|(z - \alpha)^{-r}|^2 \geq m > 0$ for z in W_ε , all $0 < \varepsilon < \delta$. Using this and the fact that $|(z - \beta)^{-s}| = \rho^{-u} e^{v\theta}$, where $s = u + iv$, the inequality of (2.4) becomes

$$\begin{aligned} \int_D \int |\phi(z)|^2 dx dy &\geq \lim_{\varepsilon \rightarrow 0} m \int_{\theta_1}^{\theta_2} \int_\varepsilon^\delta \rho^{-2u+1} e^{2v\theta} d\rho d\theta \\ &\geq \lim_{\varepsilon \rightarrow 0} mk(\theta_2 - \theta_1) \int_\varepsilon^\delta \rho^{-2u+1} d\rho , \end{aligned}$$

where $k = \text{infimum of } e^{2v\theta} \text{ on } \theta_1 \leq \theta \leq \theta_2$ which is greater than zero. But $-2u + 1 = -2(R + 1)/R + 1 = -1 - 2/R < -1$, hence the integral on the left diverges and ϕ is not square summable.

The fundamental solution for (2.3) is given by $\phi(z) = (z - \alpha)^{-r}(z - \beta)^{-s}$, where $r = (R + 1)/R - i(b - a)/R$ and $s = (R - 1)/R + i(b - a)/R$. Hence $\phi(z)$ is analytic in the open unit disc D with a singularity on the boundary at α . Let $z - \alpha = \rho e^{i\theta}$, then there exist suitable θ_1 and θ_2 such that for $0 < \varepsilon < \delta < |\alpha - \beta|$, the regions $W_\varepsilon = \{z | \varepsilon \leq \rho \leq \delta, \theta_1 \leq \theta \leq \theta_2\}$ lie within D and $\beta \in W_\varepsilon$. As before, we obtain

$$\int_D \int |\phi(z)|^2 dx dy \geq \lim_{\varepsilon \rightarrow 0} mk(\theta_2 - \theta_1) \int_{\varepsilon}^{\delta} \rho^{-2\mu+1} d\rho$$

where $|(z - \beta)^{-s}|^2 \geq m > 0$ for all z in W_ε and $0 < \varepsilon < \delta$, k is the infimum of $e^{2v\theta}$ on $\theta_1 \leq \theta \leq \theta_2$ and $r = u + iv$. But $-2u + 1 = -(\mathcal{R} + 2)/R < -1$, hence the integral on the left diverges and ϕ is not square summable.

In case 2 the coefficient of u' has a double zero at $\alpha = -a/2c$ where $|\alpha|^2 = a^2/4|c|^2 = 1$. The functions $\phi_+(z) = (z - \alpha)^{-2}e^{r(z-\alpha)^{-1}}$, $r = (b - a - i)/c$ and $\phi_-(z) = (z - \alpha)^{-2}e^{r(z-\alpha)^{-1}}$, $r = (b - a + i)/c$ are fundamental solutions for (2.2) and (2.3) respectively. Let us introduce polar coordinates at $z = \alpha$ so that $z - \alpha = \rho e^{i\theta}$ and let us agree to set $\theta = 0$ so that for $|z| < 1$, the argument of $z - \alpha$ is restricted to the intervals $0 \leq \theta < \pi/2$ and $3\pi/2 < \theta < 2\pi$. Let $r = u + iv$, then

$$\begin{aligned} |\phi_{\pm}(z)| &= |\rho^{-2}e^{-i2\theta}e^{(u+iv)(\cos\theta-i\sin\theta)/\rho}| \\ &= \rho^{-2}e^{(u\cos\theta+v\sin\theta)/\rho} . \end{aligned}$$

We note that u and v are not both zero, for then $b - a \pm i = 0$ where a and b are real. Now consider the function $F(\theta) = u \cos \theta + v \sin \theta$. If $u > 0$, $F(0) = u > 0$ and by continuity there exist θ_1 and θ_2 such that $F(\theta) \geq u/2 > 0$ for $\theta_1 \leq \theta \leq \theta_2 < \pi/2$, similarly if $v > 0$, $F(\pi/2) = v$ and there exist θ_1 and θ_2 such that $F(\theta) \geq v/2 > 0$ for $\theta_1 \leq \theta \leq \theta_2 \leq \pi/2$. If $v < 0$, $F(3\pi/2) = -v > 0$ and there exist θ_1 and θ_2 such that $F(\theta) \geq -v/2 > 0$ for $3\pi/2 < \theta_1 \leq \theta \leq \theta_2$. Hence for all $r = u + iv$, except for the case $u < 0, v = 0$, there exists a $M > 0$ and suitable θ_1 and θ_2 for which $F(\theta) \geq M, \theta_1 \leq \theta \leq \theta_2$. This case requires only a minor modification which will be provided shortly. It is easy to see that for given θ_1 and θ_2 we can find $\delta > 0$ for which the regions $W_\varepsilon = \{z | \varepsilon \leq \rho < \delta, \theta_1 \leq \theta \leq \theta_2\}$ lie entirely within the disc for $0 < \varepsilon < \delta$.

Now consider $\|\phi_{\pm}\|^2$:

$$\begin{aligned} \int_D \int |\phi_{\pm}(z)|^2 dx dy &\geq \lim_{\varepsilon \rightarrow 0} \int_{W_\varepsilon} \int |\phi_{\pm}(z)|^2 dx dy \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\theta_1}^{\theta_2} \int_{\varepsilon}^{\delta} \rho^{-3} e^{2F(\theta)/\rho} d\rho d\theta \\ &\geq \lim_{\varepsilon \rightarrow 0} (\theta_2 - \theta_1) \int_{\varepsilon}^{\delta} e^{2M/\rho} \rho^{-3} d\rho . \end{aligned}$$

Since $\int_0^\delta e^{2M/\rho} \rho^{-3} d\rho$ diverges it follows that the ϕ_{\pm} are not square summable, provided r is not a negative number. When $r = u + iv = u < 0$ we merely agree to set $\theta = 0$ so that for $|z| < 1$ the argument of $z - \alpha$ is restricted to the interval $\pi/2 < \theta < 3\pi/2$. Then $F(\pi) = -u > 0$ and the argument is the same as before.

In case 3, $a^2 > 4|c|^2$, the coefficient of u' has distinct zeros at $\alpha = (-a + R)/2c$ and $\beta = (-a - R)/2c$ where $R = (a^2 - 4|c|^2)^{1/2} > 0$. For $a > 0$,

$$|\beta| = \frac{R + a}{2|c|} > \frac{a}{2|c|} > 1,$$

and therefore $|\alpha| < 1$. For $a < 0$,

$$|\alpha| = \frac{R - a}{2|c|} > \frac{|a|}{2|c|} > 1,$$

and therefore $|\beta| < 1$. Without loss of generality we assume $|\alpha| < 1$, and $|\beta| > 1$. For $|z| < |\alpha| < 1$, the functions ϕ_+ and ϕ_- given by

$$\begin{aligned} \phi_+(z) &= (z - \alpha)^{-r}(z - \beta)^{-t} \\ \phi_-(z) &= (z - \beta)^{-s}(z - \alpha)^{-u} \end{aligned}$$

where $r = (R + b - a)/R - i/R$ and $s = (R + b - a)/R + i/R$, are fundamental solutions for $Lu = iu$ and $Lu = -iu$ respectively. Now suppose ψ is any nontrivial element of \mathcal{H} which satisfies $Lu = \pm iu$. In particular ψ is analytic for $|z| < |\alpha| < 1$. From uniqueness results this implies that $\psi(z) = c\phi_{\pm}(z)$ for $|z| < |\alpha|$, where $c \neq 0$. By the identity theorem for analytic functions this implies $\psi(z) = c\phi_{\pm}(z)$ for $|z| < 1$, hence $\phi_{\pm}(z)$ is analytic in $|z| < 1$. But $\phi_{\pm}(z)$ has a singularity at $|\alpha| < 1$, therefore, the equations $Lu = \pm iu$ have no nontrivial solutions in \mathcal{H} .

3. In this section we obtain conditions on the coefficients of L which insure that for all λ every solution of $L\phi = \lambda\phi$ is in \mathcal{H} . If L is a formally symmetric operator satisfying these conditions the defect indices of the operator T_0 are equal to the order of L and T_0 has a self-adjoint extension in \mathcal{H} .

In [2] it was shown that if $L = \sum_{k=0}^n p_k D^k$ is formally symmetric then the p_i are polynomials of degree at most $n + i$. Regarding such L with polynomial coefficients we have

THEOREM 3.1. *Let $L = \sum_{k=0}^n p_k D^k$ where $n \geq 2$, $p_n(0) \neq 0$, and $p_k = \sum_{i=0}^{n+k} a_i(k)z^k$, and*

$$\begin{aligned} A &= |a_0(n)|^{-1} \sum_{i=1}^{2n} |a_i(n)|, \\ (3.1) \quad \hat{B} &= n(n + 1)/2, \quad \text{and} \\ B &= |a_0(n)|^{-1} \sum_{i=1}^{2n} |a_i(n)n[(n + 1)/2 - i] + a_{i-1}(n - 1)|. \end{aligned}$$

If $A < 1$ or $A = 1$ and $B < \hat{B}$ then every solution of $L\phi = 0$ is in \mathcal{H} .

Proof. Since $p_n(0) = a_0(n) \neq 0$, every solution of $Lu = 0$ at the origin is analytic in some neighborhood of the origin. Let $\phi(z) = \sum_{j=0}^{\infty} b_j z^j$ be any such solution, we will show that there exists a positive constant K and positive integer p such that $|b_j| \leq K j^{-1/p}$ for j sufficiently large. Consequently the series $\sum_{j=0}^{\infty} |b_j|^2 / (j + 1)$ converges and ϕ belongs to \mathcal{H} .

We begin by obtaining a recursion formula for the b_j . Substituting $\phi(z) = \sum_{j=0}^{\infty} b_j z^j$ into the equation $L\phi(z) = 0$ we obtain

$$L\phi(z) = \sum_{j=0}^{\infty} \sum_{k=0}^n \sum_{i=0}^{n+k} \alpha_i(k) \pi_k(j - i + k) b_{j-i+k} z^j,$$

where

$$\begin{aligned} \pi_k(\lambda) &= \lambda(\lambda - 1) \cdots (\lambda - k + 1) & k \leq \lambda \\ &= 0 & k > \lambda. \end{aligned}$$

Hence $L\phi = 0$ if and only if the following relationship holds for all j .

$$(3.2) \quad \sum_{k=0}^n \sum_{i=0}^{n+k} \alpha_i(k) \pi_k(j - i + k) b_{j-i+k} = 0.$$

Hence,

$$\begin{aligned} &\sum_{k=0}^{n-1} \sum_{i=0}^{n+k} \alpha_i(k) \pi_k(j - i + k) b_{j-i+k} \\ &+ \sum_{i=1}^{2n} \alpha_i(n) \pi_n(j - i + n) b_{j-i+n} + \alpha_0(n) \pi_n(j + n) b_{j+n} = 0. \end{aligned}$$

Noting that the sums involve only the b_{j-n} thru b_{j+n-1} (where $j > n$) and $\pi_n(j + n)$ never vanishes we may solve for b_{j+n} to obtain

$$(3.3) \quad b_{j+n} = - (S_1 + S_2) / \alpha_0(n) \pi_n(j + n),$$

where

$$S_1 = \sum_{i=1}^{2n} \alpha_i(n) \pi_n(j - i + n) b_{j-i+n},$$

and

$$S_2 = \sum_{k=0}^{n-1} \sum_{i=0}^{n+k} \alpha_i(k) \pi_k(j - i + k) b_{j-i+k},$$

for $j > n$.

We now investigate the nature of S_1 and S_2 as polynomials in j . It can be shown that $\pi_n(j + n - 1)$ is a polynomial of degree n in j ,

$$(3.4) \quad \pi_n(j + n - i) = j^n + \left[\frac{n(n + 1)}{2} - in \right] j^{n-1} + \dots,$$

for $i = 1, \dots, 2n$. Using (3.4) in (3.3) we obtain

$$(3.5) \quad \begin{aligned} S_1 &= j^n \sum_{i=1}^{2n} a_i(n) b_{j-i+n} \\ &+ j^{n-1} \sum_{i=1}^{2n} a_i(n) \left[\frac{n(n + 1)}{2} - in \right] \\ &+ \text{lower powers of } j. \end{aligned}$$

Now consider S_2 . Since $\pi_k(j - i + k)$ is a polynomial of degree k in j , an examination of (3.3) shows that S_2 is a polynomial of degree $n - 1$ in j , and that the only terms which contribute to the coefficient of j^{n-1} are those corresponding to $k = n - 1$. Hence

$$(3.6) \quad \begin{aligned} S_2 &= j^{n-1} \sum_{i=0}^{2n-1} a_i(n - 1) b_{j-i+n-1} \\ &+ \text{lower powers of } j. \end{aligned}$$

Combining (3.5) and (3.6) we obtain

$$(3.7) \quad \begin{aligned} S_1 + S_2 &= j^n \sum_{i=1}^{2n} a_i(n) b_{j-i+n} \\ &+ j^{n-1} \sum_{i=1}^{2n} \left[a_i(n) \left(\frac{n(n + 1)}{2} - in \right) + a_{i-1}(n - 1) \right] b_{j-i+n} \\ &+ \dots, \quad (j > n). \end{aligned}$$

Since $\pi_n(j + n) = j^n + (n(n + 1))/2j^{n-1} + \dots$, is always positive (3.3) yields

$$(3.8) \quad |b_{j+n}| = \frac{|S_1 + S_2|}{|a_0(n)| [j^n + \hat{B}j^{n-1} + \dots]}.$$

We now estimate $|S_1 + S_2|$. Let $M(j) = \text{Max}(|b_{j-n}|, \dots, |b_{j+n-1}|)$, then it follows from (3.1) and (3.7) that $|S_1 + S_2| \leq |a_0(n)| [M(j)Aj^n + M(j)Bj^{n-1} + \dots]$. Hence

$$(3.9) \quad |b_{j+n}| \leq \frac{Aj^n + Bj^{n-1} + \dots}{j^n + \hat{B}j^{n-1} + \dots} M(j)$$

for $j > n$, where A, B , and \hat{B} are given by (3.1).

Consider the estimate (3.9) for $|b_{j+n}|$,

$$(3.10) \quad |b_{j+n}| \leq Q(j)M(j) \quad j > n,$$

where $Q(j) = (Aj^n + Bj^{n-1} + \dots)/(j^n + \hat{B}j^{n-1} + \dots)$. We note that for fixed ζ , $Q(j) \leq 1 + \zeta j^{-1}$ for j sufficiently large if and only if $Aj^n +$

$Bj^{n-1} + \dots \leq j^n + (\hat{B} + \zeta)j^{n-1} + \dots$. Hence if $A < 1$ or $A = 1$ and $B < \hat{B} + \zeta$ we have

$$(3.11) \quad Q(j) \leq 1 + \zeta j^{-1}$$

for j sufficiently large. Now consider the expression

$$(1 + \zeta(j+1)^{-1})(j-n+1)^{-1/p},$$

where $\zeta < 0$ and p a positive integer. It is not difficult to see that this is dominated by $(j+n+1)^{-1/p}$ for j sufficiently large if and only if

$$j^{p+1} + (p + p\zeta + n + 1)j^p + \dots \leq j^{p+1} + (p - n + 1)j^p + \dots,$$

for j sufficiently large. Hence, we have

$$(3.12) \quad (1 + \zeta(j+1)^{-1})(j-n+1)^{-1/p} \leq (j+n+1)^{-1/p}$$

for j sufficiently large if $p \geq -2n\zeta^{-1}$.

We now show that there exists a positive constant K and positive integer p for which $|b_j| \leq Kj^{-1/p}$, j sufficiently large. By hypothesis either $A < 1$ or $A = 1$ and $B < \hat{B}$. If $A < 1$ let $\zeta = -1$ and $p = 2n$, if $A = 1$, select ζ such that $B - \hat{B} < \zeta < 0$ and $p > -2n\zeta^{-1}$. For j sufficiently large, say $j > j_1$, (3.11) and (3.12) hold. Set

$$K = \max_{j \leq j_1+n} |b_j| j^{1/p}$$

so that $|b_j| \leq Kj^{-1/p}$ for $j \leq j_1 + n$. Using (3.10) and (3.11) it follows that

$$|b_{j_1+n+1}| \leq (1 + \zeta(j_1+1)^{-1})M(j_1+1),$$

where

$$\begin{aligned} M(j_1+1) &= \text{Max}(K(j_1-n+1)^{-1/p}, \dots, K(j_1+n)^{-1/p}) \\ &= K(j_1-n+1)^{-1/p}. \end{aligned}$$

Hence $|b_{j_1+n+1}| \leq (1 + \zeta(j_1+1)^{-1})K(j_1-n+1)^{-1/p}$, and using (3.12) this yields

$$(3.13) \quad |b_{j_1+n+1}| \leq K(j_1+n+1)^{-1/p}.$$

We now proceed inductively to establish

$$(3.14) \quad |b_{j_1+n+k}| \leq K(j_1+n+k)^{-1/p} \quad k = 2, 3, \dots$$

Let $K_1 = \max_{j \leq j_1+n+1} |b_j| j^{1/p}$, now $K_1 = \max\{K, |b_{j_1+n+1}|(j_1+n+1)^{1/p}\} \leq K$, making use of (3.13). Using (3.11) yields

$$|b_{j_1+n+2}| \leq (1 + \zeta(j_1+2)^{-1})M(j_1+2)$$

where

$$M(j_1 + 2) = \text{Max} (K(j_i - n + 2)^{-1/p}, \dots, K(j_1 + n + 1)^{-1/p}) \\ = K(j_1 - n + 2)^{-1/p} .$$

Using (3.12) it follows that

$$|b_{j_1+n+2}| \leq K(j_1 + n + 2)^{-1/p} .$$

Continuing on in this manner we establish (3.14) and the theorem is proved.

We note that the conditions (3.1) of Theorem 3.1 involve only the coefficients of the polynomials p_n and p_{n-1} , hence if L satisfies the conditions of (3.1) so do the operators $L \pm i$. Hence we have established the following.

THEOREM 3.2. *Let L be a formally symmetric operator which satisfies (3.1), then the associated operator T_0 has defect indices $n_+ = n_- = n$.*

COROLLARY 3.3. *The operator $L = (c_1z^4 + \bar{c}_1)d^2/dz^2 + (6c_1z^3 + c_3z^2 + a_2z + \bar{c}_3)d/dz + (6c_1z^2 + 2c_3z + a_3)$, where a_3 and a_2 are real and $|c_1| > |c_3| + |a_2|/2$, has self-adjoint extensions.*

Proof. Applying the algorithm given in Theorem 2.3 of [2] the general second order formally symmetric operator has coefficients

$$p_2(z) = c_1z^4 + c_2z^3 + a_1z^2 + \bar{c}_2z + \bar{c}_1 \\ p_1(z) = 6c_1z^3 + (c_3 + 3c_2)z^2 + a_2z + \bar{c}_3 \\ p_0(z) = 6c_1z^2 + 2c_3z + a_3 ,$$

where $a_1, a_2,$ and a_3 are real.

Now $A = (|c_1| + 2|c_2| + |a_1|)/|c_1| \geq 1$ and $A = 1$ if and only if $c_2 = a_1 = 0$. Now $\hat{B} = 3$ and $B = (|c_1| + |a_2| + 2|c_3|)/|c_1| < 3$ if and only if $|c_1| > |c_3| + |a_2|/2$. Hence the result follows from the previous theorem.

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