SELF-ADJOINT EXTENSIONS OF SYMMETRIC DIFFERENTIAL OPERATORS

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Let \( \mathcal{H} \) denote the Hilbert space of square summable analytic function on the unit disk, and consider those formal differential operators

\[
L = \sum_{i=0}^{n} p_i D^i
\]

which give rise to symmetric operators in \( \mathcal{H} \). This paper is devoted to a study of when these operators are actually self-adjoint or admit of self-adjoint extensions in \( \mathcal{H} \). It is shown that in the first order case the operator is always self-adjoint. For \( n > 1 \) sufficient conditions on the \( p_i \) are obtained for the existence of self-adjoint extensions. In particular a condition on the coefficients is obtained which insures that the operator has defect indices equal to the order of \( L \).

Let \( \mathcal{A} \) denote the space of functions analytic on the unit disk and \( \mathcal{H} \) the subspace of square summable functions in \( \mathcal{A} \) with inner product

\[
(f, g) = \int_{|z|<1} \int f(z) \overline{g(z)} \, dx \, dy.
\]

A complete orthonormal set for \( \mathcal{H} \) is provided by the normalized powers of \( z \),

\[
e_n(z) = \left( (n+1)/\pi \right)^{1/2} z^n, \quad n = 0, 1, \ldots
\]

From this it follows that \( \mathcal{H} \) is identical with the space of power series \( \sum_{n=0}^{\infty} a_n z^n \) which satisfy

\[
\sum_{n=0}^{\infty} |a_n|^2/(n+1) < \infty .
\]

Consider the formal differential operator

\[
L = p_n D^n + \cdots + p_1 D + p_0 ,
\]

where \( D = d/dz \) and the \( p_i \) are in \( \mathcal{H} \). We now associate two operators as follows. Let \( \mathcal{D}_0 \) denote the span of the \( e_n \) and \( \mathcal{D} \) the set of all \( f \) in \( \mathcal{H} \) for which \( Lf \) is in \( \mathcal{H} \), and define \( T_0 \) and \( T \) as

\[
T_0 f = Lf \quad f \in \mathcal{D}_0
\]

\[
T f = Lf \quad f \in \mathcal{D}.
\]

It is shown in [2] that \( T_0 \) and \( T \) are both densely defined operators.
in \( \mathcal{H} \), \( T_0 \subseteq T \) and \( T \) is closed. Moreover, \( T_0 \) is symmetric if and only if

\[(1.2) \quad (Le_n, e_m) = (e_n, Le_m), \quad n, m = 0, 1, \ldots.\]

Such a formal operator is said to be formally symmetric. Regarding symmetric \( T_0 \) we have the following result.

**Theorem 1.1.** If \( T_0 \) is symmetric, \( T_0^* = T \) and \( T^* \subseteq T \). The closure of \( T_0 \), \( S = T_0^{**} = T^* \), is self-adjoint if and only if \( S = T \).

**Proof.** See [2].

For \( f \) and \( g \) in \( \mathcal{D} \) consider the bilinear form

\[(1.3) \quad \langle f, g \rangle = (Lf, g) - (f, Lg),\]

and let \( \mathcal{D} \) be the set of those \( f \) in \( \mathcal{D} \) for which \( \langle f, g \rangle = 0 \) for all \( g \) in \( \mathcal{D} \). Since \( S = T^* \) and \( \mathcal{D}(T^*) = \mathcal{D} \), \( S \) has domain \( \mathcal{D} \).

Let \( \mathcal{D}^+ \) and \( \mathcal{D}^- \) denote the set of all solutions of the equation \( Lu = iu \) and \( Lu = -iu \) respectively, which are in \( \mathcal{H} \). It is known from the general theory of Hilbert space [1, p. 1227–1230] that \( \mathcal{D} = \mathcal{D}^- \mathcal{D}^+ \mathcal{D}^- \), and every \( f \in \mathcal{D} \) has a unique such representation. Let the dimensions of \( \mathcal{D}^+ \) and \( \mathcal{D}^- \) be \( m^+ \) and \( m^- \) respectively. Clearly, \( m^+ \) and \( m^- \) cannot exceed the order of \( L \). These integers are referred to as the deficiency indices of \( S \), and \( S \) has self-adjoint extensions if and only if \( m^+ = m^- \). Moreover, \( S \) is self-adjoint if and only if \( m^+ = m^- = 0 \).

2. In [2] it is shown that the general formally symmetric first order operator is given by

\[(2.1) \quad L = (cz^2 + az + \bar{c})D + (2cz + b)\]

where \( a \) and \( b \) are real. In this case it is possible to compute the solutions of \( Lu = \pm iu \) explicitly and show that the solutions so obtained are not in \( \mathcal{H} \). Proceeding in this manner we obtain the following result.

**Theorem 2.1.** If \( L \) is a first order formally symmetric operator, the associated operator \( T \) is self-adjoint.

**Proof.** We shall show that \( m^+ \) and \( m^- \) are both zero. When \( c = 0 \) \( L \) is just the first order Euler operator, and hence \( T \) is self-adjoint by the corollary to Theorem 1.3 of [2]. When \( c \neq 0 \) we have

\[(2.2) \quad (z^2 + (a/c)z + \bar{c}/c)w' + (2z + b/c - i/c)u = 0\]
(2.3) \((z^2 + (a/c)z + c/c)u' + (2z + b/c + i/e) = 0\).

The coefficient of \(u'\) has zeros at
\[
\alpha = -a/2c + (a^2 - 4|c|^2)^{1/2}/2c, \\
\beta = -a/2c - (a^2 - 4|c|^2)^{1/2}/2c.
\]

There are three cases to consider:

1. \(a^2 < 4|c|^2\)
2. \(a^2 = 4|c|^2\)
3. \(a^2 > 4|c|^2\).

In case 1 we have \(\alpha = -a/2c + iR/2c, \beta = -a/2c - iR/2c\) where \(R = (4|c|^2 - a^2)^{1/2}\), moreover \(|\alpha| = |\beta| = 1\). Every solution of (2.2) is a multiple of the fundamental solution \(\phi(z) = (z - a)^r(z - \beta)^s\) where \(r = (R - 1)/R - i(b - a)/R\) and \(s = (R + 1)/R + i(b - a)/R\). Hence every (nontrivial) solution of (2.2) is analytic in the open unit disc \(D\) with at least one singularity on the boundary at \(z = \beta\). We now show that \(\phi\) is not in \(\mathcal{H}\), i.e., the integral \(\int_D \int_W |\phi(z)|^2 dx dy\) diverges. Introduce polar coordinates at \(\beta\) so \(z - \beta = \rho e^{i\theta}\). Let \(\delta\) be less than \(|\beta - \alpha|\), then there exist suitable \(\theta_1\) and \(\theta_2\) such that for \(0 < \epsilon < \delta\), the regions \(W_\epsilon = \{z|\epsilon \leq \rho \leq \delta, \theta_1 \leq \theta \leq \theta_2\}\) lie within \(D\) and \(\alpha \in W_\epsilon\). Now

\[
(2.4) \int_D \int_W |\phi(z)|^2 dx dy = \lim_{\epsilon \to 0} \int_{W_\epsilon} |(z - \alpha)^{-r}|^2 |(z - \beta)^{-s}|^2 dx dy.
\]

Since \(\alpha \in W_\epsilon\) it follows from continuity that \(|(z - \alpha)^{-r}|^2 \geq m > 0\) for \(z\) in \(W_\epsilon\), all \(0 < \epsilon < \delta\). Using this and the fact that \(|(z - \beta)^{-s}| = \rho^{-s}e^{i\theta}\), where \(s = u + iv\), the inequality of (2.4) becomes

\[
\int_D \int_{W_\epsilon} |\phi(z)|^2 dx dy \geq \lim_{\epsilon \to 0} m \int_{\theta_1}^{\theta_2} \int_0^{\delta} \rho^{-2u + 1} e^{2v\theta} d\rho d\theta \\
\geq \lim_{\epsilon \to 0} mk(\theta_2 - \theta_1) \int_0^{\delta} \rho^{-2u + 1} d\rho,
\]

where \(k = \infimum\) of \(e^{2v\theta}\) on \(\theta_1 \leq \theta \leq \theta_2\) which is greater than zero. But \(-2u + 1 = -2(R + 1)/R + 1 = -1 - 2/R < -1\), hence the integral on the left diverges and \(\phi\) is not square summable.

The fundamental solution for (2.3) is given by \(\phi(z) = (z - \alpha)^{-r}(z - \beta)^{-s}\), where \(r = (R + 1)/R - i(b - a)/R\) and \(s = (R - 1)/R + i(b - a)/R\). Hence \(\phi(z)\) is analytic in the open unit disc \(D\) with a singularity on the boundary at \(\alpha\). Let \(z - \alpha = \rho e^{i\theta}\), then there exist suitable \(\theta_1\) and \(\theta_2\) such that for \(0 < \epsilon < \delta < |\alpha - \beta|\), the regions \(W_\epsilon = \{z|\epsilon \leq \rho \leq \delta, \theta_1 \leq \theta \leq \theta_2\}\) lie within \(D\) and \(\beta \in W_\epsilon\). As before, we obtain
where \(|(z - \beta)^{-r}|^2 \geq m > 0\) for all \(z\) in \(W\), and \(0 < \varepsilon < \delta\), \(k\) is the infimum of \(e^{2\rho}\) on \(\theta_1 \leq \theta \leq \theta_2\) and \(r = u + iv\). But \(-2u + 1 = -(R+2)/R < -1\), hence the integral on the left diverges and \(\phi\) is not square summable.

In case 2 the coefficient of \(u^r\) has a double zero at \(\alpha = -\alpha/2c\) where \(|\alpha|^2 = \alpha^2/4|c|^2 = 1\). The functions \(\phi_+(z) = (z - \alpha)^{-r}e^{(u+iv)(\cos \theta - i\sin \theta)/\rho}\) and \(\phi_-(z) = (z - \alpha)^{-r}e^{(u+iv)(\cos \theta + i\sin \theta)/\rho}\) are fundamental solutions for (2.2) and (2.3) respectively. Let us introduce polar coordinates at \(z = \alpha\) so that \(2 - \alpha = \varepsilon e^{i/2}\) and let us agree to set \(\theta = 0\) so that for \(|z| < 1\) the argument of \(z - \alpha\) is restricted to the intervals \(0 \leq \theta < \pi/2\) and \(3\pi/2 < \theta < 2\pi\). Let \(r = u + iv\), then

\[
|\phi_\pm(z)| = |\rho^{-2}e^{-2i\theta}e^{(u+iv)(\cos \theta - i\sin \theta)/\rho}|
\]

\[
= \rho^{-2}e^{(u \cos \theta + v \sin \theta)/\rho}.
\]

We note that \(u\) and \(v\) are not both zero, for then \(b - a \pm i = 0\) where \(a\) and \(b\) are real. Now consider the function \(F(\theta) = u \cos \theta + v \sin \theta\). If \(u > 0\), \(F(0) = u > 0\) and by continuity there exist \(\theta_1\) and \(\theta_2\) such that \(F(\theta) \geq u/2 > 0\) for \(\theta_1 \leq \theta \leq \theta_2 < \pi/2\), similarly if \(v > 0\), \(F(\pi/2) = v\) and there exist \(\theta_1\) and \(\theta_2\) such that \(F(\theta) \geq v/2 > 0\) for \(\theta_1 \leq \theta \leq \theta_2 \leq \pi/2\). If \(u < 0\), \(F(3\pi/2) = -v > 0\) and there exist \(\theta_1\) and \(\theta_2\) such that \(F(\theta) \geq -v/2 > 0\) for \(3\pi/2 < \theta_1 \leq \theta \leq \theta_2\). Hence for all \(r = u + iv\), except for the case \(u < 0\), \(v = 0\), there exists a \(M > 0\) and suitable \(\theta_1\) and \(\theta_2\) for which \(F(\theta) \geq M\), \(\theta_1 \leq \theta \leq \theta_2\). This case requires only a minor modification which will be provided shortly. It is easy to see that for given \(\theta_1\) and \(\theta_2\) we can find \(\delta > 0\) for which the regions \(W_\varepsilon = \{z| \varepsilon \leq \rho < \delta, \theta_1 \leq \theta \leq \theta_2\}\) lie entirely within the disc for \(0 < \varepsilon < \delta\).

Now consider \(||\phi_\pm||^2\):

\[
\int_{D} |\phi_\pm(z)|^2 dx dy \geq \lim_{\varepsilon \to 0} \int_{W_\varepsilon} |\phi_\pm(z)|^2 dx dy
\]

\[
= \lim_{\varepsilon \to 0} \int_{\theta_1}^{\theta_2} \rho^{-2}e^{2F(\theta)\rho} d\rho d\theta
\]

\[
\geq \lim_{\varepsilon \to 0} (\theta_2 - \theta_1) \int_{\varepsilon}^{\delta} e^{2M/\rho} \rho^{-2} d\rho.
\]

Since \(\int_{0}^{\delta} e^{2M/\rho} \rho^{-2} d\rho\) diverges it follows that the \(\phi_\pm\) are not square summable, provided \(r\) is not a negative number. When \(r = u + iv = u < 0\) we merely agree to set \(\theta = 0\) so that for \(|z| < 1\) the argument of \(z - \alpha\) is restricted to the interval \(\pi/2 < \theta < 3\pi/2\). Then \(F(\pi) = -u > 0\) and the argument is the same as before.
In case 3, \( \alpha^2 > 4|c|^2 \), the coefficient of \( u' \) has distinct zeros at 
\[ \alpha = (-a + R)/2c \quad \text{and} \quad \beta = (-a - R)/2c \]
where \( R = (\alpha^2 - 4|c|^2)^{1/2} > 0 \). For \( a > 0 \),
\[ |\beta| = \frac{R + a}{2|c|} > \frac{a}{2|c|} > 1, \]
and therefore \(|\alpha| < 1\). For \( a < 0 \),
\[ |\alpha| = \frac{R - a}{2|c|} > \frac{|a|}{2|c|} > 1, \]
and therefore \(|\beta| < 1\). Without loss of generality we assume \(|\alpha| < 1\),
and \(|\beta| > 1\). For \(|z| < |\alpha| < 1\), the functions \( \phi_+ \) and \( \phi_- \) given by
\[ \phi_+(z) = (z - \alpha)^{-r}(z - \beta)^{-i} \]
\[ \phi_-(z) = (z - \beta)^{-i}(z - \alpha)^{-u} \]
where \( r = (R + b - a)/R - i/R \) and \( s = (R + b - a)/R + i/R \), are fundamental solutions for \( Lu = iu \) and \( Lu = -iu \) respectively. Now suppose \( \psi \) is any nontrivial element of \( \mathcal{H} \) which satisfies \( Lu = \pm iu \). In particular \( \psi \) is analytic for \(|z| < |\alpha| < 1\). From uniqueness results this implies that \( \psi(z) = c\phi_\pm(z) \) for \(|z| < |\alpha| \), where \( c \neq 0 \). By the identity theorem for analytic functions this implies \( \psi(z) = c\phi_\pm(z) \) for \(|z| < 1\), hence \( \phi_\pm(z) \) is analytic in \(|z| < 1\). But \( \phi_\pm(z) \) has a singularity at \(|\alpha| < 1\), therefore, the equations \( Lu = \pm iu \) have no nontrivial solutions in \( \mathcal{H} \).

3. In this section we obtain conditions on the coefficients of \( L \)
which insure that for all \( \lambda \) every solution of \( L\phi = \lambda \phi \) is in \( \mathcal{H} \). If \( L \) is a formally symmetric operator satisfying these conditions the defect indices of the operator \( T_0 \) are equal to the order of \( L \) and \( T_0 \) has a self-adjoint extension in \( \mathcal{H} \).

In [2] it was shown that if \( L = \sum_{k=0}^{n} p_k D^k \) is formally symmetric then the \( p_i \) are polynomials of degree at most \( n + i \). Regarding such \( L \) with polynomial coefficients we have

**Theorem 3.1.** Let \( L = \sum_{k=0}^{n} p_k D^k \) where \( n \geq 2, p_n(0) \neq 0 \), and \( p_k = \sum_{i=0}^{n+k} a_i(k)z^k \), and
\[
A = |a_0(n)|^{-1} \sum_{i=1}^{2n} |a_i(n)|, \\
(3.1) \quad \hat{B} = n(n + 1)/2, \quad \text{and} \\
B = |a_0(n)|^{-1} \sum_{i=1}^{2n} |a_i(n)n[(n + 1)/2 - i] + a_{i-1}(n - 1)|.
If $A < 1$ or $A = 1$ and $B < \hat{B}$ then every solution of $L\phi = 0$ is in $\mathcal{H}$.

Proof. Since $p_n(0) = a_i(n) \neq 0$, every solution of $Lu = 0$ at the origin is analytic in some neighborhood of the origin. Let $\phi(z) = \sum_{j=0}^{\infty} b_j z^j$ be any such solution, we will show that there exists a positive constant $K$ and positive integer $p$ such that $|b_j| \leq Kj^{-l/p}$ for $j$ sufficiently large. Consequently the series $\sum_{j=0}^{\infty} |b_j|^p/(j + 1)$ converges and $\phi$ belongs to $\mathcal{H}$.

We begin by obtaining a recursion formula for the $b_j$. Substituting $\phi(z) = \sum_{j=0}^{\infty} b_j z^j$ into the equation $L\phi(z) = 0$ we obtain

$$L\phi(z) = \sum_{j=0}^{\infty} \sum_{k=0}^{n} \sum_{i=0}^{n+k} a_i(k) \pi_k(j - i + k) b_{j-i+k} z^j ,$$

where

$$\pi_k(\lambda) = \lambda(\lambda - 1) \cdots (\lambda - k + 1) \quad k \leq \lambda$$

$$= 0 \quad k > \lambda .$$

Hence $L\phi = 0$ if and only if the following relationship holds for all $j$.

(3.2) \[
\sum_{k=0}^{n} \sum_{i=0}^{n+k} a_i(k) \pi_k(j - i + k) b_{j-i+k} = 0 .
\]

Hence,

$$\sum_{k=0}^{n-1} \sum_{i=0}^{n+k} a_i(k) \pi_k(j - i + k) b_{j-i+k}$$

$$+ \sum_{i=1}^{2n} a_i(n) \pi_n(j - i + n) b_{j-i+n} + a_0(n) \pi_n(j + n) b_{j+n} = 0 .$$

Noting that the sums involve only the $b_{j-n}$ thru $b_{j+n-1}$ (where $j > n$) and $\pi_n(j + n)$ never vanishes we may solve for $b_{j+n}$ to obtain

(3.3) \[
b_{j+n} = -(S_1 + S_2)/a_0(n) \pi_n(j + n) ,
\]

where

$$S_1 = \sum_{i=1}^{2n} a_i(n) \pi_n(j - i + n) b_{j-i+n} ,$$

and

$$S_2 = \sum_{k=0}^{n-1} \sum_{i=0}^{n+k} a_i(k) \pi_k(j - i + k) b_{j-i+k} ,$$

for $j > n$.

We now investigate the nature of $S_1$ and $S_2$ as polynomials in $j$. It can be shown that $\pi_n(j + n - 1)$ is a polynomial of degree $n$ in $j$,
\[(3.4) \quad \pi_n(j + n - i) = j^n + \left[\frac{n(n + 1)}{2} - in\right]j^{n-1} + \cdots ,\]

for \(i = 1, \ldots, 2n\). Using (3.4) in (3.3) we obtain

\[S_1 = j^n \sum_{i=1}^{2n} a_i(n)b_{j-i+n}\]

\[(3.5) \quad + j^{n-1} \sum_{i=1}^{2n} a_i(n)\left[\frac{n(n + 1)}{2} - in\right]b_{j-i+n} + \text{lower powers of } j.\]

Now consider \(S_2\). Since \(\pi_k(j - i + k)\) is a polynomial of degree \(k\) in \(j\), an examination of (3.3) shows that \(S_2\) is a polynomial of degree \(n - 1\) in \(j\), and that the only terms which contribute to the coefficient of \(j^{n-1}\) are those corresponding to \(k = n - 1\). Hence

\[(3.6) \quad S_2 = j^{n-1} \sum_{i=0}^{2n-1} a_i(n - 1)b_{j-i+n-1} + \text{lower powers of } j.\]

Combining (3.5) and (3.6) we obtain

\[(3.7) \quad S_1 + S_2 = j^n \sum_{i=1}^{2n} a_i(n)b_{j-i+n} + j^{n-1} \sum_{i=1}^{2n} a_i(n)\left[\frac{n(n + 1)}{2} - in\right]b_{j-i+n} + \cdots , \quad (j > n).\]

Since \(\pi_n(j + n) = j^n + (n(n + 1))/2j^{n-1} + \cdots\), is always positive (3.3) yields

\[(3.8) \quad |b_{j+n}| = \frac{|S_1 + S_2|}{|a_o(n)||j^n + \hat{B}j^{n-1} + \cdots |}.\]

We now estimate \(|S_1 + S_2|\). Let \(M(j) = \text{Max}(|b_{j-n}|, \ldots, |b_{j+n-1}|)\), then it follows from (3.1) and (3.7) that \(|S_1 + S_2| \leq |a_o(n)||M(j)Aj^n + M(j)\hat{B}j^{n-1} + \cdots |\). Hence

\[(3.9) \quad |b_{j+n}| \leq \frac{A j^n + B j^{n-1} + \cdots }{j^n + \hat{B}j^{n-1} + \cdots } M(j)\]

for \(j > n\), where \(A, B, \text{ and } \hat{B}\) are given by (3.1).

Consider the estimate (3.9) for \(|b_{j+n}|\),

\[(3.10) \quad |b_{j+n}| \leq Q(j)M(j) \quad j > n ,\]

where \(Q(j) = (A j^n + B j^{n-1} + \cdots )/(j^n + \hat{B}j^{n-1} + \cdots )\). We note that for fixed \(\zeta\), \(Q(j) \leq 1 + \zeta j^{-1}\) for \(j\) sufficiently large if and only if \(Aj^n + \cdots\)
\[ B_j^{n-1} + \cdots \leq j^n + (\hat{B} + \zeta)j^{n-1} + \cdots. \]
Hence if \( A < 1 \) or \( A = 1 \) and \( B \leq \hat{B} + \zeta \) we have
\[
Q(j) \leq 1 + \zeta j^{-1}
\]
for \( j \) sufficiently large. Now consider the expression
\[
(1 + \zeta(j+1)^{-1})(j - n + 1)^{-1/p},
\]
where \( \zeta < 0 \) and \( p \) a positive integer. It is not difficult to see that this is dominated by \((j + n + 1)^{-1/p}\) for \( j \) sufficiently large if and only if
\[
\hat{j}^{p+1} + (p + p\zeta + n + 1)j^p + \cdots \leq \hat{j}^{p+1} + (p - n + 1)j^p + \cdots,
\]
for \( j \) sufficiently large. Hence, we have
\[
(1 + \zeta(j+1)^{-1})(j - n + 1)^{-1/p} \leq (j + n + 1)^{-1/p}
\]
for \( j \) sufficiently large if \( p \geq -2n\zeta^{-1} \).

We now show that there exists a positive constant \( K \) and positive integer \( p \) for which \(|b_j| \leq Kj^{-1/p}, j \) sufficiently large. By hypothesis either \( A < 1 \) or \( A = 1 \) and \( B < \hat{B} \). If \( A < 1 \) let \( \zeta = -1 \) and \( p = 2n \), if \( A = 1 \), select \( \zeta \) such that \( B - \hat{B} < \zeta < 0 \) and \( p > -2n\zeta^{-1} \). For \( j \) sufficiently large, say \( j > j_1 \), (3.11) and (3.12) hold. Set
\[
K = \max_{j \leq j_1 + n} |b_j| j^j^{1/p}
\]
so that \(|b_j| \leq Kj^{-1/p} \) for \( j \leq j_1 + n \). Using (3.10) and (3.11) it follows that
\[
|b_{j_1+n+1}| \leq (1 + \zeta(j_1 + 1)^{-1})M(j_1 + 1),
\]
where
\[
M(j_1 + 1) = \max (K(j_1 - n + 1)^{-1/p}, \cdots, K(j_1 + n)^{-1/p}) = K(j_1 - n + 1)^{-1/p}.
\]
Hence \(|b_{j_1+n+1}| \leq (1 + \zeta(j_1 + 1)^{-1})K(j_1 - n + 1)^{-1/p}\), and using (3.12) this yields
\[
|b_{j_1+n+1}| \leq K(j_1 + n + 1)^{-1/p}.
\]
We now proceed inductively to establish
\[
|b_{j_1+n+k}| \leq K(j_1 + n + k)^{-1/p} \quad k = 2, 3, \cdots.
\]
Let \( K_1 = \max_{j \leq j_1 + n+1} |b_j| j^{j^1/p} \), now \( K_1 = \max \{K, |b_{j_1+n+1}|(j_1 + n + 1)^{1/p}| \} \leq K \), making use of (3.13). Using (3.11) yields
\[
|b_{j_1+n+2}| \leq (1 + \zeta(j_1 + 2)^{-1})M(j_1 + 2)
\]
where
\[ M(j_i + 2) = \text{Max} (K(j_i - n + 2)^{-1/p}, \ldots, K(j_i + n + 1)^{-1/p}) \]
\[ = K(j_i - n + 2)^{-1/p}. \]

Using (3.12) it follows that
\[ |b_{j_i+n+2}| \leq K(j_i + n + 2)^{-1/p}. \]

Continuing on in this manner we establish (3.14) and the theorem is proved.

We note that the conditions (3.1) of Theorem 3.1 involve only the coefficients of the polynomials \( p_n \) and \( p_{n-1} \), hence if \( L \) satisfies the conditions of (3.1) so do the operators \( L \pm i \). Hence we have established the following.

**Theorem 3.2.** Let \( L \) be a formally symmetric operator which satisfies (3.1), then the associated operator \( T_0 \) has defect indices \( n_+ = n_- = n \).

**Corollary 3.3.** The operator \( L = (c_1 z^4 + c_2) dz^2 + (6c_2 z^3 + c_3 z^2 + a_2 z + c_4) dz + (6c_4 z^2 + 2c_3 z + a_3) \), where \( a_3 \) and \( a_2 \) are real and \( |c_1| > |c_3| + |a_2|/2 \), has self-adjoint extensions.

**Proof.** Applying the algorithm given in Theorem 2.3 of [2] the general second order formally symmetric operator has coefficients
\[ p_4(z) = c_1 z^4 + c_2 z^3 + a_1 z^2 + c_4, \]
\[ p_2(z) = 6c_1 z^3 + (3c_3 + 3c_2) z^2 + a_2 z + c_3, \]
\[ p_0(z) = 6c_3 z^2 + 2c_3 z + a_3, \]
where \( a_2, a_3, \) and \( a_4 \) are real.

Now \( A = (|c_1| + 2|c_2| + |a_1|)/|c_1| \geq 1 \) and \( A = 1 \) if and only if \( c_2 = a_1 = 0 \). Now \( \hat{B} = 3 \) and \( B = (|c_1| + |a_2| + 2|c_3|)/|c_1| < 3 \) if and only if \( |c_1| > |c_3| + |a_2|/2 \). Hence the result follows from the previous theorem.

**References**


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