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# SIERPINSKI CURVES IN FINITE 2-COMPLEXES

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**In this note certain one-dimensional continua are defined for finite 2-complexes. These continua, called $S$-curves, are a generalization of the Sierpinski plane universal curve. By a 2-complex is meant a finite connected 2-dimensional euclidean polyhedron which has a triangulation such that every 1-simplex is the face of at least one 2-simplex. It is shown that any two $S$-curves in a 2-complex are homeomorphic. In addition, it is established that two 2-complexes (with the property that every 1-simplex in a triangulation is the face of two or more 2-simplexes) are homeomorphic if and only if the corresponding $S$-curves are homeomorphic.**

In 1916 Sierpinski [4] described a one-dimensional continuum that is known as the Sierpinski plane universal curve. In 1958 Whyburn [7] defined the notion of an $S$-curve in a 2-sphere and established that an $S$-curve in a 2-sphere is homeomorphic to the Sierpinski plane universal curve. In 1966 Borsuk [1] defined an $S$-curve in a surface. He established that any two $S$-curves in a given surface are homeomorphic and that two surfaces are homeomorphic if and only if the corresponding $S$-curves are homeomorphic. In this paper the same type of theorems are established for certain 2-complexes.

In order to define an $S$-curve in a 2-complex, it is necessary to introduce some terminology from Whittlesey [5] or [6]. A point $x$ in a 2-complex $K$ is a *regular point* if it has a neighborhood in $K$ homeomorphic to the plane (euclidean 2-dimensional space). The *regular part* of $K$ is the collection of all regular points in $K$. The points of $K$ which are not regular are called *singular*; the collection of all singular points in $K$ constitute the *singular graph* of $K$. Let $D_1, D_2, \cdots$ be a sequence of mutually disjoint closed discs contained in the regular part of $K$. Then $\mathrm{A}(K) = K - \bigcup_{i=1}^{\infty} \mathrm{Int}\, D_i$ (Int = interior in the sense of manifolds) is said to be an *$S$-curve in $K$* provided that $\bigcup_{i=1}^{\infty} D_i$ is dense in $K$ and the diameters of the $D_i$ converge to zero. Note that if the 2-complex is also a surface, then this definition is precisely that of Borsuk [1, pp. 81–82].

LEMMA 1. *Let $K$ be a 2-complex and $\Sigma$ an upper semi-continuous decomposition of $K$ with the property that every nondegenerate element of $\Sigma$ is contained in the regular part of $K$ and each nondegenerate element has arbitrarily small neighborhoods (in $K$) homeomorphic with the plane. Then the decomposition space $K_\Sigma$ is homeomorphic to $K$.*

*Proof.* It follows from results of Whittlesey [5, p. 843] that there exists a finite collection of bounded surfaces (compact, connected 2-manifolds with nonempty boundary) $M_1, \cdots, M_j$ such that $K$ is an identification space of their topological sum $M_1 + \cdots + M_j$. The identification takes place on the boundaries of the surfaces. More precisely, if $f\colon M_1 + \cdots + M_j \to K$ is the identification map, then $f$ restricted to the manifold interiors of the surfaces is a homeomorphism.

Consider the following diagram:

$$\begin{array}{ccc} M_1 + \cdots + M_j & \xrightarrow{f} & K \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ M_{\Sigma(1)} + \cdots + M_{\Sigma(j)} & \xrightarrow[f_*]{} & K_\Sigma \,. \end{array}$$

The upper semi-continuous decomposition $\Sigma$ of $K$ induces an upper semi-continuous decomposition $\Sigma(i)$ of $M_i$, $i = 1, \cdots, j$. $\Sigma(i)$ has as nondegenerate elements those sets $B$ such that $B = f^{-1}(b)$ where $b$ is a nondegenerate element of $\Sigma$. Let $p_i$ be the identification map of $M_i$ onto the decomposition space $M_{\Sigma(i)}$, $i = 1, \cdots, j$. Let $p$ denote the identification map induced by the identification maps $p_i$, $i = 1, \cdots, j$, and let $q$ denote the identification map for the decomposition $\Sigma$ of $K$. The map $f$ is a relation-preserving continuous map that is an identification. Hence, the induced map $f_*$ is continuous and is also an identification [2, Theorem 4.3, p. 126].

It follows from results of Borsuk [1, Theorem 3.1, p. 76] that $M_i$ is homeomorphic to $M_{\Sigma(i)}$. For each $i$, $i = 1, \cdots, j$, the map $p_i$ restricted to Bd $M_i$ (Bd = boundary in the sense of manifolds) is a homeomorphism onto Bd $M_{\Sigma(i)}$. Furthermore, all the orientations of the boundaries are preserved by $p_i$, and so by [5, Lemma, p. 843] $p_i$ restricted to Bd $M_i$ can be extended to a homeomorphism $h_i$ mapping $M_i$ onto $M_{\Sigma(i)}$.

Next consider the diagram:

$$\begin{array}{ccc} M_1 + \cdots + M_j & \xrightarrow{f} & K \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h_*} \\ M_{\Sigma(1)} + \cdots + M_{\Sigma(j)} & \xrightarrow[f_*]{} & K_\Sigma \,. \end{array}$$

The homeomorphism $h$ is induced by the homeomorphisms $h_i$, $i = 1, \cdots, j$. As above, there exists a continuous mapping $h_*$ of $K$ onto $K_\Sigma$. Furthermore, $h_*$ is one-to-one. Since $K$ is compact and Hausdorff and $\Sigma$ is an upper semi-continuous decomposition, it follows from [3, Theorem 3-33, p. 133] that $K_\Sigma$ is Hausdorff. Thus $h_*$ is a one-to-one continuous mapping of a compact space onto a Hausdorff space and hence is a homeomorphism.

The proof of the next result closely parallels that of Borsuk [1, pp. 82–83] but is included for completeness.

THEOREM 1. *Any two S-curves in a given 2-complex are homeomorphic.*

*Proof.* Let $A = K - \bigcup_{i=1}^{\infty} \operatorname{Int} D_i$ be an $S$-curve in a 2-complex $K$. Consider the upper semi-continuous decomposition $\Sigma$ of $K$ whose nondegenerate elements are the discs $D_i$. $K$ is homeomorphic to $K_\Sigma$ by Lemma 1. The subset of the decomposition space $K_\Sigma$ consisting of points $d_i$ corresponding to the discs $D_i$ is countable and is contained in the regular part of $K_\Sigma$. If $K$ has triangulation $T_0$, there exists a "curved" triangulation $T$ of $K_\Sigma$ isomorphic to $T_0$ such that no point $d_i$, $i = 1, 2, \cdots$, belongs to the 1-dimensional skeleton $Z$ of $T$. The skeleton $Z$ may be considered as lying in the set $K - \bigcup_{i=1}^{\infty} D_i$. Thus a triangulation $T$ of $K$ is obtained that is isomorphic to $T_0$ with the property that every disc $D_i$ lies in the interior of a 2-simplex of $K$.

Similarly, if $A' = K - \bigcup_{i=1}^{\infty} \operatorname{Int} D_i'$ is another $S$-curve in $K$, it follows from the above argument that there exists another triangulation $T'$ of $K$ isomorphic to $T$ such that every disc $D_i$ lies in the interior of a 2-simplex of $T'$. Let $Z'$ denote the 1-skeleton.

Since $T$ and $T'$ are isomorphic, there is a homeomorphism $h$ mapping $K$ onto $K$ such that each 2-simplex $E$ of $T$ is mapped by $h$ onto a 2-simplex $E'$ of $T'$. Then $E \cap A$ and $E' \cap A'$ may be viewed as $S$-curves in a 2-sphere, and $h$ as a homeomorphism mapping the outer boundary of $E \cap A$ onto the outer boundary of $E' \cap A'$. Thus by a result of Whyburn [7, p. 322], $h$ restricted to Bd $E$ can be extended to a homeomorphism $h_E$ mapping $E \cap A$ onto $E' \cap A'$. The mapping $h$ can then be extended to a homeomorphism mapping $A$ onto $A'$ by defining $h(x) = h_E(x)$ for $x \in A$ and $x$ contained in the 2-simplex $E$ of $T$.

Next it is established that certain 2-complexes are completely characterized by their $S$-curves. Let $K$ be the union of all the proper faces of a 3-simplex and let $K'$ be a 2-simplex. Then $A(K)$ is homeomorphic to $A(K')$ but $K$ is not homeomorphic to $K'$. This example shows that extra conditions are needed on the 2-complexes for such a characterization. The sufficient conditions are stated in Theorem 2.

First, some terminology from Borsuk [1, p. 84] must be introduced. Let $A(K) = K - \bigcup_{i=1}^{\infty} \operatorname{Int} D_i$ be an $S$-curve associated with a 2-complex $K$. The set $\operatorname{Bd} A(K) = \bigcup_{i=1}^{\infty} \operatorname{Bd} D_i$ is said to be the *boundary of* $A(K)$. The set $\operatorname{Int} A(K) = A(K) - \operatorname{Bd} A(K)$ is said to be the *interior of* $A(K)$. *Singular interior points* of $A(K)$ are those interior points contained in the singular graph of $K$.

Let $S$ be a Sierpinski plane universal curve and $I$ an arc (a space homeomorphic to the closed interval $[0, 1]$). Let $Y$ denote the space obtained by identifying an endpoint of $I$ with an interior point $x$ of $S$. Observe that every interior point of $S$ is interior to arbitrarily small rectangular plane neighborhoods whose boundaries lie in $S$. Hence $Y$ is not embeddable in the plane. This fact will be used in the proof of the following lemma.

LEMMA 2. *Let $K$ be a 2-complex such that every 1-simplex is the face of two or more 2-simplexes, and let $A(K)$ be the associated S-curve. A point $x$ in $A(K)$ is a singular interior point if and only if no neighborhood of $x$ in $A(K)$ is embeddable in the plane.*

*Proof.* It is clear that if $x$ does not have a neighborhood in $A(K)$ embeddable in the plane, then $x$ does not have such a neighborhood in $K$. Thus $x$ belongs to the singular graph of $K$ and is a singular interior point of $A(K)$.

Conversely, suppose $x$ is a singular interior point. Then $x$ is an element of the singular graph of $K$. To show that no neighborhood of $x$ in $A(K)$ is embeddable in the plane it suffices to establish that every neighborhood of $x$ in $A(K)$ contains a subset homeomorphic to $Y$ (as defined above). Whittlesey has classified the singular points of a 2-complex. His definitions [5, p. 842] are used to consider the various cases.

*Case* 1. $x$ is a line singularity. Then $x$ has arbitrarily small neighborhoods in $K$ homeomorphic to the space obtained by identifying the $x$-axes of $n$ ($n \geqq 3$ by the hypothesis of the lemma) copies of the closed euclidean half-plane $y \geqq 0$. It follows that every neighborhood of $x$ in $A(K)$ contains a subset homeomorphic to $Y$.

*Case* 2. $x$ is a conical point. Then $x$ has arbitrarily small neighborhoods in $K$ homeomorphic to the set which is obtained if $n$ copies ($n \geqq 2$) of the plane are identified at the origin. Again every neighborhood of $x$ in $A(K)$ contains a copy of $Y$.

*Case* 3. $x$ is a node. A node is necessarily a vertex in any triangulation of $K$. Let $T$ be a triangulation of $K$. Then the regular part of the Star of $x$ falls into components each of which is a cone with $x$ at the vertex or is, topologically, an open triangle with $x$ as a vertex and with two singular edges, both edges having $x$ as a vertex, and the edges may be distinct or coincide. Since by hypothesis every 1-simplex is the face of two or more 2-simplexes, every neighborhood in $A(K)$ of a node will contain a copy of $Y$.

All possible singular interior points have been considered and the proof is completed.

THEOREM 2. *Let $K$ and $K'$ be 2-complexes such that every 1-simlex in a triangulation of $K$ or $K'$ is the face of two or more 2-simplexes. Let $A(K)$ and $A(K')$ be the S-curves associated with $K$ and $K'$ respectively. Then $A(K)$ is homeomorphic to $A(K')$ if and only if $K$ is homeomorphic to $K'$.*

*Proof*. Let $h$ mapping $A(K)$ onto $A(K')$ be a homeomorphism. Let Int $D_i$ be an open disc in $K - A(K)$ with Bd $D_i = C$ which is contained in $A(K)$. Consider $h(C) = C'$. Then $C'$ is a simple closed curve in $A(K')$. Next it is established that $C' = \text{Bd } D'_i$ where $A(K') = K' - \bigcup_{i=1}^{\infty} \text{Int } D'_i$.

If $x \in C$, then there exists a neighborhood of $x$ in $A(K)$ embeddable in the plane. By Lemma 2, $h(x)$ is not a singular interior point of $A(K')$. Furthermore, if $x \in C$ then $x$ is contained in the interior of an arc in $C$ that does not locally decompose $A(K)$. It follows from [1, p. 84] that $C'$ is contained in Bd $A(K')$. Hence $C' = \text{Bd } D'_i$ for some $i$. For each $i$ the map $h$ restricted to the Bd $D_i$ can be extended to a homeomorphism $h_i$ mapping the disc $D_i$ onto the disc $D'_i$. Since the diameters of the sets $h(\text{Bd } D_i)$ converge to zero, the diameters of the discs $D'_i$ converge to zero. Extend $h$ to a mapping $h'$ of $K$ onto a subset of $K'$ by defining $h'(x) = h(x)$ for $x$ in $A(K)$ and $h'(x) = h_i(x)$ for $x$ in Int $D_i$. Then $h'$ is a mapping of $K$ onto a subset of $K'$. But since $h(A(K)) = A(K')$, $h'$ is also onto $K'$; and $K$ is homeomorphic to $K'$.

The converse follows from Theorem 1.

The reader will able to make the necessary modifications to extend these results by himself to arbitrary finite 2-complexes.

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