

Pacific Journal of Mathematics

SIERPINSKI CURVES IN FINITE 2-COMPLEXES

GAIL ATNEOSEN

SIERPINSKI CURVES IN FINITE 2-COMPLEXES

GAIL H. ATNEOSEN

In this note certain one-dimensional continua are defined for finite 2-complexes. These continua, called S -curves, are a generalization of the Sierpinski plane universal curve. By a 2-complex is meant a finite connected 2-dimensional euclidean polyhedron which has a triangulation such that every 1-simplex is the face of at least one 2-simplex. It is shown that any two S -curves in a 2-complex are homeomorphic. In addition, it is established that two 2-complexes (with the property that every 1-simplex in a triangulation is the face of two or more 2-simplexes) are homeomorphic if and only if the corresponding S -curves are homeomorphic.

In 1916 Sierpinski [4] described a one-dimensional continuum that is known as the Sierpinski plane universal curve. In 1958 Whyburn [7] defined the notion of an S -curve in a 2-sphere and established that an S -curve in a 2-sphere is homeomorphic to the Sierpinski plane universal curve. In 1966 Borsuk [1] defined an S -curve in a surface. He established that any two S -curves in a given surface are homeomorphic and that two surfaces are homeomorphic if and only if the corresponding S -curves are homeomorphic. In this paper the same type of theorems are established for certain 2-complexes.

In order to define an S -curve in a 2-complex, it is necessary to introduce some terminology from Whittlesey [5] or [6]. A point x in a 2-complex K is a *regular point* if it has a neighborhood in K homeomorphic to the plane (euclidean 2-dimensional space). The *regular part* of K is the collection of all regular points in K . The points of K which are not regular are called *singular*; the collection of all singular points in K constitute the *singular graph* of K . Let D_1, D_2, \dots be a sequence of mutually disjoint closed discs contained in the regular part of K . Then $A(K) = K - \bigcup_{i=1}^{\infty} \text{Int } D_i$ (Int = interior in the sense of manifolds) is said to be an *S -curve in K* provided that $\bigcup_{i=1}^{\infty} D_i$ is dense in K and the diameters of the D_i converge to zero. Note that if the 2-complex is also a surface, then this definition is precisely that of Borsuk [1, pp. 81-82].

LEMMA 1. *Let K be a 2-complex and Σ an upper semi-continuous decomposition of K with the property that every nondegenerate element of Σ is contained in the regular part of K and each nondegenerate element has arbitrarily small neighborhoods (in K) homeomorphic with the plane. Then the decomposition space K_{Σ} is homeomorphic to K .*

Proof. It follows from results of Whittlesey [5, p. 843] that there exists a finite collection of bounded surfaces (compact, connected 2-manifolds with nonempty boundary) M_1, \dots, M_j such that K is an identification space of their topological sum $M_1 + \dots + M_j$. The identification takes place on the boundaries of the surfaces. More precisely, if $f: M_1 + \dots + M_j \rightarrow K$ is the identification map, then f restricted to the manifold interiors of the surfaces is a homeomorphism.

Consider the following diagram:

$$\begin{array}{ccc} M_1 + \dots + M_j & \xrightarrow{f} & K \\ p \downarrow & & \downarrow q \\ M_{\Sigma(1)} + \dots + M_{\Sigma(j)} & \xrightarrow{f_*} & K_{\Sigma} \end{array}$$

The upper semi-continuous decomposition Σ of K induces an upper semi-continuous decomposition $\Sigma(i)$ of M_i , $i = 1, \dots, j$. $\Sigma(i)$ has as nondegenerate elements those sets B such that $B = f^{-1}(b)$ where b is a nondegenerate element of Σ . Let p_i be the identification map of M_i onto the decomposition space $M_{\Sigma(i)}$, $i = 1, \dots, j$. Let p denote the identification map induced by the identification maps p_i , $i = 1, \dots, j$, and let q denote the identification map for the decomposition Σ of K . The map f is a relation-preserving continuous map that is an identification. Hence, the induced map f_* is continuous and is also an identification [2, Theorem 4.3, p. 126].

It follows from results of Borsuk [1, Theorem 3.1, p. 76] that M_i is homeomorphic to $M_{\Sigma(i)}$. For each i , $i = 1, \dots, j$, the map p_i restricted to $\text{Bd } M_i$ (Bd = boundary in the sense of manifolds) is a homeomorphism onto $\text{Bd } M_{\Sigma(i)}$. Furthermore, all the orientations of the boundaries are preserved by p_i , and so by [5, Lemma, p. 843] p_i restricted to $\text{Bd } M_i$ can be extended to a homeomorphism h_i mapping M_i onto $M_{\Sigma(i)}$.

Next consider the diagram:

$$\begin{array}{ccc} M_1 + \dots + M_j & \xrightarrow{f} & K \\ h \downarrow & & \downarrow h_* \\ M_{\Sigma(1)} + \dots + M_{\Sigma(j)} & \xrightarrow{f_*} & K_{\Sigma} \end{array}$$

The homeomorphism h is induced by the homeomorphisms h_i , $i = 1, \dots, j$. As above, there exists a continuous mapping h_* of K onto K_{Σ} . Furthermore, h_* is one-to-one. Since K is compact and Hausdorff and Σ is an upper semi-continuous decomposition, it follows from [3, Theorem 3-33, p. 133] that K_{Σ} is Hausdorff. Thus h_* is a one-to-one continuous mapping of a compact space onto a Hausdorff space and hence is a homeomorphism.

The proof of the next result closely parallels that of Borsuk [1, pp. 82-83] but is included for completeness.

THEOREM 1. *Any two S-curves in a given 2-complex are homeomorphic.*

Proof. Let $A = K - \bigcup_{i=1}^{\infty} \text{Int } D_i$ be an S-curve in a 2-complex K . Consider the upper semi-continuous decomposition \mathcal{S} of K whose nondegenerate elements are the discs D_i . K is homeomorphic to $K_{\mathcal{S}}$ by Lemma 1. The subset of the decomposition space $K_{\mathcal{S}}$ consisting of points d_i corresponding to the discs D_i is countable and is contained in the regular part of $K_{\mathcal{S}}$. If K has triangulation T_0 , there exists a "curved" triangulation T of $K_{\mathcal{S}}$ isomorphic to T_0 such that no point $d_i, i = 1, 2, \dots$, belongs to the 1-dimensional skeleton Z of T . The skeleton Z may be considered as lying in the set $K - \bigcup_{i=1}^{\infty} D_i$. Thus a triangulation T of K is obtained that is isomorphic to T_0 with the property that every disc D_i lies in the interior of a 2-simplex of K .

Similarly, if $A' = K - \bigcup_{i=1}^{\infty} \text{Int } D'_i$ is another S-curve in K , it follows from the above argument that there exists another triangulation T' of K isomorphic to T such that every disc D'_i lies in the interior of a 2-simplex of T' . Let Z' denote the 1-skeleton.

Since T and T' are isomorphic, there is a homeomorphism h mapping K onto K such that each 2-simplex E of T is mapped by h onto a 2-simplex E' of T' . Then $E \cap A$ and $E' \cap A'$ may be viewed as S-curves in a 2-sphere, and h as a homeomorphism mapping the outer boundary of $E \cap A$ onto the outer boundary of $E' \cap A'$. Thus by a result of Whyburn [7, p. 322], h restricted to $\text{Bd } E$ can be extended to a homeomorphism h_E mapping $E \cap A$ onto $E' \cap A'$. The mapping h can then be extended to a homeomorphism mapping A onto A' by defining $h(x) = h_E(x)$ for $x \in A$ and x contained in the 2-simplex E of T .

Next it is established that certain 2-complexes are completely characterized by their S-curves. Let K be the union of all the proper faces of a 3-simplex and let K' be a 2-simplex. Then $A(K)$ is homeomorphic to $A(K')$ but K is not homeomorphic to K' . This example shows that extra conditions are needed on the 2-complexes for such a characterization. The sufficient conditions are stated in Theorem 2.

First, some terminology from Borsuk [1, p. 84] must be introduced. Let $A(K) = K - \bigcup_{i=1}^{\infty} \text{Int } D_i$ be an S-curve associated with a 2-complex K . The set $\text{Bd } A(K) = \bigcup_{i=1}^{\infty} \text{Bd } D_i$ is said to be the *boundary* of $A(K)$. The set $\text{Int } A(K) = A(K) - \text{Bd } A(K)$ is said to be the *interior* of $A(K)$. *Singular interior points* of $A(K)$ are those interior points contained in the singular graph of K .

Let S be a Sierpinski plane universal curve and I an arc (a space homeomorphic to the closed interval $[0, 1]$). Let Y denote the space

obtained by identifying an endpoint of I with an interior point x of S . Observe that every interior point of S is interior to arbitrarily small rectangular plane neighborhoods whose boundaries lie in S . Hence Y is not embeddable in the plane. This fact will be used in the proof of the following lemma.

LEMMA 2. *Let K be a 2-complex such that every 1-simplex is the face of two or more 2-simplexes, and let $A(K)$ be the associated S -curve. A point x in $A(K)$ is a singular interior point if and only if no neighborhood of x in $A(K)$ is embeddable in the plane.*

Proof. It is clear that if x does not have a neighborhood in $A(K)$ embeddable in the plane, then x does not have such a neighborhood in K . Thus x belongs to the singular graph of K and is a singular interior point of $A(K)$.

Conversely, suppose x is a singular interior point. Then x is an element of the singular graph of K . To show that no neighborhood of x in $A(K)$ is embeddable in the plane it suffices to establish that every neighborhood of x in $A(K)$ contains a subset homeomorphic to Y (as defined above). Whittlesey has classified the singular points of a 2-complex. His definitions [5, p. 842] are used to consider the various cases.

Case 1. x is a line singularity. Then x has arbitrarily small neighborhoods in K homeomorphic to the space obtained by identifying the x -axes of n ($n \geq 3$ by the hypothesis of the lemma) copies of the closed euclidean half-plane $y \geq 0$. It follows that every neighborhood of x in $A(K)$ contains a subset homeomorphic to Y .

Case 2. x is a conical point. Then x has arbitrarily small neighborhoods in K homeomorphic to the set which is obtained if n copies ($n \geq 2$) of the plane are identified at the origin. Again every neighborhood of x in $A(K)$ contains a copy of Y .

Case 3. x is a node. A node is necessarily a vertex in any triangulation of K . Let T be a triangulation of K . Then the regular part of the Star of x falls into components each of which is a cone with x at the vertex or is, topologically, an open triangle with x as a vertex and with two singular edges, both edges having x as a vertex, and the edges may be distinct or coincide. Since by hypothesis every 1-simplex is the face of two or more 2-simplexes, every neighborhood in $A(K)$ of a node will contain a copy of Y .

All possible singular interior points have been considered and the proof is completed.

THEOREM 2. *Let K and K' be 2-complexes such that every 1-simplex in a triangulation of K or K' is the face of two or more 2-simplices. Let $A(K)$ and $A(K')$ be the S-curves associated with K and K' respectively. Then $A(K)$ is homeomorphic to $A(K')$ if and only if K is homeomorphic to K' .*

Proof. Let h mapping $A(K)$ onto $A(K')$ be a homeomorphism. Let $\text{Int } D_i$ be an open disc in $K - A(K)$ with $\text{Bd } D_i = C$ which is contained in $A(K)$. Consider $h(C) = C'$. Then C' is a simple closed curve in $A(K')$. Next it is established that $C' = \text{Bd } D'_i$ where $A(K') = K' - \bigcup_{i=1}^{\infty} \text{Int } D'_i$.

If $x \in C$, then there exists a neighborhood of x in $A(K)$ embeddable in the plane. By Lemma 2, $h(x)$ is not a singular interior point of $A(K')$. Furthermore, if $x \in C$ then x is contained in the interior of an arc in C that does not locally decompose $A(K)$. It follows from [1, p. 84] that C' is contained in $\text{Bd } A(K')$. Hence $C' = \text{Bd } D'_i$ for some i . For each i the map h restricted to the $\text{Bd } D_i$ can be extended to a homeomorphism h_i mapping the disc D_i onto the disc D'_i . Since the diameters of the sets $h(\text{Bd } D_i)$ converge to zero, the diameters of the discs D'_i converge to zero. Extend h to a mapping h' of K onto a subset of K' by defining $h'(x) = h(x)$ for x in $A(K)$ and $h'(x) = h_i(x)$ for x in $\text{Int } D_i$. Then h' is a mapping of K onto a subset of K' . But since $h(A(K)) = A(K')$, h' is also onto K' ; and K is homeomorphic to K' .

The converse follows from Theorem 1.

The reader will be able to make the necessary modifications to extend these results by himself to arbitrary finite 2-complexes.

REFERENCES

1. K. Borsuk, *On embedding curves in surface*, Fund. Math., **59** (1966), 73-89.
2. J. Dugundji, *Topology*, Allyn and Bacon, Inc., (1966).
3. J. G. Hocking and G. S. Young, *Topology*, Addison-Wesley, (1961).
4. W. Sierpinski, *Sur une courbe cantorienne qui contient une image biunivoque et continue de toute courbe donnée*, Comptes Rendus Acad. Sci., Paris, **162** (1916), 629-632.
5. E. F. Whittlesey, *Classification of finite 2-complexes*, Proc. Amer. Math. Soc., **9** (1958), 841-845.
6. ———, *Finite surfaces I, II, local structure and canonical form*, Math. Mag., **34** (1960), no. 1 11-22, no. 2 67-68.
7. G. T. Whyburn, *Topological characterization of the Sierpinski curve*, Fund. Math., **45** (1958), 320-324.

Received June 12, 1972 and in revised form October 16, 1972.

WESTERN WASHINGTON STATE COLLEGE

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

RICHARD ARENS (Managing Editor)
University of California
Los Angeles, California 90024

J. DUGUNDJI*
Department of Mathematics
University of Southern California
Los Angeles, California 90007

R. A. BEAUMONT
University of Washington
Seattle, Washington 98105

D. GILBARG AND J. MILGRAM
Stanford University
Stanford, California 94305

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
NAVAL WEAPONS CENTER

* C. R. DePrima California Institute of Technology, Pasadena, CA 91109, will replace J. Dugundji until August 1974.

Pacific Journal of Mathematics

Vol. 50, No. 1

September, 1974

Gail Atneosen, <i>Sierpinski curves in finite 2-complexes</i>	1
Bruce Alan Barnes, <i>Representations of B^*-algebras on Banach spaces</i>	7
George Benke, <i>On the hypergroup structure of central $\Lambda(p)$ sets</i>	19
Carlos R. Borges, <i>Absolute extensor spaces: a correction and an answer</i>	29
Tim G. Brook, <i>Local limits and tripleability</i>	31
Philip Throop Church and James Timourian, <i>Real analytic open maps</i>	37
Timothy V. Fossum, <i>The center of a simple algebra</i>	43
Richard Freiman, <i>Homeomorphisms of long circles without periodic points</i>	47
B. E. Fullbright, <i>Intersectional properties of certain families of compact convex sets</i>	57
Harvey Charles Greenwald, <i>Lipschitz spaces on the surface of the unit sphere in Euclidean n-space</i>	63
Herbert Paul Halpern, <i>Open projections and Borel structures for C^*-algebras</i>	81
Frederic Timothy Howard, <i>The numer of multinomial coefficients divisible by a fixed power of a prime</i>	99
Lawrence Stanislaus Husch, Jr. and Ping-Fun Lam, <i>Homeomorphisms of manifolds with zero-dimensional sets of nonwandering points</i>	109
Joseph Edmund Kist, <i>Two characterizations of commutative Baer rings</i>	125
Lynn McLinden, <i>An extension of Fenchel's duality theorem to saddle functions and dual minimax problems</i>	135
Leo Sario and Cecilia Wang, <i>Counterexamples in the biharmonic classification of Riemannian 2-manifolds</i>	159
Saharon Shelah, <i>The Hanf number of omitting complete types</i>	163
Richard Staum, <i>The algebra of bounded continuous functions into a nonarchimedean field</i>	169
James DeWitt Stein, <i>Some aspects of automatic continuity</i>	187
Tommy Kay Teague, <i>On the Engel margin</i>	205
John Griggs Thompson, <i>Nonsolvable finite groups all of whose local subgroups are solvable, V</i>	215
Kung-Wei Yang, <i>Isomorphisms of group extensions</i>	299