SIERPINSKI CURVES IN FINITE 2-COMPLEXES

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In this note certain one-dimensional continua are defined for finite 2-complexes. These continua, called $S$-curves, are a generalization of the Sierpinski plane universal curve. By a 2-complex is meant a finite connected 2-dimensional euclidean polyhedron which has a triangulation such that every 1-simplex is the face of at least one 2-simplex. It is shown that any two $S$-curves in a 2-complex are homeomorphic. In addition, it is established that two 2-complexes (with the property that every 1-simplex in a triangulation is the face of two or more 2-simplexes) are homeomorphic if and only if the corresponding $S$-curves are homeomorphic.

In 1916 Sierpinski [4] described a one-dimensional continuum that is known as the Sierpinski plane universal curve. In 1958 Whyburn [7] defined the notion of an $S$-curve in a 2-sphere and established that an $S$-curve in a 2-sphere is homeomorphic to the Sierpinski plane universal curve. In 1966 Borsuk [1] defined an $S$-curve in a surface. He established that any two $S$-curves in a given surface are homeomorphic and that two surfaces are homeomorphic if and only if the corresponding $S$-curves are homeomorphic. In this paper the same type of theorems are established for certain 2-complexes.

In order to define an $S$-curve in a 2-complex, it is necessary to introduce some terminology from Whittlesey [5] or [6]. A point $x$ in a 2-complex $K$ is a regular point if it has a neighborhood in $K$ homeomorphic to the plane (euclidean 2-dimensional space). The regular part of $K$ is the collection of all regular points in $K$. The points of $K$ which are not regular are called singular; the collection of all singular points in $K$ constitute the singular graph of $K$. Let $D_1, D_2, \cdots$ be a sequence of mutually disjoint closed discs contained in the regular part of $K$. Then $A(K) = K - \bigcup_{i=1}^{\infty} \text{Int} D_i$ (Int = interior in the sense of manifolds) is said to be an $S$-curve in $K$ provided that $\bigcup_{i=1}^{\infty} D_i$ is dense in $K$ and the diameters of the $D_i$ converge to zero. Note that if the 2-complex is also a surface, then this definition is precisely that of Borsuk [1, pp. 81–82].

**Lemma 1.** Let $K$ be a 2-complex and $\Sigma$ an upper semi-continuous decomposition of $K$ with the property that every nondegenerate element of $\Sigma$ is contained in the regular part of $K$ and each nondegenerate element has arbitrarily small neighborhoods (in $K$) homeomorphic with the plane. Then the decomposition space $K_\Sigma$ is homeomorphic to $K$.  

Proof. It follows from results of Whittlesey [5, p. 843] that there exists a finite collection of bounded surfaces (compact, connected 2-manifolds with nonempty boundary) $M_i, \cdots, M_j$ such that $K$ is an identification space of their topological sum $M_i + \cdots + M_j$. The identification takes place on the boundaries of the surfaces. More precisely, if $f: M_i + \cdots + M_j \to K$ is the identification map, then $f$ restricted to the manifold interiors of the surfaces is a homeomorphism.

Consider the following diagram:

\[
\begin{array}{ccc}
M_i + \cdots + M_j & \xrightarrow{f} & K \\
p & & q \\
M_{\Sigma(i)} + \cdots + M_{\Sigma(j)} & \xrightarrow{f_*} & K_x
\end{array}
\]

The upper semi-continuous decomposition $\Sigma$ of $K$ induces an upper semi-continuous decomposition $\Sigma(i)$ of $M_i$, $i = 1, \cdots, j$. $\Sigma(i)$ has as nondegenerate elements those sets $B$ such that $B = f^{-1}(b)$ where $b$ is a nondegenerate element of $\Sigma$. Let $p_i$ be the identification map of $M_i$ onto the decomposition space $M_{\Sigma(i)}$, $i = 1, \cdots, j$. Let $p$ denote the identification map induced by the identification maps $p_i$, $i = 1, \cdots, j$, and let $q$ denote the identification map for the decomposition $\Sigma$ of $K$. The map $f$ is a relation-preserving continuous map that is an identification. Hence, the induced map $f_*$ is continuous and is also an identification [2, Theorem 4.3, p. 126].

It follows from results of Borsuk [1, Theorem 3.1, p. 76] that $M_i$ is homeomorphic to $M_{\Sigma(i)}$. For each $i$, $i = 1, \cdots, j$, the map $p_i$ restricted to $\text{Bd } M_i$ ($\text{Bd } M_i = \text{boundary in the sense of manifolds}$) is a homeomorphism onto $\text{Bd } M_{\Sigma(i)}$. Furthermore, all the orientations of the boundaries are preserved by $p_i$, and so by [5, Lemma, p. 843] $p_i$ restricted to $\text{Bd } M_i$ can be extended to a homeomorphism $h_i$ mapping $M_i$ onto $M_{\Sigma(i)}$.

Next consider the diagram:

\[
\begin{array}{ccc}
M_i + \cdots + M_j & \xrightarrow{f} & K \\
h & & h_* \\
M_{\Sigma(i)} + \cdots + M_{\Sigma(j)} & \xrightarrow{h_*} & K_x
\end{array}
\]

The homeomorphism $h$ is induced by the homeomorphisms $h_i$, $i = 1, \cdots, j$. As above, there exists a continuous mapping $h_*$ of $K$ onto $K_x$. Furthermore, $h_*$ is one-to-one. Since $K$ is compact and Hausdorff and $\Sigma$ is an upper semi-continuous decomposition, it follows from [3, Theorem 3-33, p. 133] that $K_x$ is Hausdorff. Thus $h_*$ is a one-to-one continuous mapping of a compact space onto a Hausdorff space and hence is a homeomorphism.
The proof of the next result closely parallels that of Borsuk [1, pp. 82-83] but is included for completeness.

**Theorem 1.** Any two S-curves in a given 2-complex are homeomorphic.

**Proof.** Let \( A = K - \bigcup_{i=1}^{\infty} \text{Int} D_i \) be an S-curve in a 2-complex \( K \). Consider the upper semi-continuous decomposition \( \Sigma \) of \( K \) whose nondegenerate elements are the discs \( D_i \). \( K \) is homeomorphic to \( K_x \) by Lemma 1. The subset of the decomposition space \( K_x \) consisting of points \( d_i \) corresponding to the discs \( D_i \) is countable and is contained in the regular part of \( K_x \). If \( K \) has triangulation \( T_o \), there exists a “curved” triangulation \( T \) of \( K_x \) isomorphic to \( T_o \) such that no point \( d_i, i = 1, 2, \ldots, \) belongs to the 1-dimensional skeleton \( Z \) of \( T \). The skeleton \( Z \) may be considered as lying in the set \( K - \bigcup_{i=1}^{\infty} D_i \). Thus a triangulation \( T \) of \( K \) is obtained that is isomorphic to \( T_o \) with the property that every disc \( D_i \) lies in the interior of a 2-simplex of \( K \).

Similarly, if \( A' = K - \bigcup_{i=1}^{\infty} \text{Int} D_i \) is another S-curve in \( K \), it follows from the above argument that there exists another triangulation \( T' \) of \( K \) isomorphic to \( T \) such that every disc \( D_i \) lies in the interior of a 2-simplex of \( T' \). Let \( Z' \) denote the 1-skeleton.

Since \( T \) and \( T' \) are isomorphic, there is a homeomorphism \( h \) mapping \( K \) onto \( K \) such that each 2-simplex \( E \) of \( T \) is mapped by \( h \) onto a 2-simplex \( E' \) of \( T' \). Then \( E \cap A \) and \( E' \cap A' \) may be viewed as S-curves in a 2-sphere, and \( h \) as a homeomorphism mapping the outer boundary of \( E \cap A \) onto the outer boundary of \( E' \cap A' \). Thus by a result of Whyburn [7, p. 322], \( h \) restricted to \( \text{Bd} E \) can be extended to a homeomorphism \( h_E \) mapping \( E \cap A \) onto \( E' \cap A' \). The mapping \( h \) can then be extended to a homeomorphism mapping \( A \) onto \( A' \) by defining \( h(x) = h_E(x) \) for \( x \in A \) and \( x \) contained in the 2-simplex \( E \) of \( T \).

Next it is established that certain 2-complexes are completely characterized by their S-curves. Let \( K \) be the union of all the proper faces of a 3-simplex and let \( K' \) be a 2-simplex. Then \( A(K) \) is homeomorphic to \( A(K') \) but \( K \) is not homeomorphic to \( K' \). This example shows that extra conditions are needed on the 2-complexes for such a characterization. The sufficient conditions are stated in Theorem 2.

First, some terminology from Borsuk [1, p. 84] must be introduced. Let \( A(K) = K - \bigcup_{i=1}^{\infty} \text{Int} D_i \) be an S-curve associated with a 2-complex \( K \). The set \( \text{Bd} A(K) = \bigcup_{i=1}^{\infty} \text{Bd} D_i \) is said to be the boundary of \( A(K) \). The set \( \text{Int} A(K) = A(K) - \text{Bd} A(K) \) is said to be the interior of \( A(K) \). Singular interior points of \( A(K) \) are those interior points contained in the singular graph of \( K \).

Let \( S \) be a Sierpinski plane universal curve and \( I \) an arc (a space homeomorphic to the closed interval \([0, 1]\)). Let \( Y \) denote the space
obtained by identifying an endpoint of \( I \) with an interior point \( x \) of \( S \). Observe that every interior point of \( S \) is interior to arbitrarily small rectangular plane neighborhoods whose boundaries lie in \( S \). Hence \( Y \) is not embeddable in the plane. This fact will be used in the proof of the following lemma.

**Lemma 2.** Let \( K \) be a 2-complex such that every 1-simplex is the face of two or more 2-simplexes, and let \( A(K) \) be the associated \( S \)-curve. A point \( x \) in \( A(K) \) is a singular interior point if and only if no neighborhood of \( x \) in \( A(K) \) is embeddable in the plane.

**Proof.** It is clear that if \( x \) does not have a neighborhood in \( A(K) \) embeddable in the plane, then \( x \) does not have such a neighborhood in \( K \). Thus \( x \) belongs to the singular graph of \( K \) and is a singular interior point of \( A(K) \).

Conversely, suppose \( x \) is a singular interior point. Then \( x \) is an element of the singular graph of \( K \). To show that no neighborhood of \( x \) in \( A(K) \) is embeddable in the plane it suffices to establish that every neighborhood of \( x \) in \( A(K) \) contains a subset homeomorphic to \( Y \) (as defined above). Whittlesey has classified the singular points of a 2-complex. His definitions [5, p. 842] are used to consider the various cases.

**Case 1.** \( x \) is a line singularity. Then \( x \) has arbitrarily small neighborhoods in \( K \) homeomorphic to the space obtained by identifying the \( x \)-axes of \( n \) \((n \geq 3) \) by the hypothesis of the lemma) copies of the closed euclidean half-plane \( y \geq 0 \). It follows that every neighborhood of \( x \) in \( A(K) \) contains a subset homeomorphic to \( Y \).

**Case 2.** \( x \) is a conical point. Then \( x \) has arbitrarily small neighborhoods in \( K \) homeomorphic to the set which is obtained if \( n \) copies \((n \geq 2) \) of the plane are identified at the origin. Again every neighborhood of \( x \) in \( A(K) \) contains a copy of \( Y \).

**Case 3.** \( x \) is a node. A node is necessarily a vertex in any triangulation of \( K \). Let \( T \) be a triangulation of \( K \). Then the regular part of the Star of \( x \) falls into components each of which is a cone with \( x \) at the vertex or is, topologically, an open triangle with \( x \) as a vertex and with two singular edges, both edges having \( x \) as a vertex, and the edges may be distinct or coincide. Since by hypothesis every 1-simplex is the face of two or more 2-simplexes, every neighborhood in \( A(K) \) of a node will contain a copy of \( Y \).

All possible singular interior points have been considered and the proof is completed.
Theorem 2. Let $K$ and $K'$ be 2-complexes such that every 1-simplex in a triangulation of $K$ or $K'$ is the face of two or more 2-simplices. Let $A(K)$ and $A(K')$ be the S-curves associated with $K$ and $K'$ respectively. Then $A(K)$ is homeomorphic to $A(K')$ if and only if $K$ is homeomorphic to $K'$.

Proof. Let $h$ mapping $A(K)$ onto $A(K')$ be a homeomorphism. Let $\text{Int } D_i$ be an open disc in $K - A(K)$ with $\text{Bd } D_i = C$ which is contained in $A(K)$. Consider $h(C) = C'$. Then $C'$ is a simple closed curve in $A(K')$. Next it is established that $C' = \text{Bd } D_i$ where $A(K') = K' - \bigcup_{i=1}^{m} \text{Int } D_i$.

If $x \in C$, then there exists a neighborhood of $x$ in $A(K)$ embeddable in the plane. By Lemma 2, $h(x)$ is not a singular interior point of $A(K')$. Furthermore, if $x \in C$ then $x$ is contained in the interior of an arc in $C$ that does not locally decompose $A(K)$. It follows from [1, p. 84] that $C'$ is contained in $\text{Bd } A(K')$. Hence $C' = \text{Bd } D_i$ for some $i$. For each $i$ the map $h$ restricted to the $\text{Bd } D_i$ can be extended to a homeomorphism $h_i$ mapping the disc $D_i$ onto the disc $D_i$. Since the diameters of the sets $h(\text{Bd } D_i)$ converge to zero, the diameters of the discs $D_i$ converge to zero. Extend $h$ to a mapping $h'$ of $K$ onto a subset of $K'$ by defining $h'(x) = h(x)$ for $x$ in $A(K)$ and $h'(x) = h_i(x)$ for $x$ in $\text{Int } D_i$. Then $h'$ is a mapping of $K$ onto a subset of $K'$. But since $h(A(K)) = A(K')$, $h'$ is also onto $K'$; and $K$ is homeomorphic to $K'$.

The converse follows from Theorem 1.

The reader will able to make the necessary modifications to extend these results by himself to arbitrary finite 2-complexes.

REFERENCES

Received June 12, 1972 and in revised form October 16, 1972.

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* C. R. DePrima California Institute of Technology, Pasadena, CA 91109, will replace
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Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan
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