Pacific Journal of Mathematics

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Vol. 50, No. 1 September 1974

PACIFIC JOURNAL OF MATHEMATICS
Vol. 50, No. 1, 1974

# ON THE HYPERGROUP STRUCTURE OF CENTRAL $\Lambda(p)$ SETS

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**Let $G$ be a compact group and let $\Gamma$ be the set of equivalence classes of the continuous irreducible unitary representations of $G$. For $\gamma \in \Gamma$ denote by $\chi_\gamma$ the character of $\gamma$, then for $E \subset \Gamma$ any function of the form $\sum_{j=1}^{n} a_n \chi_{\gamma_n} (\gamma_1, \cdots, \gamma_n \in E$ and $a_1, \cdots, a_n \in C)$ will be called a central $E$-polynomial, and the set of all such functions will be denoted ${}^z\mathscr{T}_E$. A set $E \subset \Gamma$ is a central Sidon set when the norms $\| \|_\infty$ and $\| \|_A (\|f\|_A = \sum |a_n|$, where $f = \sum a_n \chi_{\gamma_n})$ are equivalent on ${}^z\mathscr{T}_E$, and it is a central $\Lambda(p)$ set when the norms $\| \|_1$ and $\| \|_p$ are equivalent on ${}^z\mathscr{T}_E$. When $G$ is abelian the algebraic structure of $\Lambda(p)$ and Sidon set has been studied extensively. In this paper the structure of central $\Lambda(p)$ sets is investigated in terms of the hypergroup structure of $\Gamma$. In particular it is shown that central $\Lambda(p) (p > 2)$ sets cannot contain arbitrarily long "arithmetic progressions."**

1. Preliminary remarks. Following Helgason [2] we shall say that a set $S$ is hypergroup if to any pair $(\alpha, \beta)$ of elements from $S$ there corresponds a measure $\mu_{\alpha,\beta}$ on $S$. For $\Gamma$, a hypergroup structure is induced by the decomposition of tensor products. Thus if $\alpha, \beta \in \Gamma \alpha \otimes \beta = \bigoplus_{\gamma \in \Gamma} [\gamma : \alpha \otimes \beta]\gamma$, where $[\gamma : \alpha \otimes \beta]$ is a nonnegative integer which is called the multiplicity of $\gamma$ in $\alpha \otimes \beta$, and the measure assigned to the pair $(\alpha, \beta)$ is the discrete measure whose mass at $\gamma$ is $[\gamma : \alpha \otimes \beta]$. From the elementary properties of characters we write $\chi_{\gamma_1} \cdots \chi_{\gamma_n} = \chi_{\gamma_1 \otimes \cdots \otimes \gamma_n} = \sum_{\gamma \in \Gamma} [\gamma : \gamma_1 \otimes \cdots \otimes \gamma_n] \chi_\gamma$. We shall denote by 1 the class of the trivial one dimensional representation, and by $\tilde{\gamma}$ the class containing the conjugates of representations in $\gamma$. All the basic facts about representations needed in this paper may be found in [3].

2. A necessary and sufficient condition for central $\Lambda(2s)$. Although the condition we are about to give is cumbersome, it will allow us to get both necessary conditions and sufficient conditions which are reminiscent of conditions given by Rudin [6, Thm. 4.5] for the case where $G$ is the circle group.

THEOREM. *Let $E \subset \Gamma$ and let $s$ be a natural number, then the following are equivalent.*

(a) *$E$ is a central $\Lambda(2s)$ set.*

(b) *There exists a constant $B$ depending only on $E$ and $s$ such that for every choice of positive real numbers $a_1, \cdots, a_N$ and elements*

*$\gamma_1, \cdots, \gamma_N \in E$ the inequality*

$$\sum_{\gamma \in \Gamma} (\sum a_{k_1} \cdots a_{k_s}[\gamma: \gamma_{k_1} \otimes \cdots \otimes \gamma_{k_s}])^2 \leqq \left(B\left(\sum_{k=1}^N a_k^2\right)^{1/2}\right)^{2s}$$

*holds, where the inner sum on the left is over all*

$$(k_1, \cdots, k_s) \in \{1, \cdots, N\}^s .$$

*Proof.* The logarithmic convexity of the $\| \ \|_p$ norms shows that for $p > 2$ a set $E$ is central $\Lambda(p)$ if $\| \ \|_2$ and $\| \ \|_p$ are equivalent on ${}^z\mathscr{T}_E$ [6, Thm. 1.4]. Accordingly, we will work with the $\| \ \|_2$ and $\| \ \|_{2s}$ norms.

Suppose that $E$ is a central $\Lambda(2s)$ set, and choose positive real numbers $a_1, \cdots, a_N$ and $\gamma_1, \cdots, \gamma_N \in E$. Let $\mathscr{N}$ denote the set $\{1, \cdots, N\}^s$, denote elements in $\mathscr{N}$ by $\boldsymbol{k}$, write $a_{k_1}, \cdots, a_{k_s}$ as $a(\boldsymbol{k})$, $(\gamma_{k_1}, \cdots, \gamma_{k_s}) \in E^s$ as $\gamma(\boldsymbol{k})$ and $\gamma_{k_1} \otimes \cdots \otimes \gamma_{k_s}$ as $\gamma^{(\boldsymbol{k})}$ were $(k_1, \cdots, k_s) = \boldsymbol{k}$. Define $f = \sum_{k=1}^N a_k \chi_{\gamma_k}$, then

$$\begin{aligned} f^s &= \sum_{\boldsymbol{k} \in \mathscr{N}} a(\boldsymbol{k}) \chi_{\gamma(\boldsymbol{k})} \\ &= \sum_{\boldsymbol{k} \in \mathscr{N}} a(\boldsymbol{k}) \sum_{\gamma \in \Gamma} [\gamma: \gamma^{(\boldsymbol{k})}] \chi_\gamma \\ &= \sum_{\gamma \in \Gamma} (\sum_{\boldsymbol{k} \in \mathscr{N}} a(\boldsymbol{k})[\gamma: \gamma^{(\boldsymbol{k})}]) \chi_\gamma . \end{aligned}$$

Using the fact that the irreducible characters are an orthonormal family and $E$ is a central $\Lambda(2s)$ set we have

$$\| f \|_{2s}^{2s} = \| f^s \|_2^2 = \sum_{\gamma \in \Gamma} \left( \sum_{\boldsymbol{k} \in \mathscr{N}} a(\boldsymbol{k})[\gamma: \gamma^{(\boldsymbol{k})}] \right)^2 \leqq (B \| f \|_2)^{2s} .$$

To show (b) implies (a), let $g = \sum_{k=1}^N b_k \chi_{\gamma_k}$ be any central $E$-polynomial. As before

$$\begin{aligned} \| g \|_{2s}^{2s} &= \sum_{\gamma \in \Gamma} \left| \sum_{\boldsymbol{k} \in \mathscr{N}} b(\boldsymbol{k})[\gamma: \gamma^{(\boldsymbol{k})}] \right|^2 \\ &\leqq \sum_{\gamma \in \Gamma} \left( \sum_{\boldsymbol{k} \in \mathscr{N}} | b | (\boldsymbol{k})[\gamma: \gamma^{(\boldsymbol{k})}] \right)^2 \end{aligned}$$

which by hypothesis is $\leqq (B(\sum_{k=1}^N | b_k |^2)^{1/2})^{2s} = (B \| g \|_2)^{2s}$.

COROLLARY. *Let $E \subset \Gamma$ be a central $\Lambda(2s)$ set with constant $B$ so that $\| f \|_{2s} \leqq B \| f \|_2$ for all central $E$-polynomials $f$, then for any finite subset $F \subset E$,*

$$\sum_{\gamma \in \Gamma} \left( \sum_{(\gamma) \in F^s} [\gamma: \gamma_1 \otimes \cdots \otimes \gamma_s] \right)^2 \leqq B^{2s} (\text{card } E)^s .$$

*Proof.* In the theorem set $a_1 = \cdots = a_N = 1$.

REMARK. A case where this criterion is violated in a very simple way is that of $G = SU(2)$. Here $\Gamma$ can be written as $\{\underline{1}, \underline{2}, \cdots\}$ and the Clebsch-Gordan [3, p. 135] formula shows that $\underline{n} \otimes \underline{n} = \underline{1} \oplus \underline{3} \oplus \cdots \oplus \underline{2n-1}$. So if $E$ is any set in $\Gamma$, take $F = \{n\} \subset E$, then

$$\sum_{k=1}^{\infty} [\underline{k}: \underline{n} \otimes \underline{n}]^2 = n$$

and hence $\Gamma$ cannot contain any infinite central $\Lambda(4)$ sets. This fact has already been observed by Helgason [2, p. 789].

**3. A sufficient condition for central $\Lambda(2s)$.** Let $F$ be any subset of $\Gamma$ and write as $(\gamma)$ the $s$-tuples $(\gamma_1, \cdots, \gamma_s) \in F^s$. Write $\otimes(\gamma)$ for $\gamma_1 \otimes \cdots \otimes \gamma_s$, and for $(\gamma) \in F^s$ let $M((\gamma))$ stand for the set of irreducible components of $\otimes(\gamma)$. Furthermore, define

$$r_s(F, \gamma) = \sum_{(\gamma) \in F^s} [\gamma: \otimes(\gamma)]^2 .$$

Note that when $G$ is abelian $r_s(F, \gamma)$ is the number of ways we can write $\gamma = \gamma_{k_1} \otimes \cdots \otimes \gamma_{k_s}$ where $\gamma_{k_j} \in F$ and where a permutation of the same set of $\gamma_{k_j}$'s is counted as a distinct partition of $\gamma$. The following corollary generalizes Rudin's result [6, Thm. 4.5(b)].

COROLLARY. *Let $E \subset \Gamma$ and let $s$ be a natural number. If $E$ is the union of sets $E_i (i = 1, \cdots, j)$ for which there exist constants $C_i$ and $D_i$ depending only on $E_i$ and $s$ such that*

(i) *$r_s(E_i, \gamma) \leqq C_i$ for all $\gamma \in \Gamma$ and*

(ii) *card $M((\gamma)) \leqq D_i$ for all $(\gamma) \in E_i^s$*

*then*

(a) *$E$ is a central $\Lambda(2s)$ set and*

(b) *$\| f \|_{2s} \leqq (\sum_{i=1}^{j} (C_i D_i)^{1/s})^{1/2} \| f \|_2$ for all central $E$-polynomials $f$.*

*Proof.* We show first that the $E_i$ are central $\Lambda(2s)$ sets by applying the theorem of §2. Choose positive numbers $a_1, \cdots, a_N$ and $\gamma_1, \cdots, \gamma_N \in E_i$. Then

$$\sum_{\gamma \in \Gamma} \Big( \sum_{k \in \mathscr{N}} a(k) [\gamma: \gamma^{(k)}] \Big)^2$$
$$\leqq \sum_{k \in \mathscr{N}} \Big( \sum_{k \in \mathscr{N}} [\gamma: \gamma^{(k)}]^2 \Big) \Big( \sum_{k \in \mathscr{N}} a^2(k) \phi(\gamma, k) \Big)$$

where $\phi(\gamma, k) = 1$ if $\gamma$ appears in the decomposition of $\gamma^{(k)}$ and $\phi = 0$ otherwise. Observe that $\sum_{\gamma \in \Gamma} \phi(\gamma, k) = \text{card } M(\gamma(k))$ and so by hypothesis this sum is

$$\leqq C_i D_i \sum_{k \in \mathscr{N}} a^2(k) = C_i D_i \Big( \sum_{k=1}^{N} a_k^2 \Big)^s .$$

Hence $E_i$ is a central $\Lambda(2s)$ set and $\|f\|_{2s} \leqq (C_iD_i)^{1/2s}\|f\|_2$ for any central $f$.

Now suppose that the $E_i$'s are disjoint, for if not they may be replaced by $E_i - \bigcup_{l=1}^{i-1} E_l$. If $f = \sum a_\gamma \chi_\gamma$ is a central $E$-polynomial then $f = f_1 + \cdots + f_j$ where $f_i = \sum_{\gamma \in E_i} a_\gamma \chi_\gamma$ and

$$\begin{aligned}\|f\|_{2s} &\leqq \sum_{i=1}^{j} \|f_i\|_{2s} \leqq \sum_{i=1}^{j} (C_iD_i)^{1/2s}\|f_i\|_2 \\ &\leqq \left(\sum_{i=1}^{j} (C_iD_i)^{1/s}\right)^{1/2}\left(\sum_{i=1}^{j} \|f_i\|_2^2\right)^{1/2} \\ &= \left(\sum_{i=1}^{j} (C_iD_i)^{1/s}\right)^{1/2}\|f\|_2\end{aligned}$$

since the $f_i$ are orthogonal.

REMARKS. (1) The condition (ii) of the previous corollary is also necessary. Take $F = \{\gamma_1, \cdots, \gamma_s\} \in E$ and apply the corollary in §2. Then we have

$$\begin{aligned}B^{2s}s^s &\geqq \sum_{\gamma \in \Gamma}\left(\sum_{(\gamma) \in F^s} [\gamma : \otimes (\gamma)]\right)^2 \\ &\geqq s! \sum_{\gamma \in \Gamma} [\gamma : \otimes (\gamma)] = s! \operatorname{card} M((\gamma))\end{aligned}$$

where $(\gamma)$ in the last two expressions is the $s$-tuple whose components are the elements of $F$.

(2) The condition (ii) is always satisfied when $\sup\{\deg \gamma \mid \gamma \in E\} = P < \infty$. For if $(\gamma) \in E^s$, then the degree of $\otimes(\gamma)$ is not larger than $P^s$ and hence there can be at most $P^s$ elements in $M((\gamma))$.

**4. The relationship between central Sidon and central $\Lambda(p)$ sets.** A set $E \subset \Gamma$ will be called a central $\Lambda$ set if there exists a constant $C$ depending only on $E$ such that $\|f\|_p \leqq Cp^{1/2}\|f\|_2$ for all $2 < p < \infty$ and all central $E$-polynomials $f$. In the case of abelian groups, Rudin [7, p. 128] shows that every Sidon set is a central $\Lambda$ set. Using essentially the same technique Parker [5, p. 43] extends this result to central Sidon sets which have a uniform bound on the degrees of the representations in the set. Moreover, Parker [5, p. 73] shows by an example that some sort of condition is required; he gives an example of a central Sidon set which is not even central $\Lambda(4)$. Using essentially the same technique as Rudin and Parker we will characterize those central Sidon sets which are also central $\Lambda(2s)$ or central $\Lambda$. An interesting consequence of this result is that a central Sidon set which is also central $\Lambda(p)$ for all $p$ must be a central $\Lambda$ set. It should be noted that sets which are central $\Lambda(p)$ for all $p < \infty$ need not in general be central $\Lambda$ sets, in fact such sets exist in every infinite abelian group [1, p. 788].

THEOREM. *Let $E \subset \Gamma$ be a central Sidon set.*

(i) *$E$ is central $\Lambda(2s)$ if and only if there exists a constant $B$ depending on $E$ and $s$, so that $\|\chi_\gamma\|_{2s} \leqq B$ for all $\gamma \in E$.*

(ii) *$E$ is central $\Lambda$ if and only if there exists a constant $B$ depending only on $E$ such that $\|\chi_\gamma\|_{2s} \leqq B$ for all $\gamma \in E$ and $s = 1, 2, \cdots$.*

*Proof.* Since $\|\chi_\gamma\|_2 = 1$ for all $\gamma \in \Gamma$ we clearly have the "only if" parts of (i) and (ii).

Suppose $E$ is a central Sidon set and we have a constant $B$ as in (i). Let $f = \sum_{n=1}^N a_n \chi_{\gamma_n}$ be a central $E$-polynomial. Let

$$\Omega = \Pi_1^N\{-1, 1\}$$

with the operation of coordinatewise multiplication and let $\varepsilon_n : \Omega \rightarrow \{-1, 1\}$ be projection onto the $n$th coordinate. Since $E$ is a central Sidon set, for every $\omega \in \Omega$ there exists a central measure $\mu_\omega$ on $G$ such that $\hat{\mu}_\omega(\gamma_n) = \varepsilon_n(\omega) I_{d_{\gamma_n}} (n = 1, \cdots, N)$ and $\|\mu_\omega\|_1 \leqq C$ where $C$ depends only on $E$ [5, p. 27]. We have

$$\begin{aligned} \|f\|_{2s}^{2s} &= \|\mu_\omega * \mu_\omega * f\|_{2s}^{2s} \leqq \|\mu_\omega\|_1^{2s} \|\mu_\omega * f\|_{2s}^{2s} \\ &\leqq C^{2s} \int_G \left| \sum_{n=1}^N a_n \chi_{\gamma_n}(x) \varepsilon_n(\omega) \right|^{2s} dx \,. \end{aligned}$$

Integrating both sides of the inequality over $\Omega$ and using Fubini's theorem and the inequality

$$\left( \int_\Omega \left| \sum_{n=1}^N b_n \varepsilon_n(\omega) \right|^{2s} d\omega \right)^{1/2s} \leqq 2\sqrt{s} \left( \sum_{n=1}^N |b_n|^2 \right)^{1/2}$$

whose proof is the same as that of [8, 8.4, p. 213], we have

$$\begin{aligned} \|f\|_{2s}^{2s} &\leqq C^{2s} 2^{2s} s^s \int_G \left( \sum_{n=1}^N |a_n|^2 |\chi_{\gamma_n}(x)|^2 \right)^s dx \\ &= (2\sqrt{s}\, C)^{2s} \sum |a_{n_1}{}^2| \cdots |a_{n_s}|^2 \int_G |\chi_{\gamma_{n_1}}|^2 \cdots |\chi_{\gamma_{n_s}}|^2 dx \end{aligned}$$

where the sum is over all $(n_1, \cdots, n_s) \in \{1, \cdots, N\}^s$. By Hölder's inequality this expression is

$$\begin{aligned} &\leqq (2\sqrt{s}\, C)^{2s} \sum |a_{n_1}|^2 \cdots |a_{n_s}|^2 \Pi_{j=1}^s \left( \int_G |\chi_{\gamma_{n_j}}|^{2s} dx \right)^{2/2s} \\ &\leqq (2\sqrt{s}\, C)^{2s} \left( \sum_{n=1}^N |a_n|^2 \right)^s B^{2s} \end{aligned}$$

that is, $\|f\|_{2s} \leqq (CB\sqrt{2})\sqrt{2s}\, \|f\|_2$.

REMARKS. (1) Since $\deg \gamma = \|\chi_\gamma\|_\infty = \lim_{s\to\infty} \|\chi_\gamma\|_{2s}$, (ii) is a restatement of Parker's result.

(2) The following are equivalent.

(a) There exists a constant $B$ depending only on $E$ and s so that $\|\chi_\gamma\|_{2s} \leqq B$ for all $\gamma \in E$.

(b) There exist constants $C$ and $D$ depending only on $E$ and $s$ so that

(i) $[\sigma: \otimes(\gamma)] \leqq C$ for all $\sigma \in \Gamma$ and $\gamma \in E$ where $(\gamma)$ is the $s$-tuple whose components are $\gamma$, and

(ii) card $M((\gamma)) \leqq D$ for all $\gamma \in E$.

The orthogonality of the characters gives

$$\begin{aligned}
\|\chi_\gamma\|_{2s}^{2s} &= \int_G \chi_\gamma^s \bar{\chi}_\gamma^s dx \\
&= \int_G \Big(\sum_{\sigma\in\Gamma} [\sigma: \otimes(\gamma)]\chi_\sigma\Big)\Big(\sum_{\nu\in\Gamma}[\nu: \otimes(\gamma)]\bar{\chi}_\nu\Big) dx \\
&= \sum_{(\sigma,\nu)\in\Gamma\times\Gamma} [\sigma: \otimes(\gamma)][\nu: \otimes(\gamma)] \int_G \chi_\gamma \bar{\chi}_\nu dx \\
&= \sum_{\sigma\in M((\gamma))} [\sigma: \otimes(\gamma)]^2 .
\end{aligned}$$

Since the terms in this last sum are positive we have the equivalence of (a) and (b).

## 5. Product groups and lacunary projections.

Let $G_\alpha$, $\alpha \in I$ be a collection of compact groups with dual objects $\Gamma_\alpha$. Let $G = \prod_{\alpha\in I} G_\alpha$ be the complete direct product and $\Gamma = \prod^*_{\alpha\in I}\Gamma_\alpha$ be the incomplete direct product. Then $\Gamma$ is the dual object of $G$ and the operations are all the obvious coordinatewise ones [3, p. 27]. Let $\sigma_\alpha \in \Gamma_\alpha$ and let $\pi_\alpha$: $G \to G_\alpha$ be the projection onto the $\alpha$'th coordinate, then $\sigma_\alpha \circ \pi_\alpha \in \Gamma$. Write $\sigma_\alpha^j$ for the $j$-fold tensor product of $\sigma_\alpha$ in $\Gamma_\alpha$ and let $M(\sigma_\alpha^j)$ be the set of irreducible components of $\sigma_\alpha^j$ in $\Gamma_\alpha$.

THEOREM. *Let $G$ and $\Gamma$ be as above and consider $E = \{\gamma_\alpha = \pi_\alpha \circ \sigma_\alpha \mid \alpha \in I\}$. A necessary and sufficient condition that $E$ be a central $\Lambda(2s)$ set is that there exist constants $K$ and $L$ both depending only on $s$ and the set $\{\sigma_\alpha \mid \alpha \in I\}$ so that*

(a) *$[\tau_\alpha: \sigma_\alpha^s] \leqq L$ for all $\tau_\alpha \in \Gamma_\alpha$ and $\alpha \in I$, and*

(b) *card $M(\sigma_\alpha^s) \leqq K$ for all $\alpha \in I$.*

*Proof.* Parker [5, p. 70] shows that $E$ is a central Sidon set, hence by the theorem in §4 we need a uniform bound on $\|\chi_{\gamma_\alpha}\|_{2s}$ as $\alpha$ ranges over $I$. Since Haar measure on $G$ is just the product of the Haar measures on the $G_\alpha$, we have $\|\chi_{\gamma_\alpha}\|_{2s} = \|\chi_{\sigma_\alpha}\|_{2s}$ but by remark (2) of §4 this is equivalent to the conditions (a) and (b).

REMARK. If $\sup\{\deg \sigma_\alpha \mid \alpha \in I\} = P < \infty$, then $E$ is a central $\Lambda(2s)$ set.

**6. Intersections with arithmetic progressions.** Let $\sigma \in \Gamma$ and let $N$ be a natural number, we define the arithmetic progression of length $N$ generated by $\sigma$ to be

$$A(\sigma, N) = \bigcup_{j=1}^{N} M(\sigma^j)$$

where $\sigma^j$ is the $j$-fold tensor product of $\sigma$.

THEOREM. *Let $E$ be a central $\Lambda(p)$ set ($p > 2$) with constant $B$ so that $\|f\|_p \leqq B\|f\|_2$ for all central $E$-polynomials $f$. Let $\sigma \in \Gamma$, then*

$$\operatorname{card}(A(\sigma, N) \cap E) = 0\,(N^{4(\deg\sigma)^2/p}) \quad as \quad N \longrightarrow \infty\,.$$

*Proof.* Choose $\varepsilon$ and let $D^\sigma_{2N} = \sum_{\gamma \in A(\sigma,2N)} d_\gamma \chi_\gamma$ and

$$F^\sigma_{2N} = |D^\sigma_{2N}|^2/(\sum_{\gamma \in A(\sigma,2N)} d_\gamma^2)$$

so

$$\begin{aligned} F^\sigma_{2N} &= (\sum d_\gamma \chi_\gamma)(\sum d_\nu \bar{\chi}_\nu)/(\sum d_\gamma^2) \\ &= (\sum_{\zeta \in \Gamma} (\sum d_\gamma d_\nu [\zeta\colon \gamma \otimes \bar{\nu}])\chi_\zeta)/(\sum d_\gamma^2) \end{aligned}$$

where the inner sum is over all $(\gamma, \nu) \in A(\sigma, 2N) \times A(\sigma, 2N)$. If we write $F^\sigma_{2N} = \sum_{\zeta \in \Gamma} d_\zeta \alpha(F^\sigma_{2N}, \zeta)\chi_\zeta$ then Mayer [4, p. 688] shows that for all $N$ sufficiently large and $\zeta \in A(\sigma, N)$

$$\alpha(F^\sigma_{2N}, \zeta) \geqq r_\sigma(N)/d_\zeta r_\sigma(2N)$$

where $r_\sigma$ is a polynomial of degree $\leqq d_\sigma^2$. Choose $\eta > 0$ small enough so that $(2^{-(\deg\sigma)2} - \eta)^{-1} \leqq 2^{(\deg\sigma)^2} + \varepsilon$. Then for this $\eta$ and $\zeta \in A(\sigma, N)$ we have for $N$ sufficiently large that

$$\alpha(F^\sigma_{2N}, \zeta) \geqq (2^{-(\deg\sigma)^2} - \eta)/d_\zeta\,. \tag{1}$$

We also have $\|F^\sigma_{2N}\|_2 \leqq \|F^\sigma_{2N}\|_\infty = (D^\sigma_{2N}(e))^2/(\sum d_\gamma^2)$, and since $\chi_\gamma(e) = d_\gamma$ we have

$$\|F^\sigma_{2N}\|_2 \leqq \sum_{\gamma \in A(\sigma,2N)} d_\gamma^2 = r_\sigma(N)$$

for all $N$ sufficiently large as shown in [4, p. 687]. Hence

$$\|F^\sigma_{2N}\|_2 \leqq KN^{(\deg\sigma)^2} \tag{2}$$

for all $N$ sufficiently large. Let $f = \sum_{\zeta \in E \cap A(\sigma,N)} \chi_\zeta$, define $\alpha(f, \zeta)$ so that $f = \sum_{\zeta \in \Gamma} d_\zeta \alpha(f, \zeta)\chi_\zeta$, and suppose $N$ is large enough to satisfy (1)

and (2).
Then

$$\begin{aligned}\operatorname{card}(E \cap A(\sigma, N)) &= \sum_{\zeta \in \Gamma} d_\zeta \alpha(f, \zeta) \\ &= \frac{1}{(2^{-(\deg \sigma)^2} - \eta)} \sum_{\zeta \in \Gamma} d_\zeta \alpha(f, \xi)(2^{-(\deg \sigma)^2} - \eta) \\ &\leqq \frac{1}{(2^{-(\deg \sigma)^2} - \eta)} \sum_{\zeta \in \Gamma} d_\zeta \alpha(f, \zeta) d_\zeta \alpha(F^\sigma_{2N}, \zeta) \\ &= (2^{-(\deg \sigma)^2} - \eta)^{-1} \int_G f(x) F^\sigma_{2N}(x) dx \\ &\leqq (2^{-(\deg \sigma)^2} - \eta)^{-1} \| f \|_p \| F^\sigma_{2N} \|_q .\end{aligned}$$

The logarithmic convexity of the $\| \ \|_p$ norms gives $\| \ \|_q \leqq \| \ \|_1^{(2-q)/q} \| \ \|_2^{2/p}$. Using this fact and the hypothesis that $E$ was a central $\Lambda(p)$ set, the last expression is

$$\leqq B(2^{-(\deg \sigma)^2} + \varepsilon) \| f \|_2 \| F^\sigma_{2N} \|_1^{(2-q)/q} \| F^\sigma_{2N} \|_2^{2/p} .$$

Note that $\| f \|_2 = (\operatorname{card}(A(\sigma, N) \cap E))^{1/2}$ and $\| F^\sigma_{2N} \|_1 = \hat{F}^\sigma_{2N}(1) = 1$, so that by (2) we have

$$(\operatorname{card}(A(\sigma, N) \cap E))^{1/2} \leqq B(2^{(\deg \sigma)^2} + \varepsilon)(KN^{(\deg \sigma)^2})^{2/p}$$

for all $N$ sufficiently large, the size of $N$ depending only on $\sigma$ and $\varepsilon$.

COROLLARY. *Let $E$ be a central $\Lambda$ set, and let $\sigma \in \Gamma$. Then*

$$\operatorname{card}(A(\sigma, N) \cap E) = 0(\log N) .$$

*Proof.* For a central $\Lambda$ set we may take $B = Cp^{1/2}$ where $C$ depends only on $E$. In the last inequality of the previous proof, set $p = 4 \log (KN^{(\deg \sigma)^2})$, then

$$\operatorname{card}(A(\sigma, N) \cap E) \leqq (2^{(\deg \sigma)^2} + \varepsilon)^2 C^2 e 4 \log (KN^{(\deg \sigma)^2})$$

for all $N$ sufficiently large.

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Received September 27, 1972 and in revised form January 31, 1973.

GEORGETOWN UNIVERSITY

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

# Pacific Journal of Mathematics

Vol. 50, No. 1 September, 1974