Pacific Journal of Mathematics

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PHILIP THROOP CHURCH AND JAMES TIMOURIAN

Vol. 50, No. 1 September 1974

PACIFIC JOURNAL OF MATHEMATICS
Vol. 50, No. 1, 1974

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Let R and C be the real and complex fields, respectively, and for $\zeta \in C$ let $\mathscr{R}(\zeta)$ be the real part of ζ. If $f: M^{p+1} \to N^p$ is real analytic and open with $p \geqq 1$, then there is a closed subspace $X \subset M^{p+1}$ such that $\dim f(X) \leqq p-2$ and, for every $x \in M^{p+1} - X$, there is a natural number $d(x)$ with f at x locally topologically equivalent to the map

$$\phi_{d(x)}: C \times R^{p-1} \longrightarrow R \times R^{p-1}$$

defined by $\phi_{d(x)}(z, t_1, \cdots, t_{p-1}) = (\mathscr{R}(z^{d(x)}), t_1, \cdots, t_{p-1})$.

In [7] Nathan proved: If $f: M^2 \to N^1$ is real analytic and open, then for every $x \in M^2$ there is a natural number $d(x)$ with f at x locally topologically equivalent to the map $\phi_{d(x)}: C \to R$ defined by $\phi_{d(x)}(z) = \mathscr{R}(z^{d(x)})$. This is the case $p = 1$ of the above theorem, but our proof is not a generalization of his.

Examples (see (3.3)) show that "topologically equivalent" cannot be replaced by "analytically equivalent" or even "C^1 equivalent", f real analytic cannot be replaced by f C^∞ (but see (3.1)), an exceptional set X with $\dim f(X) \geqq p-2$ is needed, and $\dim X$ may be $p-1$.

CONVENTIONS 1.2. We must assume that the reader has [2] at hand, and we follow its conventions. In particular we need [2; (2.2), (2.4), (2.5), (2.6), (2.8), (2.9), (3.1), and (3.9)]. For $f: M^n \to N^p$, B_f is the set of x in M^n at which f is not locally topologically equivalent to the projection map $\rho: R^n \to R^p$. The symbol $\approx$ is read "is diffeomorphic to".

DEFINITIONS 1.3. C-analytic sets are defined in [2]. A C-analytic set is called *C-irreducible* [9, p. 155] if it is not the sum of two C-analytic subsets distinct from itself. Whitney and Bruhat [9, p. 155, Proposition 11] prove that any C-analytic set V is uniquely the (countable) locally finite union of C-irreducible C-analytic subsets V_m, no one of which contains another. The V_m are called the *C-irreducible components* of V. Conversely, any locally finite union of C-analytic sets is a C-analytic set [9, p. 154].

DEFINITIONS 1.4. Let V be a complex analytic set of dimension v. There is a complex analytic subset $S \subset V$ such that $\dim S < v$ and $V - S$ is a complex analytic v-manifold [8, p. 500]. (The points of

$V - S$ are called *simple* or *regular*.) Let M be a complex analytic manifold, and let $T(M, p)$ for $p \in M$ be the tangent plane of M at p. Suppose that for each $p \in M$, v-plane T, and sequence $\{q_i\} \subset V - S$ with $q_i \to p$ and $T(V, q_i) \to T$, we have $T(M, p) \subset T$; then V is said to be *a-regular over* M. If V and M also satisfy another property (b-regular), then V is said to be *regular over* M [8, p. 540].

LEMMA 1.5. (Whitney [7, p. 540, Lemma 19.3].) *Suppose that* V *and* W *are complex analytic sets, and* $\dim V \gtrdot \dim W$. *Then there is a complex analytic subset* S *of* W *such that* $\dim S < \dim W$, *each point of* $W - S$ *is simple, and* V *is regular over the complex analytic manifold* $W - S$.

2. Analytic sets and maps.

LEMMA 2.1. *Let* $f: M^n \to N^p$ *be* C^ω, *and let* $V \subset M^n$ *be a non-empty C-analytic subset of* M^n *with dimension* v. *Then*

(a) (Whitney and Bruhat [9, p. 156, Proposition 13]), *there is a C-analytic subset* $S \subset V$ *such that* $\dim S \leqq v - 1$ *and* $V - S$ *is a* v-*dimensional* C^ω *submanifold of* M^n;

(b) *there is a C-analytic subset* $E \subset V$ *such that* $V - E$ *is a* v-*dimensional* C^ω *submanifold of* M^n, $f|(V - E)$ *has constant rank* r, *and* $\dim f(E) \leqq v - 1$;

(c) $\dim f(V) \leqq \max\{v - 1, r\} \leqq v$; *and*

(d) *if* V *is C-irreducible* (1.3), *then* $\dim f(V) \leqq r$.

Proof. We use induction on v; if $v = 0$, then V is discrete and the results are trivial.

Let V_m be the C-irreducible components of V (1.3). According to [2, p. 22, (3.1)] there is a C-analytic subset E_m of V_m such that $\dim E_m < \dim V_m$, $V_m - E_m$ is a C^ω submanifold of V_m with dimension $\dim V_m$, and $f|(V_m - E_m)$ has constant rank r_m. Let r be the maximum r_m for those m with $\dim V_m = v$.

If (1) $\dim V_m < v$, or (2) $\dim V_m = v$ and $r_m < r$, let $F_m = V_m$; (3) otherwise, let $F_m = E_m$. Since the V_m are locally finite, the F_m are also. Let $S \subset V$ be the C-analytic subset given by (a). Then by inductive hypothesis (c), $\dim f(S) < v$ and $\dim f(F_m) < v$ in cases (1) and (3). In case (2) $\dim f(E_m) < v$ also, and, from the Rank Theorem [1, p. 155] applied to $f|(V_m - E_m)$, $\dim(f(V_m - E_m)) \leqq r_m < r \leqq v$. Since each of E_m and $(V_m - E_m)$ is the countable union of compact sets, $\dim f(V_m) < v$. Let $E = S \cup (\bigcup_m F_m)$. Then $\dim f(E) < v$; (b) results from the local finiteness of the F_m and (1.3); and (again from the Rank Theorem) (c) is a corollary.

Now suppose that V is C-irreducible. Let W be the set E of (b), let W_m be its C-irreducible components, and let E_m be as given by (b) for W_m. If each $f|(W_m - E_m)$ has rank at most r, then $\dim f(W_m) \leqq r$ by inductive hypothesis, and (d) follows. Thus we may suppose that for some W_m and E_m, call them W and E, $f|(W-E)$ has rank greater than r.

Let M^*, N^*, f^*, V^*, W^*, and $(W-E)^*$ be complexifications (see e.g. [2, (2.4), (2.5), (2.6)]), where M^* is small enough that V^* is irreducible in M^* [9, p. 155, Proposition 11 and p. 151, Corollary 2]. Let $E' \subset V^*$ be as given by [2, (3.1)] for V^* and f^*, so that $f^*|(V^* - E')$ has constant rank k. By definition of r, V has a simple point x at which $f|V$ has rank r; thus $f^*|V^*$ has rank r at x also, so that $k \geqq r$. Since $\dim E' < \dim V^* = v$ [9, p. 155, Proposition 12], $\dim(E' \cap M^n) < v$; thus $k = r$.

Let S^* be the analytic subset of $(W-E)^*$ given by (1.5) such that V^* is regular over the manifold $X^* = (W-E)^* - S^*$ and let $q \in X^*$. Since V^* is irreducible, the simple points of V^* are dense in V^* [5, p. 68, Corollary 2]. Thus $V^* = Cl[V^* - E']$, so there exist $q_i \in V^* - E'$ with $q_i \to q$. Let T_i and T be the tangent planes of $V^* - E'$ at q_i and of X^* at q, respectively. Since the Grassman manifold G of v-planes in C^n is compact, there are $T' \in G$ and a subsequence $T_{i(j)} \to T'$, and since V^* is a-regular over X^*, $T \subset T'$. Now $f|(V^* - E')$ has rank r, while $f^*|X^*$ has rank greater than r, and a contradiction results.

Substantially the same proof yields the complex analog, where C-analytic is replaced by analytic. There is a unique minimal set E satisfying (b), viz. the intersection of all sets E satisfying (b).

LEMMA 2.2. *Let $f\colon K^k \times R^{p-1} \to R \times R^{p-1}$ $(p \geqq 1)$ be a C^ω layer map* (i.e., $f(K^k \times \{t\}) \subset R \times \{t\}$), *let $f_t\colon K^k \to R$ be defined by $(f_t(x), t) = f(x, t)$, and let $\Gamma \subset R_{p-1}(f)$ be a C-analytic subset with* $\dim \Gamma \leqq p-1$. *Then there is a C-analytic subset $\Delta \subset \Gamma$ such that* $\dim f(\Delta) \leqq p-2$ *and*

$$\dim((\Gamma - \Delta) \cap (K^k \times \{t\})) \leqq 0$$

for each $t \in R^{p-1}$.

Proof. Let $E \subset \Gamma$ and r be as given by (2.1(b)). If $r < p-1$, then let $\Delta = \Gamma$; if $r = p-1$, let $\Delta = E$. In either case, $\dim f(\Delta) \leqq p-2$ (2.1(c)). If $\Gamma - \Delta \neq \emptyset$, it is a $C^\omega(p-1)$-manifold, and $f|(\Gamma - \Delta)$ has rank $p-1$. Since $\Gamma \subset R_{p-1}(f)$, $R_{p-1}(f) \cap (K^k \times \{t\}) = R_0(f_t)$, and $\dim(f_t(R_0(f_t))) \leqq 0$ (by Sard's Theorem [1, p. 156]), $\Gamma - \Delta$ is transverse to each $K^k \times \{t\}$ at each point of intersection. In other words, the inclusion map $i\colon \Gamma - \Delta \to K^k \times R^{p-1}$ is transverse regular on $K^k \times \{t\}$, so by Thom's Transversality Theorem [1, p. 165] $i^{-1}(K^k \times \{t\}) =$

$(\Gamma - \Delta) \cap (K^k\{t\})$ is a 0-dimensional manifold.

LEMMA 2.3. *If* $f\colon R^2 \times R^{p-1} \to R \times R^{p-1}$ *is an open* C^ω *layer map, then there is a closed subset* $X \subset R^2 \times R^{p-1}$ *such that* $\dim f(X) \leqq p-2$ *and* $\dim((B_f - X) \cap f^{-1}(y, t)) \leqq 0$ *for each* $(y, t) \in R \times R^{p-1}$.

Proof. By the Rank Theorem [1, p. 155] $B_f \subset R_{p-1}(f)$. (*) It suffices to prove the theorem locally, i.e., to show that for each $(x, t) \in R_{p-1}(f)$, there are neighborhoods $P \approx R^2$ of x and $Q \approx R^{p-1}$ of t such that $f \mid P \times Q$ satisfies the conclusion.

Now $R_{p-1}(f)$ is a C-analytic set [2, (2.9)], and since $\dim(f(R_{p-1}(f))) \leqq p-1$ by Sard's Theorem [1, p. 156] and f is open, $\dim(R_{p-1}(f)) \leqq p$. It is the union of its C-irreducible components V_m with dimension v_m; let E_m and r_m be as given by (2.1(b)) (or [2, (3.1)]). Let E be the union of the C-analytic subset $S \subset R_{p-1}(f)$ given by (2.1(a)), the V_m for $v_m = r_m = p-1$, and the E_m for $v_m = p$ and $r_m = p-1$, and let F be the union of the V_m with $r_m \leqq p-2$. Let $G \subset E$ be the C-analytic subset Δ given by (2.2) for $\Gamma = E$. Then $\dim(f(F \cup G)) \leqq p-2$ (2.1(d)), so we may define X to contain $F \cup G$. Thus (see (*)) we may consider only neighborhoods $P \times Q$ disjoint from $F \cup G$, i.e., it suffices to prove the lemma in case $F = G = \varnothing$. By (2.2) $\dim(E \cap (R^2 \times \{t\})) \leqq 0$ for each $t \in R^{p-1}$, so (see (*)) it suffices to prove the lemma at the points of $R_{p-1}(f) - E$, i.e., to assume $E = \varnothing$. Thus $R_{p-1}(f)$ is a p-manifold (or is $\varnothing$) and $f \mid R_{p-1}(f)$ has rank $p-1$.

We now apply [2, (3.9)] to each component Γ of $R_{p-1}(f)$. Since f is open, each $k(\Gamma)$ is odd and B_f is contained in the at most $(p-1)$-dimensional analytic set $A = \bigcup_\Gamma A(\Gamma)$. Let $\Delta \subset A$ be the C-analytic subset given by (2.2). We may take $\Delta = X$, and the conclusion results.

3. Proof of the theorem.

THEOREM 3.1. ([3, (1.1) and (4.1)].) *Let* $f\colon M^n \to N^p$ *be a* C^3 *open map with* $p \geqq 1$, *and let* $\dim(B_f \cap f^{-1}(y)) \leqq 0$ *for each* $y \in N^p$. *Then there is a closed set* $X \subset M^{p+1}$ *such that* $\dim f(X) \leqq p-2$ *and, for every* $x \in M^{p+1} - X$, *there is a natural number* $d(x)$ *with* f *at* x *locally topologically equivalent to the map*

$$\phi_{d(x)}\colon C \times R^{p-1} \longrightarrow R \times R^{p-1}$$

defined by $\phi_{d(x)}(z, t_1, \cdots, t_{p-1}) = (\mathscr{R}(z^{d(x)}), t_1, \cdots, t_{p-1})$.

Proof of (1.1) 3.2. Let $X = X(f)$ be the complement of the set on which f has the desired structure; then $X \subset B_f$ is closed. We suppose that $\dim f(X) \geqq p - 1$, and will obtain a contradiction.

Since f is C^3, $\dim(f(R_{p-2}(f))) \leqq p - 2$ [1, p. 156]. If, for every $x \in M^{p+1} - f^{-1}(f(R_{p-2}(f)))$, there is an open neighborhood $U_x \subset M^{p+1} - f^{-1}(f(R_{p-2}(f)))$ of x with $\bar{U}_x$ compact and $\dim(f(U_x \cap X)) \leqq p - 2$, it follows from the fact that $\{U_x\}$ has a countable subcover that $\dim(f(X)) \leqq p - 2$. Thus, there is an $\bar{x} \in M^{p+1} - f^{-1}(f(R_{p-2}(f)))$ such that, for every open neighborhood $U \subset M^{p+1} - f^{-1}(f(R_{p-2}(f)))$ of $\bar{x}$, $\dim(f(U \cap X)) \geqq p - 1$.

By [1, p. 156, Layering Lemma] there are open neighborhoods U of $\bar{x}$ and V of $f(\bar{x})$ and C^r diffeomorphisms $\sigma: R^2 \times R^{p-1} \approx U$ and $\rho: V \approx R \times R^{p-1}$ such that $\rho \circ f \circ \sigma = g$ is a C^r layer map and $\sigma(\bar{x}) = (0, 0)$. Thus $\dim g(X(g)) \geqq p - 1$. By (2.3) there is a closed set $Y \subset R^2 \times R^{p-1}$ such that $\dim g(Y) \leqq p - 2$ and $\dim((B_g - Y) \cap g^{-1}(y)) \leqq 0$ for each $y \in R \times R^{p-1}$.

Let h be the restriction $g \mid [(R^2 \times R^{p-1}) - Y]$; then $X(h) = X(g) - Y$, $\dim h(X(h)) = p - 1$, and $\dim(B_h \cap h^{-1}(y)) \leqq 0$ for each $y \in R \times R^{p-1}$, contradicting (3.1).

EXAMPLES 3.3. Open maps $f: M^2 \to R$ with $\dim(B_f \cap f^{-1}(y)) = 1$ are given in [4, p. 341] and [6, p. 329]; the latter example may be assumed to be C^∞ except on one point inverse, and thus [1, p. 151] may be assumed to be C^∞. As a result, "f real analytic" may not be replaced by "$f C^\infty$" in (1.1).

The maps f and g defined by $f(z) = \mathscr{R}(z)$ and $g(z) = (\mathscr{R}(z))^3$ are locally topologically equivalent at 0, but are not locally C^1 equivalent, since g has rank 0 at the origin.

There are examples [2, (4.7)(b)] with $X = B_f$, $\dim B_f = p - 1$, and $\dim f(B_f) = p - 2$.

REMARK 3.4. A real analytic open map $f: M^p \to N^p$ is light [2, p. 28, (4.2)], and thus for $p \geqq 2$ satisfies a structure theorem [1, p. 155] similar to (1.1).

REFERENCES

1. P. T. Church, *Differentiable monotone mappings and open mappings*, pp. 145–183 in The Proceedings of the First Conference on Monotone Mappings and Open Mappings, edited by L. F. McAuley, Oct. 8–11, 1970, SUNY at Binghamton, Binghamton, N. Y., 1971.

2. P. T. Church and W. D. Nathan, *Real analytic maps on manifolds*, J. Math. Mech., **19** (1969), 19–36.

3. P. T. Church and J. G. Timourian, *Differentiable open maps of $(p+1)$-manifolds to p-manifold*, Pacific J. Math., **48** (1973), 35–45.

4. W. G. Fox, *The critical points of peano interior functions defined on 2-manifolds*, Trans. Amer. Math. Soc., **83** (1956), 338-370.
5. R. Narasimhan, *Introduction to the Theory of Analytic Spaces*, Lecture Notes in Mathematics No. 25, Springer Verlag, Berlin, 1966.
6. W. D. Nathan, *Open mappings into a 1-manifold*, pp. 322-342 in The Proceedings of the First Conference on Monotone Mappings and Open Mappings, edited by L. F. McAuley, Oct. 8-11, 1970, SUNY at Binghamton, Binghamton, N. Y., 1971.
7. ———, *Open mappings on 2-manifolds*, Pacific J. Math., (to appear).
8. H. Whitney, *Tangents to an analytic variety*, Ann. of Math., (2) **81** (1965), 495-549.
9. W. D. Nathan and F. Bruhat, *Quelques proprietes fondamentales des ensembles analytiques-rèels*, Comment. Math. Helv., **33** (1959), 132-160.

Received August 21, 1972. Work of the first author supported in part by N. S. F. Grant GP-6871, and that of the second author by N. S. F. Grant GP-8888 and N. R. C. Grant A7357.

SYRACUSE UNIVERSITY
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Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics

Vol. 50, No. 1 September, 1974