OPEN PROJECTIONS AND BOREL STRUCTURES FOR C*-ALGEBRAS

HERBERT PAUL HALPERN
In this paper the relationships existing among the Boolean σ-algebra generated by the open central projections of the enveloping von Neumann algebra $\mathcal{B}$ of a $C^*$-algebra $\mathcal{A}$, the Borel structure induced by a natural topology on the quasi-spectrum of $\mathcal{A}$, and the type of $\mathcal{A}$ are discussed. The natural topology is the hull-kernel topology. It is shown that this topology is induced by the open central projections and is the quotient topology of the factor states of $S\mathcal{A}$ (with the relativized $\omega^*$-topology) under the relation of quasi-equivalence. The Borel field is shown to be Borel isomorphic with the Boolean σ-algebra multiplied by the least upper bound of all minimal central projections. Finally, it is shown that $\mathcal{A}$ is GCR if and only if the Boolean σ-algebra (resp. algebra) contains all minimal projections in the center of $\mathcal{B}$, or equivalently, if and only if every point in the quasi-spectrum is a Borel set.

T. Digernes and the present author [10] showed that $\mathcal{A}$ is CCR if and only if the open projections are strongly dense in the center of $\mathcal{B}$. They also showed that the complete Boolean algebra generated by the open central projections is equal to the set of all central projections in $\mathcal{B}$ whenever $\mathcal{A}$ is GCR. Recently, T. Digernes [9] obtained the converse of this result for separable $C^*$-algebras.

2. The Boolean algebra of open projections. Let $\mathcal{B}$ be a von Neumann algebra with center $\mathcal{K}$ and let $\mathcal{A}$ be a uniformly closed *-subalgebra of $\mathcal{B}$. A projection $P$ in $\mathcal{K}$ is said to be open relative to $\mathcal{A}$ if there is a two-sided ideal $\mathcal{I}$ in $\mathcal{A}$ whose strong closure is $\mathcal{B}P$. In the sequel all ideals (unless specifically excluded) will be assumed to be closed two-sided ideals. The definition corresponds to the definition of Akemann [1, Definition II.1] for $C^*$-algebras with identity. The set $\mathcal{P}(\mathcal{B}, \mathcal{A})$ of all open central projections of $\mathcal{B}$ relative to $\mathcal{A}$ contains $0, 1$ and the least upper bound (resp. greatest lower bound) of any (resp. any finite) subset [1, Proposition II.5, Theorem II.7].

Now let $\mathcal{B}$ be the enveloping von Neumann algebra of $\mathcal{A}$. The algebra $\mathcal{A}$ will be identified with its embedded image in $\mathcal{B}$. In this case the set $\mathcal{P}(\mathcal{B}, \mathcal{A})$ will be denoted simply by $\mathcal{P}$ and the projection in $\mathcal{P}$ will be called open projections. The smallest Boolean algebra (resp. σ-algebra) containing $\mathcal{P}$ will be denoted by $\langle \mathcal{P} \rangle$ (resp. $\mathcal{P}$).
Let $\mathcal{A}$ be the set of all unitary equivalence classes of irreducible representations of $\mathcal{A}$. The set $\mathcal{A}$ is called the spectrum of $\mathcal{A}$. For every irreducible representation $\rho$ of $\mathcal{A}$ on the Hilbert space $H(\rho)$, let $[\rho]$ denote the class in $\mathcal{A}$ of which $\rho$ is the representative. If $X$ is a subset of $\mathcal{A}$, let $\mathcal{A}(X) = \cap \{\ker \tau \mid \tau \in X\}$. Here $\ker \tau$ is uniquely defined by $\ker \tau = \ker \rho$ for $\rho \in \tau$. Then setting $X^- = \{\tau \in \mathcal{A} \mid \ker \tau \supset\mathcal{A}(X)\}$ for $X \neq \emptyset$ and $X^- = \emptyset$, we obtain a closure operation $\mathcal{A}$. The topology defined by this closure operation is called the hull-kernel topology and is the family of subsets of $\mathcal{A}$ given by

$\{\{\tau \in \mathcal{A} \mid \ker \tau \supset \mathcal{A}\} \mid \mathcal{A} \text{ is an ideal of } \mathcal{A}\}$ (cf. [12, § 3]). Let $\rho$ be a representation of $\mathcal{A}$ and let $\rho^-$ denote the unique extension of $\rho$ to a $\sigma$-weakly continuous representation of $\beta$ on $H(\rho)$ such that $\rho^-(\beta)$ is the $\sigma$-weak closure of $\rho(\mathcal{A})$. If $A \in \mathcal{A}$ and $\tau \in \mathcal{A}$ there is a unique scalar $A^-(\tau)$ such that $A^-(\tau)1_{H(\rho)} = \rho^-(A)$ for all $\rho \in \tau$. Here $1_{H(\rho)}$ is the identity operator on $H(\rho)$. With this notation, the hull-kernel topology of $\mathcal{A}$ is given by $\{\{\tau \in \mathcal{A} \mid P^-(\tau) = 1\} \mid P \in \mathcal{P}\}$.

Let $S(\mathcal{A})$ (resp. $S(\mathcal{A})$) denote the ring (resp. $\sigma$-ring) generated by the open subsets of $\mathcal{A}$. Then there is a projection-valued measure $\gamma$ of $S(\mathcal{A})$ onto $\langle \mathcal{P} \rangle$ such that $\gamma([\tau \in \mathcal{A} \mid P^-(\tau) = 1]) = P$ ([19, Theorem 1.9], cf. [12, 5.7.6]).

**Lemma 1.** Let $\mathcal{A}$ be a $C^*$-algebra, let $\beta$ be the enveloping von Neumann algebra of $\mathcal{A}$ and let $\mathcal{P}$ be the set of all open projections of the center $\mathcal{Z}$ of $\beta$. Let $Q$ be a minimal projection of $\mathcal{Z}$ and let $P_\pi$ be the least upper bounded of all minimal projections of $\mathcal{Z}$. Then $\beta Q$ is a type I factor whenever $Q$ is in the Boolean $\sigma$-algebra $\langle \mathcal{P} \rangle P_\pi$.

**Proof.** If $X_1$ and $X_2$ are open subsets of $\mathcal{A}$ with $X_1 \supset X_2$, then $\gamma(X_1 - X_2)^-(\tau) = \gamma(X_1)^-(\tau) - \gamma(X_2)^-(\tau) = 1$ for every $\tau \in X_1 - X_2$ and $\gamma(X_1 - X_2)^-(\tau) = 0$ for every $\tau \notin X_1 - X_2$. Since every set $X$ in $S(\mathcal{A})$ is the union of a finite number of mutually disjoint sets of the form $X_1 - X_2$, where $X_1, X_2$ are open in $\mathcal{A}$ and $X_1 \supset X_2$, we see that $\gamma(X)^-(\tau) = 1$ if and only if $\tau \in X$. Since every element $X$ in $S(\mathcal{A})$ is the union of a monotonally increasing sequence of sets $\{X_n\}$ in $S(\mathcal{A})$, we get that $\gamma(X)^-(\tau) = 1$ for every $\tau \in X$.

Now there is a set $X \in S(\mathcal{A})$ with $\gamma(X)^- P_\pi = Q$. If $\tau \in X$ then $Q^-(\tau) = 1$, and so $\rho^-(Q) = 1$ for $\rho \in \tau$. This means that the kernel of $\rho^-$ is $\beta (1 - Q)$. Since $\rho$ is irreducible on $\mathcal{A}$ and since $\rho^-(\beta)$, which is

---

1 This proof was suggested by the referee. My original proof was based on the results of [10].
isomorphic to $\mathcal{B}Q$, is equal to the weak closure of $\rho(\mathcal{A})$, we conclude that $\mathcal{B}Q$ is a type I factor.

The next result characterizes a GCR algebra in terms of the open central projections of its enveloping algebra.

**Theorem 2.** Let $\mathcal{A}$ be a C*-algebra, let $\mathcal{B}$ be the enveloping von Neumann algebra of $\mathcal{A}$, and let $\mathcal{P}$ be the set of open projections of the center $\mathcal{Z}$ of $\mathcal{B}$. Then the following statements are equivalent:

1. $\mathcal{A}$ is GCR;
2. $\langle \mathcal{P} \rangle$ contains all minimal projections $\mathcal{Z}$; and
3. $\langle \mathcal{P} \rangle$ contains all minimal projections of $\mathcal{Z}$.

**Proof.** $(1) \Rightarrow (2)$. We apply the fact that the set of open central projections in the enveloping von Neumann algebra of a CCR algebra is strongly dense in the set of central projection [10, Theorem 2].

There is a set $\{P_i \mid 0 \leq i \leq k\}$ of projections in $\mathcal{P}$ indexed by the ordinals such that (i) $P_0 = 0, P_k = 1$, (ii) $P_i < P_{i+1}(i < k)$, (iii) $V \{P_i \mid i < j\} = P_j$ if $j$ is a limit ordinal with $j \leq k$; and (iv) $\mathcal{B}_i = \mathcal{B}(P_{i+1} - P_i)$ is the strong closure of a CCR ideal $\mathcal{J}_i$ in $\mathcal{A}(1 - P_i)$ [10, proof of Theorem 3]. Let $Q$ be a minimal projection in $\mathcal{Z}$. There is an ordinal $i < k$ such that $Q \leq P_{i+1} - P_i$. Let $\mathcal{J}$ be the ideal in $\mathcal{A}$ given by $\mathcal{J} = \{A \in \mathcal{A} \mid AP_i = A\}$. Setting $\mathcal{J}' = \{A \in \mathcal{A} \mid A(1 - P_i) \in \mathcal{A}\}$, we obtain an ideal $\mathcal{J}'$ of $\mathcal{A}$ containing $\mathcal{J}$ such that $\mathcal{J}'/\mathcal{J}$ is isomorphic to $\mathcal{J}$. Let $\rho$ be the unique extension of the representation $A + \mathcal{J} \rightarrow A(1 - P_i)$ of $\mathcal{J}'/\mathcal{J}$ onto $\mathcal{A}$ to a $\sigma$-weakly continuous representation of the enveloping von Neumann algebra $\mathcal{B}$ of $\mathcal{J}'/\mathcal{J}$ onto the strong closure $\mathcal{B}_i$ of $\mathcal{J}_i$ on the subspace of the Hilbert space of $\mathcal{B}$ corresponding to the projection $1 - P_i$ (cf. [12, 12.1.5]). Now, if $P \in \mathcal{P}(\mathcal{B}, \mathcal{J}'/\mathcal{J})$, we show that $\rho(P) + P_i$ is in $\mathcal{J}$. Indeed, there is an ideal $\mathcal{K}$ in $\mathcal{J}'/\mathcal{J}$ such that $CP$ is the strong closure of $\mathcal{K}$ in $\mathcal{B}$. Let $\mathcal{K}'$ be an ideal in $\mathcal{J}'$ with $\mathcal{K}' \supset \mathcal{J}$ such that $\mathcal{K}'/\mathcal{J} = \mathcal{K}$. Then we have that the strong closure of $\mathcal{K}'(1 - P_i) = \rho(\mathcal{K}')$ in $\mathcal{B}(1 - P_i)$ is equal to $\rho(CP) = \mathcal{B}_i\rho(P) = \mathcal{B}\rho(P)$. This means that the strong closure of $\mathcal{K}'$ in $\mathcal{B}$ is equal to $\mathcal{B}(\rho(P) + P_i)$. Hence $\rho(P) + P_i$ is in $P$. Because $\mathcal{A}$ is CCR, the set $\mathcal{P}(\mathcal{B}, \mathcal{J}'/\mathcal{J})$ is strongly dense in the set of central projections of $\mathcal{B}$ [10, Theorem 2]. Recalling that $\rho$ maps the center of $\mathcal{B}$ onto the center of $\mathcal{A}$ [14, III, § 5, Problem 7], we obtain a net $\{R_n\}$ of projections in $\mathcal{P}$ which majorizes $P_i$ and is majorized by $P_{i+1}$ and which converges strongly to $P_{i+1} - Q$. Since $Q$ is a minimal projection, there is an $n_0$ such that $R_nQ = 0$ whenever $n \geq n_0$. This means that the open projection $R = V \{R_n \mid n \geq n_0\}$ is majorized by $P_{i+1} - Q$. But it is also clear that $P_{i+1} - Q \leq R$. Hence, we get that $P_{i+1} - Q = R$ and consequently
that $Q \in \langle \mathcal{F} \rangle$.

(2) $\Rightarrow$ (3). Obvious.

(3) $\Rightarrow$ (4). If $\mathcal{A}$ is not a GCR algebra, then $\mathcal{A}$ has a type III factor representation [24]. This means that there is a minimal projection $Q \in \mathcal{A}$ such that $\mathcal{A}Q$ is a type III factor. This is impossible by Lemma 1. Hence $\mathcal{A}$ is a GCR algebra.

3. Borel structure on the quasi-spectrum. Throughout this section let $\mathcal{A}$ be a C*-algebra, let $\mathcal{B}$ be the enveloping von Neumann algebra of $\mathcal{A}$, and let $\mathcal{P}$ be the set of open projections of the center $\mathcal{Z}$ of $\mathcal{B}$. The weak (resp. strong) topology of subalgebras of $\mathcal{B}$ will refer to the weak-operator (resp. strong-operator) topology. If $\rho$ is a representation of $\mathcal{A}$ on a Hilbert space $H(\rho)$, let $\rho^\sigma$ be the unique extension of $\rho$ to a $\sigma$-weakly continuous representation of $\mathcal{B}$ on $H(\rho)$ so that the weak closure of $\rho(\mathcal{A})$ is equal to $\rho^\sigma(\mathcal{B})$ [12, 12.1.5]. If $\rho$ is nondegenerate (i.e., the identity of $H(\rho)$ lies in the weak closure of $\rho(\mathcal{A})$), then $\rho(\mathcal{B})$ is the von Neumann algebra generated by $\rho(\mathcal{A})$ [14, I, §3, Theorem 2].

Now two nondegenerate representations $\rho_1$ and $\rho_2$ of $\mathcal{A}$ are said to be quasi-equivalent (notation: $\rho_1 \sim \rho_2$) if $\rho_1^\tau$ and $\rho_2^\tau$ have the same kernel. The relation of quasi-equivalence partitions the set of (non-degenerate) representations of $\mathcal{A}$ into quasi-equivalence classes. The class containing $\rho$ is denoted by $[\rho]$. If $\rho_1 \in [\rho]$, then ker $\rho = \ker \rho_1$ and thus for every class $[\rho]$, there is a uniquely associated ideal ker $[\rho] = \ker \rho$ of $\mathcal{A}$. Furthermore, if $\rho$ is a factor representation of $\mathcal{A}$ (i.e., $\rho^\sigma(\mathcal{B})$ is a factor von Neumann algebra), then so is every $\rho_1$ in the class $[\rho]$ (cf. [12, §5]).

Let $\mathcal{N}$ be the set of all quasi-equivalence classes of factor representations. The set $\mathcal{N}$ is called the quasi-spectrum of $\mathcal{A}$. If $A \in \mathcal{Z}$ and $\tau \in \mathcal{N}$, then there is a unique scalar $A^\tau(\tau)$ such that $\rho^\tau(A) = A^\tau(\tau)1_{H(\rho)}$ for every $\rho \in \tau$. Here $1_{H(\rho)}$ is the identity operator on $H(\rho)$. So every $A \in \mathcal{Z}$ defines a complex-valued function $A^\tau$ on $\mathcal{N}$ (cf. [7, §4]). Now it is clear that the map $A \mapsto A^\tau$ is a bounded *-homomorphism of $\mathcal{Z}$ into the C*-algebra $F(\mathcal{N})$ of bounded complex-valued functions on $\mathcal{N}$. For each $\tau \in \mathcal{N}$ there is a unique minimal projection of the algebra $\mathcal{Z}$ such that $Q^\tau(\tau) = 1$. Conversely, if $Q$ is a minimal projection of $\mathcal{Z}$, there is a unique $\tau \in \mathcal{N}$ such that $Q^\tau(\tau) = 1$. Thus there is a one-to-one map of the set of minimal projections of $\mathcal{Z}$ onto $\mathcal{N}$. Therefore, if $P^\tau$ denotes the least upper bound of all minimal projections in $\mathcal{Z}$, then $P^\tau = 1$. Furthermore, if $\mathcal{I}$ is an ideal of $\mathcal{A}$ and $P \in \mathcal{P}$ is such that $\mathcal{B}P$ is the strong closure of $\mathcal{I}$, then

\[(1) \quad \{ \tau \in \mathcal{N} \mid \ker \tau \not\supset \mathcal{I} \} = \{ \tau \in \mathcal{N} \mid P^\tau(\tau) = 1 \} .\]
Now let \( \tau \in \mathcal{S} \). The ideal \( \ker \tau \) is a prime ideal in the sense that \( \ker \tau \) contains the intersection of two ideals \( \mathcal{J} \) and \( \mathcal{I} \) in \( \mathcal{A} \) if and only if it contains one of them. Indeed, if \( \rho \in [\tau] \) and \( \rho(\mathcal{J}) \neq (0) \), then the strong closure of \( \rho(\mathcal{J}) \) is \( \rho(\mathcal{I}) \); otherwise, \( \rho(\mathcal{I}) \) would have a nontrivial center (cf. [11]). There is a net \( \{A_n\} \) in \( \mathcal{J} \) with \( \lim \rho(A_n) = 1 \) (strongly). Hence, for any \( A \in \mathcal{I} \), we have that

\[
\rho(A) = \rho(A) = \lim \rho(AA_n) = \lim \rho(AA_n) = 0.
\]

This means \( \rho(\mathcal{J}) = (0) \). Thus \( \ker \tau \) is a prime ideal. For any nonvoid subset \( X \) of \( \mathcal{A} \), we let \( \mathcal{J}(X) = \cap \{ \ker \tau \mid \tau \in X \} \) and we let

\[
X^- = \{ \tau \in \mathcal{A} \mid \ker \tau \supset I(X) \}.
\]

Setting \( X^- = \emptyset \), we get a unique topology on \( \mathcal{A} \), called the hull-kernel topology, such that the closure of a subset \( X \) of \( \mathcal{A} \) is \( X^- \) (cf. [12, 3.1]). The hull-kernel topology on \( \mathcal{A} \) generates a Borel structure \( S(\mathcal{A}) \) on \( \mathcal{A} \).

Thus the construction of the hull-kernel topology for the quasi-spectrum is analogous to that of the hull-kernel topology of the spectrum. We shall see further parallels in Propositions 3 and 9. However, the greater size of the quasi-spectrum allows us to prove Theorem 11.

**Proposition 3.** Let \( \mathcal{A} \) be a C*-algebra, let \( \mathcal{B} \) be the enveloping von Neumann algebra of \( \mathcal{A} \), let \( \mathcal{Z} \) be the center of \( \mathcal{B} \), and let \( P_m \) be the least upper bound of all minimal projections in \( \mathcal{Z} \). Let \( \mathcal{C} \) be the weak (\( \omega \))-sequential closure of the *-subalgebra of \( \mathcal{Z} P_m \) generated by \( \mathcal{P} P_m \). Then \( \mathcal{C} \) is the C*-algebra generated by \( \langle \mathcal{P} \rangle P_m \). Also there is an isomorphism \( \lambda \) of \( \mathcal{C} \) onto the C*-algebra \( B(\mathcal{A}) \) of bounded \( S(\mathcal{A}) \)-Borel functions on the quasi-spectrum \( \mathcal{A} \) such that the image of \( \langle \mathcal{P} \rangle P_m \) is the set of all characteristic functions in \( B(\mathcal{A}) \). Furthermore, the map \( \lambda \) is bi-continuous in the sense that \( \{\lambda(C_n)\} \) converges pointwise to \( \lambda(C) \) if and only if \( \{C_n\} \) is a sequence in \( \mathcal{C} \) that converges weakly to \( C \).

**Remark.** On \( \mathcal{Z} P_m \) the notions of strong and weak sequential convergence coincide.

**Proof.** The restriction \( \lambda \) of \( A \rightarrow A^\sim \) to \( \mathcal{C} \) is a *-homomorphism of \( \mathcal{C} \) into \( F(\mathcal{A}) \). If \( \{C_n\} \) is a sequence in \( \mathcal{C} \) that converges weakly to \( C \), then \( \{C_nQ\} \) converges uniformly to \( CQ \) for each minimal projection \( Q \) of \( \mathcal{Z} \) and so \( \lim \lambda(C_n) = \lambda(C) \) in the topology of pointwise convergence of \( F(\mathcal{A}) \). Hence \( \lambda \) is continuous. If \( \lambda(C) = 0 \) for some \( C \in \mathcal{C} \), then \( C^\sim(\tau) = 0 \) for all \( \tau \in \mathcal{A} \) and so \( CQ = 0 \) for all minimal projections.
This means \( C = CP_m = 0 \) and so \( \lambda \) is an isomorphism. Clearly, the inverse is continuous. We also have that

\[
\| \lambda(C) \| = \text{lub} \{ \| \lambda(C)(\tau) \| : \tau \in \mathcal{F} \}
\]

\[
= \text{lub} \{ \| CQ \| : Q \text{ minimal} \} = \| C \| ,
\]

for every \( C \in \mathcal{C} \). Furthermore, the image of \( \mathcal{SP}_m \) under \( \lambda \) is the set of all characteristic functions of open subsets of \( \mathcal{F} \) by relation (1). Hence \( \lambda \) maps the \(*\)-algebra generated by \( \mathcal{SP}_m \) into \( B(\mathcal{F}) \). By the continuity of \( \lambda \) and the norm preserving property (2), the map \( \lambda \) takes \( \mathcal{S} \) into \( B(\mathcal{F}) \).

Now we show that \( \lambda(\mathcal{S}) \) is sequentially closed in \( B(\mathcal{F}) \). Let \( \{C_n\} \) be a sequence in \( \mathcal{S} \) such that \( \{\lambda(C_n)\} \) converges pointwise to a function \( f \in B(\mathcal{F}) \). Since \( \lambda \) is a \(*\)-isomorphism, we may assume that \( f \) and each \( C_n \) is self-adjoint. Now if \( C \) and \( D \) are self-adjoint in \( \mathcal{S} \) there is a projection \( P \) in \( \mathcal{S} \) with \( PC + (1 - P)D = C \lor D \) in the lattice of self-adjoint elements in \( \mathcal{SP}_m \). In fact, the spectral projections \( \{E(\alpha)\} \) and \( \{F(\alpha)\} \) of \( C \) and \( D \) respectively are in \( \mathcal{S} \). For example, let \( \alpha \) be given and let \( g_n \) be the function of a real variable given by \( g_n(t) = 0 \) if \( t \geq \alpha \), \( g_n(t) = 1 \) if \( t \leq \alpha - n^{-1} \), and \( g_n \) linear on \([\alpha - n^{-1}, \alpha]\]. Then \( \{g_n(C)\} \) is a monotonally increasing sequence in \( \mathcal{S} \) whose least upper bound is \( E(\alpha) \). Let \( \{r_n\} \) be an enumeration of the rationals. Then \( P \) is the least upper bound of the sequence of projections \( \{F(r_n)(1 - E(r_n)) : r_m < r_n; n, m = 1, 2, \ldots\} \). Indeed, if \( Q \) is a minimal projection with \( Q \leq P \), then \( Q \leq F(r_n)(1 - E(r_n)) \) for some \( r_m < r_n \). This means that \( Q \leq F(r_m) \) and \( Q \leq 1 - E(r_m) \), and thus that \( DQ \leq r_n Q < r_m Q \leq CQ \). Conversely, let \( Q \) be a minimal projection with \( DQ < CQ \). Then there are \( r_m \) and \( r_n \) with \( DQ < r_m Q < r_n Q < CQ \). This means that \( Q \leq F(r_m)(1 - E(r_n)) \). Since \( P_m \) is the least upper bound of minimal projections, the projection \( P \) satisfies the requirements. We notice that \( \lambda(C \lor D) = \lambda(CP) + \lambda((1 - P)D) = \lambda(C) \lor \lambda(D) \) since \( \lambda \) preserves order and since \( \lambda(P) \) and \( \lambda(1 - P) \) are characteristic functions of disjoint sets whose union is \( \mathcal{F} \). The analogous statements hold for \( C \land D \). These facts allow us to assume that \( \{C_n\} \) is bounded since we may replace each \( C_n \) by \( C_n \land \| f \| P_m \). Now let \( D_n = \bigvee \{C_k \mid k \geq n\} \). We have that \( D_n \) lies in \( \mathcal{S} \) since \( D_n \) is the strong limit of the sequence \( \{\bigvee \{C_k \mid p \geq k \geq n\}\} \) in \( \mathcal{S} \). Since \( \lambda \) is continuous, we get that

\[
\lambda(D_n) = \lim \lambda(\bigvee \{C_k \mid p \geq k \geq n\}) = \bigvee \{\lambda(C_k) \mid k \geq n\} .
\]

By the same reasoning we get that

\[
\lambda(\bigwedge D_n) = \bigwedge \bigvee \{\lambda(C_k) \mid k \geq n\} .
\]

Now \( C = \bigwedge D_n \in \mathcal{S} \) and \( f = \lim \lambda(C_k) = \limsup \lambda(C_k) \). Hence we have
that \( f = \lambda(C) \). This proves that \( \lambda \) maps \( \mathcal{C} \) onto a sequentially closed subalgebra of \( B(\mathcal{A}) \) containing the characteristic functions of all open sets. Hence \( \lambda(\mathcal{C}) \) maps onto \( B(\mathcal{A}) \).

We now show \( \lambda(\mathcal{P}P_0) \) is the set of all characteristic functions of Borel sets. However, a proof similar to the one we have already given shows that \( \lambda(\mathcal{P}P_0) \) is a \( \sigma \)-complete Boolean algebra of characteristic functions. This Boolean algebra contains all characteristic functions of open sets and hence it coincides with the set of characteristic functions in \( B(\mathcal{A}) \).

Finally, we show that \( \mathcal{C} \) is the \( C^* \)-algebra \( \mathcal{C}_0 \) generated by \( \langle P \rangle P_0 \). Let \( f \in B(\mathcal{A}) \) be real-valued and let \( n \) be a natural number. Then there is a partition \( \{X_k \mid k = 0, \pm 1, \ldots, \pm n\} \) of \( \mathcal{A} \) into disjoint Borel sets such that each \( X_k \) is contained in the set

\[
\{\tau \in \mathcal{A} \mid kn^{-1} \leq f(\tau) \leq (k + 1)n^{-1} \}.
\]

If we set \( g_k \in B(\mathcal{A}) \) equal to the characteristic function of \( X_k \) for every \( k \), we get \( \| \sum \alpha_k g_k - f \| \leq n^{-1} \) for suitable scalars \( \alpha_k \). Because \( \sum \alpha_k g_k \in \lambda(\mathcal{C}_0) \) and because \( \lambda \) is an isometry, we get that \( f \in \lambda(\mathcal{C}_0) \).

Due to the fact \( \lambda \) is an *-isomorphism, we get that \( \mathcal{C}_0 = \mathcal{C} \).

For the spectrum of a \( C^* \)-algebra we have the following result.

**Proposition 4.** Let \( \mathcal{A} \) be a \( C^* \)-algebra, let \( \mathcal{B} \) be the enveloping von Neumann algebra of \( \mathcal{A} \), and let \( \mathcal{P} \) be the set of open projections of the center \( \mathcal{Z} \) of \( \mathcal{B} \). Let \( \mathcal{A} \) be the set of equivalence classes of irreducible representations of \( \mathcal{A} \) with the hull-kernel topology. Then there is an isomorphism \( \phi \) of the \( C^* \)-algebra \( \mathcal{R} \) generated by \( \langle \mathcal{P} \rangle \) onto the algebra \( B(\mathcal{A}) \) of all bounded complex-valued Borel functions on \( \mathcal{A} \) such that the image of \( \langle \mathcal{P} \rangle \) is the set of all characteristic functions in \( B(\mathcal{A}) \). Furthermore, \( \phi \) is continuous in the sense that \( \{\phi(A_n)\} \) converges to \( \phi(A) \) whenever \( \{A_n\} \) is a sequence in \( \mathcal{R} \) that converges strongly to \( A \) in \( \mathcal{B} \).

**Proof.** Let \( P_0 \) be the least upper bound of all minimal projections \( Q \) in \( \mathcal{Z} \) such that \( \mathcal{B}Q \) is type I. There is an isomorphism \( \check{\phi} \) of the smallest weakly sequentially closed *-subalgebra \( \mathcal{D} \) of \( \mathcal{Z}P_0 \) containing \( \langle \mathcal{P} \rangle P_0 \) onto \( B(\mathcal{A}) \) such that \( \langle \mathcal{P} \rangle P_0 \) maps onto the set of all characteristic functions of \( B(\mathcal{A}) \). Also \( \mathcal{D} \) is the \( C^* \)-algebra generated by \( \langle \mathcal{P} \rangle P_0 \). This follows in the same way as Proposition 3.

We also have that the map \( A \rightarrow AP_0 \) is a homomorphism of \( \mathcal{R} \) onto \( \mathcal{D} \). Setting \( \phi(A) = \check{\phi}(AP_0) \), we obtain a homomorphism of \( \mathcal{R} \) onto \( B(\mathcal{A}) \) that is continuous in the specified sense.
We show that $\phi$ is an isomorphism. There is a projection-valued
operator $\gamma$ defined on the Borel sets $S(\hat{\mathcal{A}})$ of $\hat{\mathcal{A}}$ such that $\gamma(\{\tau \in \hat{\mathcal{A}} | P^*(\tau) = 1\}) = P$ for every open projection $P$ ([19, Theorem 1.9], cf. [12, 5.7.6]). Identifying the characteristic functions of $B(\mathcal{A})$ with their supports, we get that $\gamma \cdot \psi(PP_0) = P$ for every $P \in \mathcal{P}$ and so $\gamma \cdot \phi(P) = P$ for every $P \in \mathcal{P}$. This means that $\gamma \cdot \phi(P) = P$ for every $P \in \mathcal{P}$. Now suppose $\phi(A) = 0$ for some $A \in \mathcal{A}$. Given $\varepsilon > 0$, there exist orthogonal projections $P_1, \ldots, P_n$ in $\mathcal{P}$ and positive scalars $\alpha_i, \ldots, \alpha_n$ such that $\| \sum \alpha_i P_i - A^* A \| < \varepsilon$. This means that $\| \sum \alpha_i \phi(P_i) \| < \varepsilon$. Since the $\phi(P_i)$ are disjoint characteristic functions, we have that $\phi(P_i) = 0$ for every $i$ with $\alpha_i \geq \varepsilon$. This means $P_i = \gamma \cdot \phi(P_i) = 0$ for all such $i$. Hence we have that $\| \sum \alpha_i P_i \| < \varepsilon$ and so that $\| A \| = \| A^* A \| < 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have that $A = 0$. Hence $\phi$ is an isomorphism.

**COROLLARY 5.** Let $\mathcal{A}$ be a $C^*$-algebra, let $\hat{\mathcal{A}}$ be the spectrum of $\mathcal{A}$, and let $\mathcal{A}$ be the quasi-spectrum of $\mathcal{A}$. Suppose that both $\hat{\mathcal{A}}$ and $\mathcal{A}$ have the hull-kernel topology. Then there is a pointwise continuous isomorphism of the algebra $B(\mathcal{A})$ of bounded Borel functions on $\mathcal{A}$ onto the algebra $B(\hat{\mathcal{A}})$ of bounded Borel functions on $\hat{\mathcal{A}}$.

**Proof.** Let $\mathcal{P}$ be the set of open projections in the center $\mathcal{Z}$ of the enveloping von Neumann algebra $\mathcal{B}$ of $\mathcal{A}$. Let $P_0$ be the least upper bound of all minimal projections $Q$ in $\mathcal{Z}$ such that $\mathcal{B}Q$ is type I and let $P_m$ be the least upper bound of all minimal projections in $\mathcal{B}$. Then the $C^*$-algebra $\mathcal{D}$ generated by $\langle \mathcal{P} \rangle P_0$ is isomorphic to $B(\hat{\mathcal{A}})$ under a bi-continuous map for the strong and the pointwise topology (Proposition 4), and the $C^*$-algebra $\mathcal{D}$ generated by $\langle \mathcal{P} \rangle P_m$ is isomorphic to $B(\mathcal{A})$ under a bi-continuous map for the strong and pointwise topology (Proposition 3). But the $C^*$-algebra $\mathcal{B}$ generated by $\langle \mathcal{P} \rangle$ is isomorphic to $D$ under the map $A \to AP_0$. Hence the map $A \to AP_0$ is an isomorphism of $\mathcal{D}$ onto $\mathcal{B}$. This isomorphism is certainly strongly continuous. Hence, there is a pointwise continuous isomorphism of $B(\mathcal{A})$ onto $B(\hat{\mathcal{A}})$.

**REMARK.** The set of bounded continuous complex-valued functions on $\hat{\mathcal{A}}$ has been described recently ([5], [13]). Due to the fact that $\hat{\mathcal{A}}$ need not be separated, the continuous functions do not approximate the Borel functions.

We describe a class of elements that lie in $C^*$-algebra $\mathcal{B}$ generated by $\langle \mathcal{P} \rangle$. Let $Z$ be the spectrum of $\mathcal{Z}$. For every $A \in \mathcal{B}$ and $\zeta$ in $Z$, let $A(\zeta)$ denote the image of $A$ under the canonical map of $\mathcal{B}$ onto the algebra $\mathcal{B}$ reduced modulo the ideal generated by $\zeta$. There is
an element $\psi(A) \in \mathcal{X}$ such that $\psi(A)^\vee(\zeta) = \|A(\zeta)\|$ for all $\zeta \in Z$. Here $\psi(A)^\vee(\zeta)$ is the Gelfand transform of $\psi(A)$ evaluated at $\zeta$ [18, Lemma 10].

**Proposition 6.** Let $\mathcal{A}$ be a C*-algebra, let $\mathcal{B}$ be its enveloping von Neumann algebra, let $\mathcal{P}$ be the set of open projections of the center $\mathcal{Z}$ of $\mathcal{B}$, and let $\mathcal{E}$ be the uniformly closed *-subalgebra of $\mathcal{Z}$ generated by $\mathcal{P}$. Then, for every $A \in \mathcal{A}$, the element $\psi(A)$ lies in $\mathcal{E}$.

**Proof.** Since $\mathcal{E}$ is a C*-algebra and since $\psi(A) = \psi(A^*A)^{1/2}$, it is sufficient to show $\psi(A) \in \mathcal{E}$ for every $A$ in $\mathcal{A}$. We have that there is a projection $P$ in $\mathcal{Z}$ such that $\{\zeta \in Z \mid P^\vee(\zeta) = 1\} = \text{clos} \{\zeta \in Z \mid \psi(A)^\vee(\zeta) > 0\}$

since $Z$ is extremally disconnected. But it is clear that $P$ is an open projection since $\mathcal{B}P$ is the strong closure of the principal ideal generated by $A$. Now, for any $\alpha > 0$, let $f_\alpha$ be the continuous function of a real-variable given by $f_\alpha(t) = 0$ if $t \leq \alpha$ and $f_\alpha(t) = t - \alpha$ for $t > \alpha$. Then there is an open projection $P$ with $\{\zeta \in Z \mid P^\vee(\zeta) = 1\} = \text{clos} \{\zeta \in Z \mid \psi(f_\alpha(A))^\vee(\zeta) > 0\}$

and so $\{\zeta \in Z \mid P^\vee(\zeta) = 1\} = \text{clos} \{\zeta \in Z \mid \psi(A)^\vee(\zeta) > \alpha\}$.

Now let $n$ be a natural number. Let $P_{k}(k = 0, 1, \ldots, n - 1)$ be the open projections given by $\{\zeta \in Z \mid P_{k}^\vee(\zeta) = 1\} = \text{clos} \{\zeta \in Z \mid \psi(A)^\vee(\zeta) > n^{-k} \|A\|\}.$

Let $Q_{k} = P_{k-1} - P_{k}$ for $1 \leq k \leq n - 1$ and $Q_{n} = P_{n-1}$. Then we have that $\|\psi(A) - \sum n^{-k} \|A\|Q_{k}\| = \text{lub} (\|\psi(A)^\vee(\zeta) - \sum n^{-k} \|A\|Q_{k}\| \mid Q_{k} \in Z \mid \zeta \in Z) \leq n^{-1} \|A\|.$

Hence, the element $\psi(A)$ is in $\mathcal{E}$.

For a separable C*-algebra, we have a better result. We preserve the same notation as the preceding proposition.

**Corollary 7.** Let $\mathcal{A}$ be a separable C*-algebra, then the C*-algebra $\mathcal{R}$ in $\mathcal{Z}$ generated by $\langle \mathcal{P} \rangle$ is equal to the weak sequential closure of the C*-algebra generated by $\{\psi(A) \mid A \in \mathcal{A}\}$.

**Proof.** Let $P \in \mathcal{P}$ and let $\mathcal{I}$ be an ideal in $\mathcal{A}$ whose strong closure is $\mathcal{B}$P. The ideal $\mathcal{I}$ is a principal ideal generated by an
element $A$ of $\mathcal{A}$ [23, 6.5, Corollary]. This means that $P$ is smallest projection in $\mathcal{K}$ with $P\psi(A) = \psi(A)$. Hence $P$ is in the weak sequential closure $\mathcal{B}_0$ of the $C^*$-algebra generated by $\psi(\mathcal{A})$. This proves that $\mathcal{P}$ and thus $\mathcal{R}$ is contained in $\mathcal{B}_0$.

Conversely, each element $\psi(A)$ is contained in $\mathcal{F}$ (Proposition 6). Let $P_0$ be the least upper bound of all projections $Q$ in $\mathcal{A}$ such that $\mathcal{B}Q$ is a type I factor. The map $A \mapsto AP_0$ of the weak sequential closure $\mathcal{S}^\sim$ of $\mathcal{A}$ in $\mathcal{B}$ is a weak sequentially continuous isomorphism onto the weak sequential closure of $\mathcal{A}P_0$ [6, Theorem 3.10]. Since $\mathcal{R}_0 \subset \mathcal{A}^\sim$ and $\mathcal{B} \subset \mathcal{A}^\sim$ and since $\mathcal{R}P_0 = \mathcal{B}$ is weakly sequentially closed (cf. Proposition 4), we may find, for each $A \in \mathcal{R}_0$, a $B \in \mathcal{B}$ such that $AP_0 = BP_0$. This means that $A = B$. Hence $\mathcal{R}_0 \subset \mathcal{B}$. Thus we get that $\mathcal{R} = \mathcal{R}_0$.

Now let $\mathcal{A}$ be a separable $C^*$-algebra and let $\mathcal{A}^\sim$ be the weak sequential closure of $\mathcal{A}$ in its enveloping algebra $\mathcal{B}$. The center $\mathcal{Z}(\mathcal{A}^\sim)$ is contained in the center $\mathcal{Z}$ of $\mathcal{B}$. As is pointed out by E. B. Davies (cf. [6, p. 154] for the analogous statement for $\mathcal{A}$) each open projection in $\mathcal{Z}$ is in $\mathcal{Z}(\mathcal{A}^\sim)$. This means that $\mathcal{B}(\mathcal{A}^\sim)$ is contained in the algebra $\{A^* | A \in \mathcal{Z}(\mathcal{A}^\sim)\} \subset F(\mathcal{A})$. Thus the Davies Borel structures on $A$ (i.e., the weakest Borel structure such that all functions $\{A^* | A \in \mathcal{Z}(\mathcal{A}^\sim)\}$ are Borel on $\mathcal{A}$) is finer than the structure $S(\mathcal{N})$ induced by the hull-kernel topology. In fact the Davies Borel structure separates points whereas the Borel structure $S(\mathcal{N})$ does not in certain cases (for example, a separable uniformly hyperfinite $C^*$-algebra). The $C^*$-algebra $\mathcal{B}$ generated by the Boolean $\sigma$-algebra $\langle \mathcal{P} \rangle$ is contained in $\mathcal{Z}(\mathcal{A}^\sim)$. In order that $\mathcal{Z}(\mathcal{A}^\sim) = \mathcal{B}$, a necessary and sufficient condition is that the Davies and hull-kernel Borel structure on $\mathcal{A}$ coincide. Now, if $\mathcal{A}$ is a GCR algebra, then all the Borel structures on $\mathcal{A}$ coincide [12, 3.8.3] and so $\mathcal{Z}(\mathcal{A}^\sim) = \mathcal{B}$. We note that a special case of this result is mentioned by Glimm [19, p. 899]. Conversely, if the Davies and the hull-kernel Borel structure coincide on $\mathcal{A}$, then $\mathcal{A}$ is GCR. Indeed, it is sufficient to show that two irreducible representations $\rho_1$ and $\rho_2$ with the same kernels are equivalent [20]. It is this result, which is unavailable in the nonseparable case, that O. Dignerness [9] used to characterize a separable GCR algebra. We have that $P^\sim([\rho_1]) = P^\sim([\rho_2])$ for every open projection $P$ in $\mathcal{Z}$. Indeed, if $\mathcal{J}$ is an ideal in $\mathcal{A}$ whose strong closure is $\mathcal{B}P$, then $P^\sim([\rho_1]) = 0$ if and only if $\mathcal{J}$ is contained in the kernel of $\rho_2$. Thus we have $P^\sim([\rho_1]) = P^\sim([\rho_2])$ for all $P$ in $\langle \mathcal{P} \rangle$ and thus the Davies Borel structure fails to separate $[\rho_1]$ and $[\rho_2]$. This implies that $[\rho_1] = [\rho_2]$ [8, Theorem 2.9]. Hence the algebra $\mathcal{A}$ is GCR. It is to be noted that Effros [15] proved that $A$ is GCR if and only if the Mackey
Davies Borel structure coincides on $\mathcal{A}$.

We now examine the hull-kernel topology of the quasi-spectrum more closely. We show that this topology is induced by the canonical mapping of the factor states into the quasi-spectrum.

Let $\mathcal{A}$ be a C*-algebra and let $f$ be a state of $\mathcal{A}$. Let $L(f)$ be the left ideal of $\mathcal{A}$ given by $L(f) = \{ A \in \mathcal{A} \mid f(A^*A) = 0 \}$, let $H(f)$ be the completion of the residue class $\mathcal{A} - L(f)$ with the inner product $(A - L(f), B - L(f)) = f(B^*A)$, and let $\rho_f$ be the (nondegenerate) representations of $\mathcal{A}$ on the Hilbert space $H(f)$ induced by left multiplication of $\mathcal{A}$ on $\mathcal{A} - L(f)$. The representation $\rho_f$ is called the canonical representation of $\mathcal{A}$ induced by $f$. There is a cyclic unit vector $x_f$ under $\rho_f(\mathcal{A})$ for $H(f)$ (equal to $1 - L(f)$ if $\mathcal{A}$ has identity 1 or equal to $\lim A_n - L(f)$ if $\{A_n\}$ is an increasing approximate identity in the positive part of the unit sphere of $\mathcal{A}$ if $\mathcal{A}$ has no identity) such that $\omega_{x_f} \cdot \rho_f(A) = (\rho_f(A)x_f, x_f) = f(A)$ for all $A \in \mathcal{A}$. The state $f$ is called a factor (or primary) state if $\rho_f$ is a factor representation of $\mathcal{A}$. Let $\mathcal{F}(\mathcal{A})$ be the space of all factor states of $\mathcal{A}$ with its relativized $w^*$-topology. We write $f \sim g$ for $f, g$ in $\mathcal{F}(\mathcal{A})$ to denote $\rho_f \sim \rho_g$.

Now suppose that $\mathcal{A}$ is a C*-algebra without an identity. Then an identity 1 may be adjoined to $\mathcal{A}$ to obtain a C*-algebra $\mathcal{A}_e$ with identity so that $\mathcal{A}_e$ is a maximal ideal of $\mathcal{A}$ (cf. [12, 1.2.3]). Each state $f$ on $\mathcal{A}$ has a unique extension $f_e$ to a state of $\mathcal{A}_e$ obtained by setting $f_e(1) = 1$. The Hilbert spaces $H(f)$ and $H(f_e)$ can be identified with each other so that $\rho_{f_e}$ restricted to $\mathcal{A}$ is precisely $\rho_f$. Furthermore, the identity of $\mathcal{A}_e$ gets carried into the identity operator on $H(f)$ (cf. [12, 2.1.4]). Therefore, the state $f_e$ is a factor state if and only if $f$ is. Furthermore, if $f$ and $g$ are factor states of $\mathcal{A}_e$, then $f \sim g$ if and only if $f_e \sim g_e$. Now let $f_0$ be the unique factor state of $\mathcal{A}_e$ that vanishes on $\mathcal{A}$. If $f$ be a factor state of $\mathcal{A}_e$ not equal to $f_0$, then the ideal $\rho_f(\mathcal{A}_e)$ of $\rho_f(\mathcal{A}_e)$ is nonzero and therefore is strongly dense in $\rho_f(\mathcal{A}_e)$ (cf. [11]). For any $\varepsilon > 0$ there is a net $\{B_n\}$ in $\mathcal{A}$ with $\lim \|B_n\| \leq 1 + \varepsilon$ such that $\{\rho_f(B_n)\}$ converges strongly to the identity [22]. Hence, the restriction $g$ of $f$ to $\mathcal{A}_e$ has norm not less than $(1 + \varepsilon)^{-1}$ since

$$\| g \| \geq (1 + \varepsilon)^{-1} \lim \sup |g(B_n)|$$

$$= (1 + \varepsilon)^{-1} \lim \sup |(\rho_f(B_n)x_f, x_f)| \geq (1 + \varepsilon)^{-1}.$$
Since $\mathcal{F}'(\mathcal{A})$ is open in $\mathcal{F}(\mathcal{A})$, we may conclude that $e(\mathcal{V})$ is open in $\mathcal{F}(\mathcal{A})$. So the map $e$ is also an open map.

We now prove that quasi-equivalence is an open relation in the space $\mathcal{F}(\mathcal{A})$ by showing the saturation $\mathcal{U}^\sim$ of an open subset $\mathcal{U}$ of $\mathcal{F}(\mathcal{A})$ given by $\mathcal{U}^\sim = \{ f \in \mathcal{F}(A) \mid f \sim g \in \mathcal{U} \}$ is open.

**Lemma 8.** The saturation under the relation of quasi-equivalence of an open subset of the space of factor states of a $C^*$-algebra is open.

**Proof.** Let $\mathcal{V}$ be an open subset of the space $\mathcal{F}(\mathcal{A})$ of factor states of the $C^*$-algebra $\mathcal{A}$. We assume that $\mathcal{A}$ has an identity, and later we remove this assumption. Let $g$ be a factor state in the saturation $\mathcal{V}^\sim$ of $\mathcal{V}$. We construct a neighborhood $\mathcal{W}$ of $g$ such that $\mathcal{W} \subset \mathcal{V}^\sim$. There is an element $h \in \mathcal{V}$ with $g \sim h$. There are elements $C_1, C_2, \ldots, C_n$ in $\mathcal{A}$ and a $\delta$ with $0 < \delta < 1$ such that

$$\{ f \in \mathcal{F}(\mathcal{A}) \mid |f(C_i) - h(C_i)| < \delta, i = 1, \ldots, n \}$$

is contained in $\mathcal{V}$. Without loss of generality we may assume that $C_1 = 1$. Due to the fact that $g \sim h$, there is an isomorphism $\phi$ of the von Neumann algebra $\rho_g(\mathcal{A})''$ generated by $\rho_g(\mathcal{A})$ on $H(g)$ onto the von Neumann algebra $\rho_h(\mathcal{A})''$ generated by $\rho_h(\mathcal{A})$ on $H(h)$ such that $\phi(\rho_g(\mathcal{A})) = \rho_h(\mathcal{A})$ for every $A \in \mathcal{A}$ (cf. [12, § 5]). Since an isomorphism of von Neumann algebras is $\sigma$-weakly continuous, [14, I, § 4, Theorem 2, Corollary 1], the functional $\omega_{x_k} \cdot \phi$ is a $\sigma$-weakly continuous state of $\rho_g(\mathcal{A})''$ such that $\omega_{x_k} \cdot \rho_g = h$. This means that there is a sequence $\{x_k\}$ in $H(g)$ such that $\sum ||x_k||^2 < +\infty$ and such that $\sum \omega_{x_k} \cdot \omega_{x_k} \cdot \phi$ on $\rho_g(\mathcal{A})''$ [14, I, § 3, Theorem 2]. Setting $\eta = \delta(6\max \{|C_i||1 \leq i \leq n\})^{-1}$, we may find a natural number $m$ such that

$$||\sum \omega_{x_i} | m + 1 \leq i < +\infty || < \eta .$$

Since each $x_i$ lies in the closure of $\rho_g(\mathcal{A})x_k$, there are $A_1, A_2, \ldots, A_m$ in $\mathcal{A}$ such that the vectors $\rho_g(A_i)x_k = y_i$ in $H(g)$ satisfy

$$||\omega_{x_i} - \omega_{y_i}|| < m^{-1}\eta$$

for $i = 1, \ldots, m$.

Now let $\varepsilon = m^{-1}\eta$. We show every $f$ in the neighborhood $\mathcal{W}$ of $g$ given by

$$\mathcal{W} = \{ f \in \mathcal{F}(\mathcal{A}) \mid |f(A_i^*C_jA_i) - g(A_i^*C_jA_i)| < \varepsilon \}
\text{ for all } i = 1, \ldots, m; j = 1, \ldots, n$$

is contained in $\mathcal{V}^\sim$. Setting $f'$ equal to
\[ f'(A) = \sum \{ f(A_i^* A_i) \mid 1 \leq i \leq m \] 
for all \( A \in \mathcal{A} \), we obtain a positive functional on \( \mathcal{A} \) whose norm is given by \( \| f' \| = f'(1) = \sum f(A_i^* A_i) \). Because \( C_i = 1 \), we get 
\[ |f'(1) - \sum g(A_i^* A_i)| \leq \sum |f(A_i^* A_i) - g(A_i^* A_i)| < \eta. \]
But we have that 
\[ |\sum g(A_i^* A_i) - 1| = |\sum g(A_i^* A_i) - \sum \omega_{x_i}(1)| 
= |\sum \{\omega_{x_i}(1) \mid 1 \leq i \leq m\} - \sum \{\omega_{x_i}(1) \mid 1 \leq i < +\infty\}| 
\leq \sum \{|\omega_{x_i} - \omega_{x_i}| \mid 1 \leq i \leq m\} 
+ |\sum \{\omega_{x_i} \mid m + 1 \leq i < +\infty\}| < 2\eta \]
by relations (3) and (4). This means that 
\[ f'(1) - 1 < 3\eta < 1. \]
Hence, we have \( f'(1) \neq 0 \). Setting \( f'' = f'/\| f' \| \), we obtain a state \( f'' \) of \( \mathcal{A} \) such that \( f'' \sim f \) ([4] and [12, 5.3.6]).

We shall now show that \( f'' \in \mathcal{V} \). First we have that 
\[ |f''(C_i)| \leq f'(1)^{i/2} f'(C_i C_i)^{i/2} \leq f'(1) \| C_i \| \]
for all \( i = 1, \ldots, m \). By relation (5) this yields 
\[ |f''(C_i)| = |1 - f'(1)| f'(1)^{-1} f'(C_i)| 
\leq |1 - f'(1)| \| C_i \| < \delta/2, \]
for every \( i = 1, \ldots, n \). Furthermore, for all \( i \), we get 
\[ |f''(C_i) - h(C_i)| \leq \sum \{|f(A_i^* C_i A_j) - g(A_i^* C_i A_j)| \mid 1 \leq j \leq m\} 
+ \sum \{|\omega_{x_j}(\rho_{x}(C_i)) - \omega_{x_j}(\rho_{x}(C_i))| \mid 1 \leq j \leq m\} 
+ |\sum \{\omega_{x_j}(\rho_{x}(C_i)) \mid m + 1 \leq j < +\infty\}| 
< me + \eta \| C_i \| + \eta \| C_i \| \leq \delta/2 \]
by relations (3) and (4). Combining (6) and (7), we obtain 
\[ |f''(C_i) - h(C_i)| \leq |f''(C_i) - f'(C_i)| 
+ |f'(C_i) - h(C_i)| < \delta/2 + \delta/2 = \delta, \]
for all \( i = 1, \ldots, n \). This proves that \( f'' \in \mathcal{V} \). Hence, the lemma is true for \( C^* \)-algebras with identity.

Suppose \( \mathcal{A} \) is a \( C^* \)-algebra without identity. Let \( \mathcal{A} \) be the \( C^* \)-algebra obtained from \( \mathcal{A} \) by adjoining the identity. We use the notation developed in the paragraph preceding this lemma. If \( \mathcal{V} \) is an open subset of \( \mathcal{F}(\mathcal{A}) \), then \( e(\mathcal{V}) \) is open in \( \mathcal{F}(\mathcal{A}) \). But the
saturation \( e(\mathcal{H})^* \) of \( e(\mathcal{H}) \) in \( \mathcal{F}(\mathcal{A}) \) is \( e(\mathcal{H})^* \). By the first part of the proof \( e(\mathcal{H})^* \) is open. Thus the set \( \mathcal{H}^* = e^{-1}(e(\mathcal{H})^*) = e^{-1}(e(\mathcal{H})^*) \) is open in \( \mathcal{F}(\mathcal{A}) \).

**Proposition 9.** Let \( \mathcal{A} \) be a C*-algebra. The map \( f \rightarrow [\rho_f] \) is a continuous open mapping of the space \( \mathcal{F}(\mathcal{A}) \) of factor states of \( \mathcal{A} \) onto the quasi-spectrum \( \mathcal{A} \) of \( \mathcal{A} \) with its hull-kernel topology.

**Proof.** Let \( \phi \) denote the map \( f \rightarrow [\rho_f] \). Let \( \rho \) be any nondegenerate factor representation of \( \mathcal{A} \) on a Hilbert space \( H \). There is a unit vector \( x \in H \) such that \( f(A) = (\rho(A)x, x) \) is a state of \( \mathcal{A} \). There is an isometric isomorphism \( U \) of \( H(f) \) onto the invariant subspace \( K = \text{closure } \rho(A)x \) of \( H \) defined by \( U(A - L(f)) = \rho(A)x \) that carries \( \rho \) onto the subrepresentation \( \rho \upharpoonright K \) of \( \rho \). Since \( [\rho \upharpoonright K] = [\rho] \) [12, 5.3.5], we get that \( [\rho_f] = [\rho] \). Hence, the image of \( \phi \) is equal to \( \mathcal{A} \).

Now let \( \{f_n\} \) be a net in \( \mathcal{F}(\mathcal{A}) \) that converges to \( f \) in the \( \omega^* \)-topology. Let \( X \) be an open subset of \( \mathcal{A} \) containing \( [\rho_f] \). There is an ideal \( I \) in \( \mathcal{A} \) with \( X = \{\tau \in \mathcal{A} \mid \ker \tau \supseteq I \} \). This means there is an \( A \in \mathcal{A} \) such that \( f(A) = 0 \). There is an \( n_0 \) such that \( f_n(A) \neq 0 \) whenever \( n \geq n_0 \). Hence, the classes \( [\rho_{f_n}] \) are in \( X \) whenever \( n \geq n_0 \). This means \( \{[\rho_{f_n}]\} \) converges to \( [\rho_f] \). Thus \( \phi \) is continuous.

For the proof that \( \phi \) is an open map, we consider two cases: (1) \( \mathcal{A} \) has an identity, and (2) \( \mathcal{A} \) has no identity. First assume \( \mathcal{A} \) has an identity. Let \( Y \) be an open subset of \( \mathcal{F}(\mathcal{A}) \). We prove \( \phi(Y) \) open in \( \mathcal{A} \). By Lemma 8, we may assume that \( Y \) is saturated. The complement \( Y' \) of \( Y \) in \( \mathcal{F}(\mathcal{A}) \) is also saturated. It is sufficient to show that \( \phi(Y') \) is closed in \( \mathcal{A} \) since \( \phi(Y) = \mathcal{A} - \phi(Y') \). In fact, we shall show that \( \phi(Y') = \{\tau \in \mathcal{A} \mid \ker \tau \supseteq I \} \). This means \( \phi(Y') \) contains \( [\rho_f] \) whenever \( f \) is a pure state with \( \ker \rho_f \supseteq I \). Now let \( f \) be an arbitrary factor state of \( \mathcal{A} \) with \( \ker \rho_f \supseteq I \). Then we have that \( \mathcal{A} = \ker \rho_f \) is a prime ideal containing \( I \) (cf. introductory paragraphs of § 3). Let \( g \) be the state of the C*-algebra \( \mathcal{A}/I = \mathcal{C} \) given by \( g(A + I) = f(A) \). Let \( \mathcal{H} \) be the maximal GCR ideal of \( \mathcal{C} \). Then the state space and the pure state space of \( \mathcal{C} \) coincide [25, Theorem 2]. There is a net \( \{g_i\} \) of pure states of \( \mathcal{C} \) that converges
in the $w^*$-topology to $g$. Setting $f_4(A) = g(A + \mathcal{J})$ for all $A \in \mathcal{A}$, we get a net $\{f_4\}$ of pure states in $\mathcal{A}$ that converges to $f$ in the $w^*$-topology. Since each $f_4 \in \mathcal{W}$ by the first part of the proof, we get $f \in \mathcal{W}$ and thus $[\rho_f] \in \phi(\mathcal{W})$. Now let $\mathcal{W}_f \neq (0)$. We then have that the representation $\rho_f$ of $\mathcal{A}$ is quasi-equivalent to an irreducible representation. Indeed, we have that $\rho_f(\mathcal{W})x_0$ is dense in $H(\mathcal{A})$ since $\rho_f$ is a factor representation of $\mathcal{A}$. But the von Neumann algebra $\rho_f(\mathcal{W})''$ generated by $\rho_f(\mathcal{W})$ on $H(\mathcal{A})$ is a type I algebra (cf. [12, 5.5.2]). This means that $\rho_f(\mathcal{W})''$ has a nonzero abelian projection $E$. However, the projection $E$ is also an abelian projection for the von Neumann algebra generated by $\rho_f(\mathcal{W})$. Hence $\rho_f$ is quasi-equivalent to an irreducible representation (cf. [12, 5.4.11]). Since the representation $\rho_f$ defined by $\rho_f(A) = \rho_f(A + \mathcal{J})$ is unitarily equivalent to $\rho_f$, we see that $\rho_f$ is quasi-equivalent to an irreducible representation. Now let $\mathcal{W}$ be the $C^*$-algebra obtained from $C^*$ by the adjunction of the identity. Let $\Phi'$ be the map of $\mathcal{W}$ onto $\mathcal{W}$ given by $\Phi'(\rho) = [\rho_f]$. Let $\mathcal{W}$ be open in $\mathcal{W}(\mathcal{A})$. By using Lemma 8, we may assume that $\mathcal{W}$ is saturated. We have that $e(\mathcal{W})$ is an open saturated set in $\mathcal{W}(\mathcal{A})$, whose image $\phi(e(\mathcal{W}))$ is an open subset in $\mathcal{A}$. There is an ideal $\mathcal{J}$ in $\mathcal{A}$ with $\phi(e(\mathcal{W})) = \{\tau \in \mathcal{A} | \ker \tau \not\subseteq \mathcal{J}\}$. We show that $\phi(\mathcal{W}) = \{\tau \in \mathcal{A} | \ker \tau \not\subseteq \mathcal{J} \cap \mathcal{A}\}$. Indeed, let $f \in \mathcal{W}$; then $\ker \rho_f \not\subseteq \mathcal{J}$ and so there is an $A \in \mathcal{J}$ with $g(A) \neq 0$. If $\rho_f$ is an increasing approximate identity in the unit sphere of $\mathcal{A}$, we have that $\lim f(A_n) = \lim g(A_n) = g(A)$ because $A_n A \in \mathcal{J} \cap \mathcal{A}$ for all $n$. This means that $\ker [\rho_f] \not\subseteq \mathcal{J} \cap \mathcal{A}$. Conversely, if $\ker [\rho_f] \not\subseteq \mathcal{J} \cap \mathcal{A}$, then $f(\mathcal{J} \cap \mathcal{A}) \neq 0$ and $\ker [\rho_f] \not\subseteq \mathcal{J}$. There is an $h \in \mathcal{A}$ such that $e(h) \sim g$. This implies that $h \sim f$ and $[\rho_f] \in \phi(\mathcal{W})$. So $\phi(\mathcal{W}) = \{\tau \in \mathcal{A} | \ker \tau \not\subseteq \mathcal{J} \cap \mathcal{A}\}$. We can interpret Proposition 9 in terms of representations. An infinite dimensional Hilbert space $H$ is said to have sufficiently high dimension for the factor states of $\mathcal{A}$, if there is a faithful represen-
tation $\rho_0$ of $\mathcal{A}$ on $H$ such that, for any factor state $f$ of $\mathcal{A}$, there is a unit vector $x \in H$ with $f = \omega_x \cdot \rho_0$. Now let $H$ be a Hilbert space of sufficiently high dimension. (If $\mathcal{A}$ is separable, any infinite dimensional space has sufficiently high dimension.) Let CFac $(\mathcal{A}, H)$ be the family of all representations $\rho$ on $H$ for which there is a unit vector $x \in H$ such that $f = \omega_x \cdot \rho$ is a factor state and such that $\rho$ vanishes on the orthogonal complement of the closure of the linear manifold
A topology may be defined on $\text{CFac}(\mathcal{A}, H)$ by allowing a net \( \{\rho_n\} \) converge to \( \rho \) if and only if \( \{\rho_n(A)\} \) converges to \( \rho(A) \) in the strong topology on \( H \) for every \( A \in \mathcal{A} \).

**Proposition 10.** Let \( \mathcal{A} \) be a $C^*$-algebra, let \( H \) be a Hilbert space of sufficiently high dimension for the factor representations of \( \mathcal{A} \). Let \( \psi \) be the map that carries each \( \rho \in \text{CFac}(\mathcal{A}, H) \) into its class \([\rho]\) in \( \mathcal{A} \). Then \( \psi \) is a continuous open map of \( \text{CFac}(\mathcal{A}, H) \) onto \( \mathcal{A} \).

**Proof.** It is clear that \( \phi \) maps \( \text{CFac}(\mathcal{A}, H) \) continuously onto \( \mathcal{A} \).

We show that \( \psi \) is an open mapping. Let \( \mathcal{U} \) be an open subset of \( \text{CFac}(\mathcal{A}, H) \). Using virtually the same proof as K. Bichteler [3, Proposition 2.4(i)], we can find an open subset \( \mathcal{V} \) of \( \mathcal{A} \) such that \( \psi(\mathcal{U}) = \phi(\mathcal{V}) \). However, we have shown that \( \phi(\mathcal{V}) \) is open in \( \mathcal{A} \) (Proposition 9). Thus \( \psi(\mathcal{U}) \) is open in \( \mathcal{A} \) and \( \psi \) is an open map.

**Remark.** An infinite dimensional Hilbert space \( K \) is said to have sufficiently high dimension for the irreducible representations of \( \mathcal{A} \) if there is a faithful representation \( \rho_0 \) of \( \mathcal{A} \) on \( K \) such that, for every pure state \( f \) of \( \mathcal{A} \), there is a unit vector \( x \in K \) for which \( f = \omega_x \cdot \rho_0 \). A space \( H \) that has sufficiently high dimension for the factor representations certainly has sufficiently high dimension for the irreducible representations. Then let \( K \) have sufficiently high dimension for the irreducible representations. Let \( \text{Irr}(\mathcal{A}, K) \) be the family of all representations \( \rho \) of \( \mathcal{A} \) on \( K \) for which there is a unit vector \( x \) in \( K \) such that \( \omega_x \cdot \rho \) is a pure state and \( \rho \) vanishes on the orthogonal complement of the closure of \( \rho(\mathcal{A})x \). Then L. T. Gardner [17] proved \( \rho \to [\rho] \) is a continuous open map of \( \text{Irr}(\mathcal{A}, K) \) onto the spectrum of \( \mathcal{A} \) (with the hull-kernel topology). Notice that \( \text{Irr}(\mathcal{A}, H) \subset \text{CFac}(\mathcal{A}, H) \).

We now characterize a GCR algebra in terms of the Borel structure on the quasi-spectrum.

**Theorem 11.** Let \( \mathcal{A} \) be a $C^*$-algebra. The following are equivalent:

1. \( \mathcal{A} \) is a GCR algebra; and
2. every point of the quasi-spectrum \( \mathcal{A} \) of \( \mathcal{A} \) is a Borel set in the Borel structure induced by the hull-kernel topology.

**Proof.** (1) \( \Rightarrow \) (2). If \( \tau \in \mathcal{A} \), let \( Q \) be the unique minimal projection of the center \( \mathcal{Z} \) of the enveloping von Neumann \( \mathcal{B} \) algebra of \( \mathcal{A} \) such that \( Q^*(\tau) = 1 \). By Theorem 2, the projection \( Q \) is in the Boolean algebra generated by the open central projections \( \mathcal{P} \) of \( \mathcal{B} \). By Proposition 3 we conclude that the characteristic function of the set
\{\tau\} is in the algebra of bounded Borel function on \mathcal{A}. Hence, the set \{\tau\} is a Borel set of \mathcal{A}.

(2) \Rightarrow (1). Let Q be an arbitrary minimal projection in \mathcal{A}. The image of Q under the map \lambda defined in Proposition 3 is the characteristic function of a point set in \mathcal{A}. If \(P_m\) is the least upper bound of the minimal projection of \mathcal{A}, then \(Q \in \mathcal{P}P_m\) (Proposition 3). By Lemma 1 we have that \mathcal{B}Q is type I. Because Q is arbitrary, the algebra \mathcal{A} must be GCR [24].

Added May 1, 1973. For separable C*-algebra \mathcal{A}, I have proved that the quotient Borel structure on \mathcal{A} induced by the map \(f \mapsto [\rho_f]\) of the factor states of \mathcal{A} with the relativized \(w^*\)-topology into \mathcal{A} is the Mackey Borel structure of \mathcal{A}.

Acknowledgment. The author wishes to thank T. Digernes for discussions concerning the material in § 2.

REFERENCES


Received September 25, 1972. This research was supported by the National Science Foundation.

UNIVERSITY OF CINCINNATI
Gail Atneosen, *Sierpinski curves in finite 2-complexes* ........................................ 1
Bruce Alan Barnes, *Representations of B*-algebras on Banach spaces* ........ 7
George Benke, *On the hypergroup structure of central Λ(p) sets* ............ 19
Carlos R. Borges, *Absolute extensor spaces: a correction and an answer* ......................... 29
Tim G. Brook, *Local limits and tripleability* .................................................. 31
Philip Throop Church and James Timourian, *Real analytic open maps* ...... 37
Timothy V. Fossum, *The center of a simple algebra* ...................................... 43
Richard Freiman, *Homeomorphisms of long circles without periodic points* ...................................................... 47
B. E. Fullbright, *Intersectional properties of certain families of compact convex sets* ..................... 57
Harvey Charles Greenwald, *Lipschitz spaces on the surface of the unit sphere in Euclidean n-space* ........................................ 63
Herbert Paul Halpern, *Open projections and Borel structures for C*-algebras* ........................................ 81
Frederic Timothy Howard, *The number of multinomial coefficients divisible by a fixed power of a prime* ..................... 99
Lawrence Stanislaus Husch, Jr. and Ping-Fun Lam, *Homeomorphisms of manifolds with zero-dimensional sets of nonwandering points* .......... 109
Joseph Edmund Kist, *Two characterizations of commutative Baer rings* .... 125
Lynn McLinden, *An extension of Fenchel’s duality theorem to saddle functions and dual minimax problems* ........................................ 135
Leo Sario and Cecilia Wang, *Counterexamples in the biharmonic classification of Riemannian 2-manifolds* ..................... 159
Saharon Shelah, *The Hanf number of omitting complete types* ............ 163
Richard Staum, *The algebra of bounded continuous functions into a nonarchimedean field* ........................................ 169
James DeWitt Stein, *Some aspects of automatic continuity* ................. 187
Tommy Kay Teague, *On the Engel margin* .................................................... 205
John Griggs Thompson, *Nonsolvable finite groups all of whose local subgroups are solvable, V* ......................... 215
Kung-Wei Yang, *Isomorphisms of group extensions* ...................................... 299