THE HANF NUMBER OF OMITTING COMPLETE TYPES

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It is proved in this paper that the Hanf number $m^c$ of omitting complete types by models of complete countable theories is the same as that of omitting not necessarily complete type by models of a countable theory.

Introduction. Morley [3] proved that if $L$ is a countable first-order language, $T$ a theory in $L$, $p$ is a type in $L$, and $T$ has models omitting $p$ in every cardinality $\lambda < \aleph_1$, then $T$ has models omitting $p$ in every infinite cardinality. He also proved that the bound $\aleph_1$ cannot be improved, in other words the Hanf number is $\aleph_1$. He asked what is the Hanf number $m^c$ when we restrict ourselves to complete $T$ and $p$. Clearly $m^c \leq \aleph_1$. Independently several people noticed that $m^c \geq \aleph_1$ and J. Knight noticed that $m^c > \aleph_1$.

Malitz [2] proved that the Hanf number for complete $L_{\omega_1,\omega}$-theories with one axiom $\forall \in L_{\omega_1,\omega}$ is $\aleph_1$. We shall prove

**Theorem 1.** $m^c = \aleph_1$.

**Notation.** Natural numbers will be $i, j, k, l, m, n$, ordinals $\alpha, \beta, \delta$; cardinals $\lambda, \mu$. $|A|$ is the cardinality of $A$, $\beth_\alpha = \sum_{\beta<\alpha} 2^{2^{\beta}} + \aleph_0$.

$M$ will be a model with universe $|M|$, with corresponding countable first-order language $L(M)$. For a predicate $R \in L(M)$, the corresponding relation is $R^s$ or $R(M)$, and if there is no danger of confusion just $R$. Every $M$ will have the one place predicate $P$ and individual constants $c_\alpha$ such that $P = P^s = \{c_\alpha; n < \omega\}$, $n \neq m \Rightarrow c_\alpha \neq c_m$ (we shall not distinguish between the individual constants and their interpretation). A type $p$ in $L$ is a set of formulas $\phi(x_0) \in L$; $p$ is complete for $T$ in $L$ if it is consistent and for no $\varphi(x_0) \in L$ both $T \cup p \cup \{\varphi(x_0)\}$ and $T \cup p \cup \{\neg \varphi(x_0)\}$ are consistent.

An element $b \in |M|$ realizes $p$ if $\varphi(x_0) \in p$ implies $M \models \varphi[b]$ ($\models$ -satisfaction sign), and $M$ realizes $p$ if some $a \in |M|$ realizes it. A complete theory in $L$ is a maximal consistent set of sentences of $L$. For every permutation $\theta$ of $P$, model $M$, and sublanguage $L$ of $L(M)$ we define an Ehrenfeucht game $EG(M, L, \theta)$ between player I and II with $\omega$ moves as follows: in the $n$th move first player I chooses $i \in \{0, 1\}$ and $a^n_i \in |M|$ and secondly player II chooses $a^n_{i^{-1}} \in |M|$. Player II wins if the extension $\theta^* \circ \theta$ defined by $\theta^*(a^n_i) = a^n_{i^{-1}}$ preserves all atomic formulas of $L$. That is if $R(x_1, \ldots, x_n)$ is an atomic formula in $L$, $\theta^*(b_i)$ is defined then $M \models R[b_i, \ldots, b_n]$ iff $M \models R[\theta^*(b_i), \ldots, \theta^*(b_n)]$. 

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REMARK. So if I chooses \( a^i_n \in P \), II should choose \( a^i_n = \theta(a^i_n) \).

Define \( \Gamma(n_0) = \{ \theta : \theta \text{ a permutation of } P, n < n_0 \Rightarrow \theta(c_n) = c_n \} \) and only for finitely many \( n \) \( \theta(c_n) \neq c_n \).

\( M \mid L \) is the reduct of \( M \) to the language \( L \subseteq L(M) \), that is \( M \mid L \) is \( M \) without the relations \( R^\omega, R \in L(M), R \notin L \), and constants \( c_n \in L(M), c_n \notin L \).

**Theorem 2.** For every ordinal \( \alpha < \omega \) there is a countable first-order language \( L_\alpha \) a complete theory \( T_\alpha \) in \( L \), such that

1. \( \varphi = \{ P(x_0) \} \cup \{ x_0 \neq c_n : n < \omega \} \) is a complete type for \( T_\alpha \).
2. \( T_\alpha \) has a model of cardinality \( \beth_\alpha \) omitting \( p \).
3. \( T_\alpha \) has no model of cardinality \( > \beth_\alpha \) omitting \( p \).

**Remark.** Clearly Theorem 2 implies Theorem 1.

**Proof.** We shall define by induction on \( \alpha < \omega \) models \( M_\alpha \) such that

1. \( \| M_\alpha \| \), the cardinality of \( | M_\alpha | \), is, \( \beth_\alpha \), and of course \( P = P(M_\alpha) = \{ c_n : n < \omega \} \) and except for the \( c_n \)'s \( L(M_\alpha) \) has only predicates.
2. There is no model elementarily equivalent to \( M_\alpha \) of cardinality \( > \beth_\alpha \) which omits \( p \).
3. If \( (\exists \beta)(\alpha = \beta + 2) \) then \( Q_\alpha \in L(M_\alpha) \) and \( | Q_\alpha(M_\alpha) \| = \beth_\alpha \).
4. For every finite sublanguage \( L \) of \( L(M_\alpha) \) there is \( n_L = n(L) < \omega \), such that for every permutation \( \theta \in \Gamma(n_L) \) player II has a winning strategy in \( EG(M_\alpha, L, \theta) \).
5. In (4) if \( (\exists \beta)(\alpha = \beta + 2) \) then in the winning strategy of II, if I chooses \( a^i_n \in Q_\alpha(M_\alpha) \) then II chooses \( a^i_n = \theta^*(a^i_n) \).

The induction will go as follows. First we define \( M_0, M_1, \) and \( M_2 \); later we define \( M_{\alpha+1} \) by \( M_\alpha \) when \( (\exists \beta)(\alpha = \beta + 2) \); last for limit ordinal \( \delta \) we define \( M_\delta, M_{\delta+1}, M_{\delta+2} \) by \( M_\alpha \alpha < \delta \).

But before defining the \( M_\alpha \)'s, let us show how this will finish the proof. We choose \( L_\alpha = L(M_\alpha) \). \( T_\alpha \) is the set of sentences of \( L_\alpha \) that \( M_\alpha \) satisfies. Clearly (ii), (iii) are satisfied. To prove (i) let \( \varphi(x_0) \in L_\alpha \), so for some finite sublanguage \( L \) of \( L_\alpha \) \( \varphi(x_0) \in L \). By possibly interchanging \( \varphi(x_0) \) and \( \neg \varphi(x_0) \) we can assume \( M_\alpha \models \varphi[c_n(L)] \). For \( k \geq n(L) \) let \( \theta_k \) be the permutation of \( P \) interchanging \( c_n(L) \) and \( c_k \), and leaving the other elements fixed.

Clearly \( \theta \in \Gamma(n_L) \), hence player II has a winning strategy in \( EG(M_\beta, L, \theta) \). By Ehrenfeucht [1] this implies \( c_n(L) \) and \( c_k = \theta(c_n(L)) \) satisfy the same formulas of \( L \). Hence \( M_\beta \models \varphi[c_n(L)] \equiv \varphi[c_k] \), hence \( M_\beta \models \varphi[c_i] \). As this holds for any \( k \geq n(L) \) \( M_\beta \models (\forall x)[P(x) \land \bigwedge_{i<n(L)} x \neq c_i \rightarrow \varphi(x)] \). Hence \( T_\alpha \cup p \cup \{ \neg \varphi(x_0) \} \) is inconsistent. So \( p \) is complete
(for $T_a, L_a$) and we finish.

So let us define

\textit{Case I.} $\alpha = 0, 1, 2$

(A) Let us define $M_\alpha$:

$$| M_\alpha | = P, \text{ and its only predicate is } P \text{ (and of course the individual constants } c_n, \text{ which we will not mention in later cases). Clearly (1), (2) are immediate. (3) and (5) are satisfied vacuously. As for (4), let } n_L = \max \{n + 1: c_n \in L\}. \text{ Clearly } \theta \text{ is an automorphism of } M_\alpha | L \text{ (the reduct of } M_\alpha \text{ to } L).$$

So player II will play by the automorphism: if I chooses $a^n_0$, II will choose $a^n_0 - \theta(a^n_0)$, and if I chooses $a^n_1$, II will choose $a^n_1 = \theta^{-1}(a^n_0)$.

(B) $| M_1 | = | M_0 | \cup P_1(M_1)$, where $P_1(M_1) = \mathcal{P}(| M_0 |)$, where $\mathcal{P}(A)$ = the power set of $A = \{B: B \subseteq A\}$.

The predicates of $M_1$ are those of $M_0, P_1$ and $\varepsilon_1$

$$\varepsilon_1(M_1) = \{\langle c, A \rangle: c \in | M_0 |, A \in P_1, c \in A\}.$$

As in (A) it is clear that $M_1$ satisfies the induction conditions, as if $\theta \in \Gamma(n_L)$ $L \subseteq L(M_1)$, $L$ finite, then $\theta$ can be extended to an automorphism of $M_1$ by

$$\theta(A) = \{\theta(c): c \in A\}.$$

(C) Let us define an equivalence relation $E_1$ on $P_1(M_1)$: $AE_1B$ iff for some $\theta \in \Gamma(0)$ $A = \theta(B)[ = \{\theta(c): c \in B\}]$.

This is an equivalence relation, as $\Gamma(0)$ is a group of permutations, and as $| \Gamma(0) | = \aleph_0$, each equivalence class is countable. Define

$$| M_1 | = | M_0 | \cup Q_1(M_0)$$

$$Q_1(M_0) = \{S: S \subseteq P_1(M_1), A, B \in P_1, AE_1B \Rightarrow A \in S \iff B \in S\}$$

$$\varepsilon_1(M_1) = \{\langle A, S \rangle: A \in P_1, S \in Q_1, A \in S\}.$$

The relations of $M_1$ will be the relations of $M_0$, and $Q_1, \varepsilon_2$. By the definition of $Q_1$, each $\theta \in \Gamma(n_L)$ $[L$ a finite sublanguage of $L(M_0)]$ can be extended to an automorphism $\theta^*$ of $M_1 | L$, which is the identity over $Q_2$. As before (1), (2), (4) hold, and as $\theta^*$ is the identity over $Q_2$, also (5) holds. As for (3) each $E_1$-equivalence class is countable, and $| P_1(M_1) | = 2^{2^{\aleph_0}} = 2^{\aleph_0}$, the number of $E_1$-equivalence classes is $\aleph_1$, so $| Q_2 | = 2^{\aleph_1} = \aleph_2$.

\textit{Case II.} We define $M_{\alpha+1}$, where $M_\alpha$ is defined, $(\exists \beta)(\alpha = \beta + 2)$.

Let

$$| M_{\alpha+1} | = | M_\alpha | \cup \mathcal{P}(Q_\alpha(M_\alpha)).$$
The relations of $M_{\alpha+1}$ will be those of $M_\alpha$ and in addition $Q_{\alpha+1}(M_{\alpha+1}) = \mathcal{P}(Q_\alpha(M_\alpha))$

$$\varepsilon_{\alpha+1}(M_{\alpha+1}) = \{\langle a, A \rangle : a \in Q_\alpha(M_\alpha), A \in Q_{\alpha+1}(M_{\alpha+1}), a \in A \}.$$ 

Clearly Conditions (1), (2), (3) are satisfied. As for (4), (5) the winning strategy of player II in $EG(M_{\alpha+1}, L, \theta)[\theta \in \Gamma(n_M)]$ will be as follows: when I chooses elements in $|M_\alpha|$ he will pretend all the game is in $|M_\alpha|$ and play accordingly; and if player I chooses $a^i \in Q_{\alpha+1}(M_{\alpha+1})$, then player II will choose $a_n^i = a_n^i$. As $M_\alpha$ satisfies (5) this is a winning strategy, and trivially it satisfies (5).

**Case III.** $\delta$ a limit ordinal, $M_\alpha$ is defined for $\alpha < \delta$; and we shall define $M_\delta, M_{\delta+1}, M_{\delta+2}$.

**PART A.** By changing, when necessary, names of elements and relations, we can assume that for $\alpha < \beta < \delta$,

$$|M_\alpha| \cap |M_\beta| = P, \text{ and } L(M_\alpha) \cap L(M_\beta) = \{P, c_n : n < \omega\},$$

but that if $(\exists \beta)(\alpha = \beta + 2)$ then still $Q_\alpha \in L(M_\alpha)$. Choose an increasing sequence of ordinals $\alpha_n n < \omega$, $\delta = \bigcup_{n<\omega} \alpha_n$ and $(\exists \beta)(\alpha_n = \beta + 2)$. Define $M_\delta$ as follows

$$|M_\delta| = \bigcup_{n<\omega} M_{\alpha_n}.$$ 

The relations of $M_\delta$ will be those of $M_{\alpha_n}$ for each $n < \omega$ and $R_\delta^{\nu_\delta}$

$$R_\delta^{\nu_\delta} = \{\langle c, a \rangle : c = c_n \in P, a \in (M_{\alpha_n} - P)\}.$$ 

It is easy to check that Conditions (1), (2) are satisfied. Conditions (3) and (5) are vacuous. So let us prove Condition (4) holds. Let $L$ be a finite sublanguage of $L(M_\delta)$; then $L \subseteq \bigcup_{j<n_0} L_j \cup \{R\}$, where $L_j = L \cap L(M_{\alpha_j})$ is a finite sublanguage of $L(M_{\alpha_j})$. Define $n_L = \max\{|n_{L_j} : j < n_0\} \cup \{n_0\}$. Let $\theta \in \Gamma(n_L)$. We shall describe now the winning strategy of player II in $EG(M_\delta, L, \theta)$. When player I will choose $i \in \{0, 1\}, a^i_n \in M_{\alpha_j}, j < n_0$, player II will pretend all the game is in the model $M_{\alpha_j}$, and so play his winning strategy for $EG(M_{\alpha_j}, L \cap L(M_{\alpha_j}), \theta)$. If player I chooses $i \in \{0, 1\}, a^i_n \in M_{\alpha_j}, j \geq n_0$ then player II will choose $a_n^{i-1} \in M_{\alpha_k}$ [where $i = 0 \implies k = \theta(j), i = 1 \implies j = \theta(k)$] such that for any $m < n$ $a_n^{i-m} = a_n^{i-1} \iff a_n^{i-m} = a_n^{i-1}$. 

**Note** that for $j \geq n_0$, in $M_\delta | L$, every permutation of elements of $M_{\alpha_j}$ is an automorphism, as the only relation an $a \in |M_{\alpha_j}|$ satisfies is $R_\delta(c_j, a)$.

**PART B.** Here we define $M_{\delta+1}$. Let $A^* = \bigcup_{n<\omega} Q_{\alpha_n}(M_{\alpha_n})$, and $\vert M_{\delta+1} \vert = \vert M_\delta \vert \cup \mathcal{P}(A^*)$. 
The relations of $M_{\delta+1}$ will be those of $M_{\delta}$, and in addition

$$P_\delta(M_{\delta+1}) = |M_\delta|, \ P_{\delta+1}(M_{\delta+1}) = \mathcal{P}(A^*)$$

$$\varepsilon_{\delta+1}(M_{\delta+1}) = \{\langle b, B \rangle : b \in A^*, B \in \mathcal{P}(A^*), b \in B \}.$$ 

It is easy to see that Conditions (1), (2) are satisfied, and (3), (5) are vacuous. So let us prove (4) — let $L$ be a finite sublanguage of $L(M_{\delta+1})$. So

$$L \subseteq \bigcup_{i<n_0} L_i \cup \{R_\delta, P_\delta, P_{\delta+1}, \varepsilon_{\delta+1}\}, \ L_i = L \cap L(M_{\alpha_i}).$$

Define again

$$n_L = \max \{|n_{L_j} : j < n_0| \cup \{n_0\} \}.$$ 

Let $\theta \in I'(n_L)$ and we should describe player II's winning strategy in $EG(M_{\delta+1}, L, \theta)$. When player I chooses an element in $M_{\alpha_j}$, $j < n_0$, player II will ignore all elements chosen outside $M_{\alpha_j}$, and play by his winning strategy in $EG(M_{\alpha_j}, L_j, \theta)$. In the other cases player II will play so that the following conditions are satisfied for every $n$

$$P(1) \ a^i_n \in P_{\delta+1}(M_{\delta+1}) \iff a^i_n \in P_{\delta+1}(M_{\delta+1})$$

$$P(2) \ \text{if } c_j = \theta(c_k), \ \text{then } a^i_0 \in |M_{\alpha_k}| \iff a^i_n \in |M_{\alpha_j}|$$

$$P(3) \ \text{if } m < n \ \text{then } a^i_m = a^i_0 \iff a^i_m = a^i_n$$

$$P(4) \ \text{if } m, l \leq n \ \text{and } a^i_m \in A^*, a^i_l \in P_{\delta+1} \ \text{then } a^i_m = a^i_l \iff a^i_m = a^i_l$$

$$P(5) \ \text{if } a^i_m \in P_{\delta+1}, l < \omega, c_l = \theta(c_i) \ \text{then } a^i_m \in Q_{\alpha_l}(M_{\alpha_l}) = a^i_m \in Q_{\alpha_l}(M_{\alpha_l})$$

$$P(6) \ \text{if } c_j = \theta(c_k) \neq k < \omega, \ \text{then } \langle a^i_m : m \leq n, a^i_m \in P_{\delta+1} \rangle \ \text{and} \ \langle a^i_m : m \leq n, a^i_m \in P_{\delta+1} \rangle \ \text{generate corresponding finite Boolean algebras of subsets of } Q_{\alpha_k}(M_{\alpha_k}) \ \text{and} \ Q_{\alpha_j}(M_{\alpha_j}) \ \text{correspondingly; then the corresponding atoms in those algebras are both infinite, or have the same power.}$$

It is easy to see that this can by done, and it is a winning strategy.

PART C. Here we define $M_{\delta+2}$.

Define equivalence relations $E_{\delta+1}^n, E_{\delta+1}^{n+1}$ on $P_{\delta+1}(M_{\delta+1})$: if $A, B \in P_{\delta+1}(M_{\delta+1})$, then $A, B \subseteq A^* = \bigcup_{n<\omega} Q_{\alpha_n}(M_{\alpha_n})$; define $AE_{\delta+1}^n B$ iff $A \cap \bigcup_{n>m>n} Q_{\alpha_n}(M_{\alpha_m}) = B \cap \bigcup_{n>m>n} Q_{\alpha_n}(M_{\alpha_m})$; $AE_{\delta+1} B$ iff for some $n$ $AE_{\delta+1}^n B$.

Clearly each $E_{\delta+1}^n$ is an equivalence relation, $E_{\delta+1}^{n+1}$ refines $E_{\delta+1}^n$, hence $E_{\delta+1}$ is an equivalence relation.

It is clear that

$$|P_{\delta+1}(M_{\delta+1})| = \aleph_{\delta+1}$$

but for every $n < \omega, A \in P_{\delta+1}(M_{\delta+1})$
hence

\[ \left\{ B: B \in P_{\delta+1}(M_{\delta+1}), BE_{\delta+1}\alpha \right\} \subseteq \mathcal{P}\left( \bigcup_{m \leq n} Q_{\alpha_m}(M_{\alpha_m}) \right) \]

\[ = 2^\sum_{\alpha} = \sum_{\alpha \in \mathfrak{b}} = \mathfrak{b}_\delta \]

So each \( E_{\delta+1} \) - equivalence class has cardinality \( \leq \mathfrak{b}_\delta \), hence there
are \( \mathfrak{b}_{\delta+1} \) \( E_{\delta+1} \)-equivalence classes.

Define \( \mathcal{M}_{\delta+2} \):

\[ |\mathcal{M}_{\delta+2}| = |\mathcal{M}_{\delta+1}| \cup Q_{\delta+2}(M_{\delta+2}) \]

where

\[ Q_{\delta+2}(M_{\delta+2}) = \{ S: S \subseteq P_{\delta+1}(M_{\delta+1}), A, B \in S, AE_{\delta+1}B \rightarrow A \in S \rightarrow B \in S \} \]

Clearly \( |Q_{\delta+2}(M_{\delta+2})| = \mathfrak{b}_{\delta+2} \).

The relations of \( \mathcal{M}_{\delta+2} \) will be those of \( \mathcal{M}_{\delta+1} \), and \( Q_{\delta+2} \), and

\[ \varepsilon_{\delta+2}(M_{\delta+2}) = \{ \langle A, S \rangle: A \in P_{\delta+1}(M_{\delta+1}), S \in P_{\delta+2}(M_{\delta+2}), A \in S \} \]

It is easy to prove all conditions are satisfied as in Case II, if we notice that by Condition \( P (5) \) if for any instance of any game \( EG(M_{\delta+1}, L, \theta)[\theta \in \Gamma(n_L)] \) in which player II plays his strategy, if \( a_{n}, a_{n}^{i} \) are chosen for some \( n \) and they belong to \( P_{\delta+1}(M_{\delta+1}) \) then they are \( E_{\delta+1} \)-equivalent (as \( \{n: \theta(c_n) \neq n\} \) is finite).

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