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ON THE ENGEL MARGIN

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The marginal subgroup for any outer commutator word has been characterized by R. F. Turner-Smith. This paper considers the marginal subgroup E(G) of G for the Engel word $e_2(x, y) = [x, y, y]$ of length two. The principal result is that an element a of G is in E(G) if and only if [x, y, a][a, y, x] is a law in G. The method of proof relies upon properties of Engel elements established by W. Kappe.

Among other results are the following: (a) $E(G)/Z_2(G)$ is an elementary Abelian 3-group of central automorphisms on the commutator subgroup G'. (b) If $Z(G) \cap \gamma_3(G)$ has no elements of order 3 or if G' is Černikov complete, then E(G) = $Z_2(G)$. (c) If [G:E(G)] = m is finite, then the verbal subgroup $e_2(G)$ is finite with order dividing a power of m.

1. Notation and assumed results. Let $\phi(x_1, \dots, x_n)$ be any word in the variables x_1, \dots, x_n . The verbal subgroup $\phi(G)$ is the subgroup of G generated by all elements of the form $\phi(a_1, \dots, a_n)$ with a_1, \dots, a_n in G. We say ϕ is a law in G, or that G is in the variety determined by ϕ , if $\phi(G) = 1$.

The associated marginal subgroup $\phi^*(G)$ of G consists of all a in G such that $\phi(g_1, \dots, ag_i, \dots, g_n) = \phi(g_1, \dots, g_i, \dots, g_n)$ for every g_i in $G, 1 \leq i \leq n$. We also refer to $\phi^*(G)$ as the ϕ -margin of G.

For x, y, a_i in G, define $[x, y] = x^{-1}y^{-1}xy = x^{-1}x^y$, $[a_1, \dots, a_n] = [[a_1, \dots, a_{n-1}], a_n]$, and [x, (n + 1)y] = [[x, ny], y]. Similarly, for subgroups H and K of G, [H, K] denotes the subgroup generated by all elements of the form [h, k], where $h \in H, k \in K$. We define [H, (n + 1)K] = [[H, nK], K]. If H_1, \dots, H_n are subgroups, then $[H_1, \dots, H_n] = [[H_1, \dots, H_{n-1}], H_n]$.

The word $\gamma_1 = d_0 = x$ is an outer commutator word of weight one. If $\theta = \theta(x_1, \dots, x_m)$, $\lambda = \lambda(y_1, \dots, y_n)$ are outer commutator words of weights m and n respectively, then $\phi = \phi(x_1, \dots, x_{m+n}) =$ $[\theta(x_1, \dots, x_m), \lambda(x_{m+1}, \dots, x_{m+n})]$ is an outer commutator word of weight m + n. We write $\phi = [\theta, \lambda]$. Particular examples are the derived (or solvable) words, defined by $d_n = [d_{n-1}, d_{n-1}]$, and the nilpotent (or lower central) words, defined by $\gamma_{n+1} = [\gamma_n, \gamma_1]$.

The following two theorems appear in [15]:

THEOREM 1.1. For any group G and word ϕ ,

(a) $\phi(G)$ is fully invariant in G and $\phi^*(G)$ is characteristic in G.

(b) $\phi(\phi^*(G)) = 1.$

(c) if $K/\phi^*(G)$ is the center of $G/\phi^*(G)$, then $[K, \phi(G)] = 1$. In particular, $[\phi^*(G), \phi(G)] = 1$.

(d) if H is a subgroup such that $G = H\phi^*(G)$, then $\phi^*(H) = H \cap \phi^*(G)$ and $\phi(G) = \phi(H)$.

THEOREM 1.2. Let θ and λ be two words in independent variables and $\phi = [\theta, \lambda]$. Then, in any group G,

(a) $\phi(G) = [\theta(G), \lambda(G)].$

(b) if $U = C_G(\theta(G))$, $V = C_G(\lambda(G))$, $L/U = \lambda^*(G/U)$, and $M/V = \theta^*(G/V)$, then $\phi^*(G) = L \cap M$.

An immediate result of Theorem 1.2(b) is that $\gamma_{n+1}^*(G) = Z_n(G)$, the *n*th center of *G*. It is this theorem which makes possible a classification of marginal subgroups for all outer commutator words, since the variables in θ and λ are independent of each other (see [16, p. 328]).

An element x of G is called a left (right) Engel element of G if for every y in G there is a positive integer n such that [y, nx] = 1([x, ny] = 1). The Engel word of length n is $e_n(x, y) = [x, ny]$. We note that Theorem 1.2(b) can not be used to determine $e_n^*(G)$, since $e_{n-1}(x, y)$ and y are not independent.

For *H* a subgroup of *G*, [*G*: *H*] is the index of *H* in *G*. If *H* is a proper (normal) subgroup of *G*, write $H < G(H \triangleleft G)$. If *G* is isomorphic to a subgroup of a group *K*, write $G \subseteq K$. $C_G(H)$ is the centralizer of *H* in *G*. For *x* in *G*, x^G denotes the subgroup generated by all conjugates of *x* in *G*.

2. The Engel margin. In this section "Engel word" will mean "Engel word of length two". We write $M(G) = d_z^*(G)$ and $E(G) = e_z^*(G)$ for the metabelian and Engel margins of G respectively.

Recall that $[Z_n(G), \gamma_m(G)] \subseteq Z_{n-m}(G)$ for all positive integers m and n.

LEMMA 2.1. In any group G,

(a) $d_n^*(G)/C_G(d_{n-1}(G)) = d_{n-1}^*(G/C_G(d_{n-1}(G)))$. In particular, $M(G) = \{a \in G \mid [[a, x], [y, z]] \text{ is a law in } G\}.$

(b) $Z_{n(n+1)/2}(G) \subseteq d_n^*(G)$. In particular, $Z_3(G) \subseteq M(G)$.

Proof. Part (a) follows from Theorem 1.2(b) with $\theta = \lambda = d_{n-1}$. We prove (b) by induction on n. For n = 1, $Z_1(G) \subseteq d_1^*(G) = Z(G)$. For n > 1, let $\overline{G} = G/C_G(d_{n-1}(G))$. Then

$$\overline{d_n^*(G)} = d_{n-1}^*(\overline{G}) \supseteq Z_{n(n-1)/2}(\overline{G})$$

by part (a) and the induction hypothesis. Furthermore,

 $[Z_{n(n+1)/2}(G), n(n-1)/2(G)] \subseteq Z_{n(n+1)/2-n(n-1)/2}(G) = Z_n(G)$

and $[Z_n(G), d_{n-1}(G)] \subseteq [Z_n(G), \gamma_n(G)] = 1$ so that

$$[Z_{n(n+1)/2}(G), n(n-1)/2(G)] \subseteq C_G(d_{n-1}(G))$$
.

Consequently,

$$\overline{Z_{n(n+1)/2}(G)} \subseteq Z_{n(n-1)/2}(\overline{G}) \subseteq \overline{d_n^*(G)}$$

and $Z_{n(n+1)/2}(G) \subseteq d_n^*(G)C_G(d_{n-1}(G)) = d_n^*(G)$, as desired.

We define $E_1(G) = \{a \in G \mid [ax, y, y] = [x, y, y] \text{ for all } x, y \in G\}$ and $L(G) = \{a \in G \mid [a, x, x] \text{ is a law in } G\}$ to be the subgroup of right Engel elements of length two. It is not difficult to show that $E(G) \subseteq E_1(G)$ and $E_1(G)$ is a characteristic subgroup of G.

The following properties of L(G) were established by W. Kappe in [6]:

LEMMA 2.2. In any group G, where $a \in L(G)$, $g, h, \in G$,

(a) L(G) is a characteristic subgroup of G.

- (b) $[a, g, h] = [a, h, g]^{-1}$.
- (c) $[a, [g, h]] = [a, g, h]^2$.
- (d) [a, g, [h, g]] = 1.
- (e) $a^4 \in Z_3(G)$.

THEOREM 2.3. In any group G,

(a) $Z_2(G) \subseteq E(G) \subseteq L(G)$.

(b) $E_1(G) = \{a \in G \mid [a, x] \in C_G(x^G) \text{ for all } x \in G\} = L(G).$

(c) $[a, x] \in C_{g}(x^{g}) \cap C_{g}(a)$ for all $a \in E_{1}(G)$, $x \in G$. Furthermore, $[a, x]^{rs} = [a^{r}, x^{s}]$ for all integers r and s.

- (d) a^{G} and $x^{L(G)}$ are Abelian for all a in L(G), x in G.
- (e) $E_1(G) \subseteq C_G((x^{\alpha})') \triangleleft G$ for all x in G.

Proof. Part (a) follows immediately from the definitions.

(b) Let $a \in E_1(G)$. Then [ay, x, x] = [y, x, x] for all x, y in G. This is equivalent to saying that $1 = [[ay, x][y, x]^{-1}, x] = [[a, x]^y \times [y, x][y, x]^{-1}, x] = [[a, x]^y, x]$ for all x, y in G. Since x and y are independent, we may conclude that a is in $E_1(G)$ if and only if $1 = [a, x, x^y]$ for all x, y in G or, equivalently, $[a, x] \in C_G(x^G)$ for all x.

That $E_1(G) \subseteq L(G)$ follows from $[a, x, x^y] = 1$ by letting y = 1. Finally, let $a \in L(G)$. We have for x, y in G that

$$[a, x, x^{y}] = [a, x, x [x, y]] = [a, x, [x, y]][a, x, x]^{[x,y]}$$

From the definition of L(G) we must have that [a, x, x] = 1. By Lemma 2.2(d) we also have that [a, x, [x, y]] = 1. Hence $[a, x, x^{y}] = 1$ and $a \in E_1(G)$.

(c) Since a is a right Engel element, we have that [a, x] is in $C_{a}(a)$ by [6, Lemma 2.1]. Part (b) says that $[a, x] \in C_{a}(x^{a})$ for all x in G. The remainder of part (c) follows from [13, Theorem 3.4.4].

(d) From part (c) we see that $a^x = a[a, x] \in C_a(a)$, since a and [a, x] are in $C_a(a)$. This implies that a^a is Abelian.

The proof that $x^{L(G)}$ is Abelian follows similarly from the observation that $x^a = x[x, a], [x, a] \in C_G(x^G) \subseteq C_G(x)$.

(e) By part (c) we may conclude that $[a, x^y] \in C_G((x^y)^G) = C_G(x^G)$ for all a in $E_1(G)$, x, y in G.

Let $a \in E_1(G)$. By Lemma 2.2(c), we have $[a, [x^w, x^z]] = [[a, x^w], x^z]^2 = 1$. This implies that $a \in C_G((x^G)')$.

THEOREM 2.4. In any group G, $E(G) = \{a \in G \mid [x, a, y] | x, y, a\} = 1$ for all x, y in G}.

Proof. Set $E_2(G) = \{a \in G \mid [x, ay, ay] = [x, y, y] \text{ for all } x, y \text{ in } G\}$. We see then that $E(G) = E_1(G) \cap E_2(G)$. Let S be the set described on the right in the statement of the theorem. Suppose $a \in S, x \in G$. Then 1 = [x, a, x][x, x, a] = [x, a, x]. This implies that $a \in E_1(G) = L(G)$. Since also $E(G) \subseteq E_1(G)$, it suffices to show that $E(G) \cap E_1(G) = E_1(G) \cap E_2(G) = E_1(G) \cap S$. Then, for x, y in $G, a \in E_1(G) \cap E_2(G)$ if and only if

$$\begin{split} [x, y, y] &= [x, ay, ay] \\ &= [x, ay, y][x, ay, a]^{y} \\ &= [[x, y][x, a]^{y}, y][[x, y][x, a]^{y}, a]^{y} \\ &= [x, y, y]^{[x,a]^{y}}[[x, a]^{y}, y][x, y, a]^{[x,a]^{y}y}[[x, a]^{y}, a]^{y} \,. \end{split}$$

By assumption, $[a, x] \in C_{d}(x^{d})$. Since $C_{d}(x^{d}) \triangleleft G$, we also have that $[a, x]^{y} \in C_{d}(x^{d})$. Consequently, conjugation by $[x, a]^{y}$ is irrelevant in the last statement above because all the commutators are in x^{d} . Therefore, the above is equivalent to

$$[x, y, y] = [x, y, y][[x, a]^{y}, y][x, y, a]^{y}[[x, a]^{y}, a]^{y}$$

or

$$1 = [x, a, y][x, y, a][[x, a]^{y}, a]$$

for all $x, y \in G, a \in E(G)$.

Now a and $[x, a]^{y}$ are elements of a^{c} . By Theorem 2.3(d), a^{c} is Abelian. This implies that $[[x, a]^{y}, a] = 1$. Therefore, E(G) is contained in the set S.

We have already shown that S is a subset of $E_1(G) = L(G)$. Consequently, all the above arguments are reversible and we may conclude that S = E(G). LEMMA 2.5. (a) $E(G) \cap C_G(G') = Z_2(G)$. (b) [x, a, y] = [a, y, x] for all x, y in G, a in L(G).

Proof. (a) We need only verify that $E(G) \cap C_{c}(G') \subseteq Z_{2}(G)$ by Theorem 2.3(a) and the remark before Lemma 2.1. Let $a \in E(G) \cap C_{c}(G')$. By Theorem 2.4, 1 = [x, a, y][x, y, a] for all x, y in G. But $a \in C_{c}(G')$ implies that [x, y, a] = 1 and thus that [x, a, y] = 1 for all x, y in G. Hence $a \in Z_{2}(G)$.

(b) $[a, y, x] = [a, x, y]^{-1}$ by Lemma 2.2(b), $= [[x, a]^{-1}, y]^{-1} = (([x, a, y]^{-1})^{-1})^{[a,x]} = [x, a, y]$ since $[a, x] \in C_{G}(x^{G})$ by Theorem 2.3(c).

From Theorem 2.4 and Lemma 2.5(b) we have our characterization of E(G):

THEOREM 2.6. For any group $G, E(G) = \{a \in G \mid [x, y, a] | a, y, x\}$ is a law in $G\}$.

COROLLARY 2.7. For any $a \in E(G)$, $[a, G, G]^3 = [a^3, G, G] = 1$.

Proof. Let $x, y \in G$. By Theorem 2.6, [x, y, a][a, y, x] = 1. Then $[x, y, a] = [a, [x, y]]^{-1} = ([a, x, y]^2)^{-1}$ by Lemma 2.2(c), $= [a, y, x]^2$ by Lemma 2.2(b). Hence $1 = [x, y, a][a, y, x] = [a, y, x]^2[a, y, x] = [a, y, x]^3$.

By Theorem 2.3(d) we have that a^{σ} is Abelian. Hence $[a, x, y]^3 = 1$ for all $x, y \in G$ implies [a, G, G] has exponent dividing three, and $[a, x, y]^3 = [a^3, x, y] = 1$.

COROLLARY 2.8. For any group G, $E(G) \subseteq Z_{\mathfrak{z}}(G) \subseteq M(G)$.

Proof. Let $a \in E(G)$. By Lemma 2.2(e) we have that $a^4 \in Z_3(G)$. Since also $a^3 \in Z_2(G) \subseteq Z_3(G)$ by Corollary 2.7, it follows that $a \in Z_3(G)$.

We recall a theorem of F. W. Levi (see [12]): If e_2 is a law in a group G, then G is nilpotent of class at most three and $\gamma_3(G)$ has exponent dividing three. This, together with Theorem 1.1(b), yields the first statement in the following:

THEOREM 2.9. E(G) is nilpotent of class no greater then three and metabelian, and $\gamma_{s}(E(G))$ has exponent dividing three. If $C_{G}(G') \subseteq E(G)$, then $M(G) = Z_{s}(G)$.

Proof. Suppose $C_G(G') \subseteq E(G)$. By Lemma 2.5(a) this implies that $C_G(G') = Z_2(G)$. From Lemma 2.1(a), $M(G)/C_G(G') = Z(G/C_G(G'))$. Hence $M(G) = Z_3(G)$.

THEOREM 2.10. Let G be a group, M = M(G), $E_1 = E_1(G) = L(G)$. Then

(a) $[G', M, E_1] = [G', E_1, M] = [M, G, G'] = 1.$ (b) $[G, M', E_1] = [M', E_1, G] = [G', M'] = 1.$ In particular, $[M', E_1] \subseteq Z(G).$

Proof. (a) By Lemma 2.1(a), $[M, G] \subseteq C_{d}(G') \cap G' = Z(G')$ so that 1 = [M, G, G']. Now let $a \in E_{1}, m \in M, x \in G'$. By Lemma 2.2(c), $[a, [m, x]] = [a, m, x]^{2} = 1$. This implies $[G', M, E_{1}] = 1$. Consequently $[G', E_{1}, M] = 1$ by [13, Theorem 3.4.8(i)].

(b) As in the proof of part (a), we have $M' \subseteq Z(G')$ so that 1 = [G', M']. Let $a \in E_1, x \in M', g \in G$. Then $[a, [g, x]] = [a, g, x]^2 = 1$. Hence $[M', G, E_1] = 1$ and, as above, $[M', E_1, G] = 1$.

3. Central automorphisms on G'. It follows from Theorem 2.10(a) that $[M(G), G'] \subseteq Z(G')$. This implies that $M(G)/C_G(G')$ acts as an Abelian group of central automorphisms on G'. Then

$$(E_1(G) \cap M(G))/(E_1(G) \cap C_G(G')) \subseteq M(G)/C_G(G')$$

is also such a group. Denote the corresponding group of automorphisms on G' by \mathfrak{A}_2 . Furthermore,

$$E(G)/Z_2(G) = (E(G) \cap M(G))/(E(G) \cap C_G(G')) \subseteq \mathfrak{A}_2$$

by Lemma 2.5(a) and Corollary 2.8. Let $\mathfrak{A}_1 \subseteq \mathfrak{A}_2$ denote the corresponding group of automorphisms. From Corollary 2.7 we see that $E(G)/Z_2(G)$ has exponent 3. Hence \mathfrak{A}_1 is an elementary Abelian 3-group of central automorphisms on G'.

THEOREM 3.1. (a) If the exponent Exp(Z(G')) = n is finite, then $\text{Exp}(\mathfrak{A}_2)$ divides n.

(b) If G' is a p-group, $\mathfrak{A} \subseteq \mathfrak{A}_2$ is periodic, then \mathfrak{A} is a p-group. (c) Assume G' is polycyclic; that is, G' has a finite ascending normal series with cyclic factors. Then $E(G)/Z_2(G)$ is finite.

Proof. (a) Suppose Z(G') has exponent *n*. Then, for $x \in G'$, $a \in \mathfrak{A}_2$, $1 = [x, a]^n = [x, a^n]$ by Theorem 2.3(c). Consequently, $a^n = 1$ and \mathfrak{A}_2 has exponent dividing *n*.

(b) Now assume \mathfrak{A} is periodic. By Theorem 2.10(a) we may conclude that $[G', M(G), E_1(G)] = [G', \mathfrak{A}, \mathfrak{A}] = 1$. Thus \mathfrak{A} stabilizes the normal series $1 \triangleleft [G', \mathfrak{A}] \triangleleft G'$ of G'. By [1, Corollary 5.3.3] we have that \mathfrak{A} is a *p*-group.

(c) Smirnov [14] has shown that a solvable group of automorphisms of a polycyclic group is polycyclic. Since then \mathfrak{A}_1 is finitely generated, it must be finite.

THEOREM 3.2. If $\mathfrak{A}_2 \neq 1$ is not torsionfree, then G' has a proper subgroup of finite index and Z(G') is not torsionfree.

Proof. For $1 \neq \alpha \in \mathfrak{A}_2$, the homomorphism from G' into Z(G')defined by $f_{\alpha}(x) = [x, \alpha]$ for each x in G' is nontrivial. We choose $a \in E_1(G) \cap M(G) \setminus E_1(G) \cap C_G(G')$ such that $[x, \alpha] = [x, \alpha]$ for all x in G'. If α has finite order, then there is an integer n such that $a^n \in C_G(G')$. Thus $1 = [x, \alpha]^n = [x, a^n]$ and $G'/\operatorname{Ker} f_{\alpha} \subseteq Z(G')$ is a nontrivial direct sum of cyclic groups each of order bounded by n. In particular, there are subgroups H and C of G' such that $G'/\operatorname{Ker} f_{\alpha} = H/\operatorname{Ker} f_{\alpha} + C/\operatorname{Ker} f_{\alpha}$ and $C/\operatorname{Ker} f_{\alpha}$ is nontrivial and finite. Consequently H < G'and $G'/H \cong C/\operatorname{Ker} f_{\alpha}$ is finite.

Let $1 \neq \alpha \in \mathfrak{A}_2$, $o(\alpha) = n < \infty$. Then there is an $x \in G'$ such that $1 \neq [x, \alpha] \in Z(G')$. But $[x, \alpha]^n = [x, \alpha^n] = 1$ so that the order of $[x, \alpha]$ divides n.

COROLLARY 3.3. If $E(G) > Z_2(G)$, then G' has a proper subgroup of finite index.

Proof. If $E(G) > Z_2(G)$, then \mathfrak{A}_1 is a nontrivial torsion subgroup of \mathfrak{A}_2 . Hence $\mathfrak{A}_2 \neq 1$ is not torsionfree and the theorem applies.

It is known that no complete, or even Cernikov complete, group can have a proper subgroup of finite index (see [7, p. 234]). From this fact we derive part of the following:

COROLLARY 3.4. If G' is Cernikov complete, or if $Z(G) \cap \gamma_3(G)$ has no elements of order three, then $E(G) = Z_2(G)$.

Proof. We shall show that $\mathfrak{A}_1 = 1$. By Corollary 2.8, $E(G) \subseteq Z_{\mathfrak{z}}(G)$. Hence $[G', E(G)] = [G', \mathfrak{A}_1] \subseteq Z(G) \cap \gamma_{\mathfrak{z}}(G)$.

Let $a \in \mathfrak{A}_1$, $x \in G'$. Then, by Corollary 2.7 and Theorem 2.3(c), $1 = [x, a^3] = [x, a]^3$. By hypothesis, this implies that 1 = [x, a]. Consequently a = 1.

EXAMPLE 3.5. We now construct a group G such that $Z_2(G) < E(G) < Z_3(G)$.

Let $H = \langle a_1, a_2, a_3; x^3 \rangle$. Levi and van der Waerden [8] have shown that H has nilpotence class exactly three and is in the variety determined by e_2 . Hence $E(H) = H = Z_3(H) > Z_2(H)$. Let K be any group of nilpotence class at least three having no elements of order three (see for example [12, p. 198]). By Corollary 3.4, $E(K) = Z_2(K) < Z_3(K) \subseteq K$. Letting $G = H \times K$, we see that $E(G) = E(H) \times E(K) =$ $H \times Z_2(K)$. Hence $Z_2(G) < E(G) < Z_3(G)$. REMARK 3.6. Define $N_{A}(G) = \bigcap \{N_{G}(H) \mid H \text{ maximal Abelian sub$ $group of } G\}$ to be the A-Norm of G. Kappe [6] has shown that $a \in N_{A}(G)$ if and only if [g, h] = 1 for g, h in G implies that [a, g, h] = 1. From Theorem 2.6 it follows immediately that $E(G) \subseteq N_{A}(G) \subseteq E_{1}(G)$.

4. Finiteness conditions. We shall say that a word ϕ satisfies the Schur-Baer property if $[G: \phi^*(G)] = m$ finite implies $\phi(G)$ finite with order which divides a power of m for all groups G.

Schur showed that γ_2 satisfies the Schur-Baer property; Baer extended this result to any outer commutator word ϕ (see [15]).

Recall that a group G is residually finite if for every x in G, $x \neq 1$, there is a normal subgroup N_x of G such that $x \notin N_x$ and G/N_x is finite. A group is locally residually finite if every finitely generated subgroup is residually finite.

We shall need the following theorem. For a proof (due to P. Hall), see [15, Theorem 2].

THEOREM 4.1. If ϕ generates a locally residually finite variety, then ϕ satisfies the Schur-Baer property.

THEOREM 4.2. If $\phi \in \{e_2, e_3\}$, then ϕ satisfies the Schur-Baer property.

Proof. Suppose $\phi = e_2$. A group in the variety generated by ϕ is nilpotent by Levi's Theorem. A finitely generated nilpotent group is residually finite by P. Hall [4]. Therefore, a finitely generated group in the variety generated by ϕ is residually finite and Theorem 4.1 applies.

Let $\phi = e_3$. Heineken [5] has shown that a group in the variety generated by ϕ is locally nilpotent. Hence a finitely generated group in this variety is also residually finite and the theorem follows as above.

Recall that a group is an SN^* group if it possesses an ascending normal series with Abelian factors (see [7]). Also, the unique maximum locally nilpotent normal subgroup of a group is called its Hirsch-Plotkin radical (see [12]).

We note that in P. Hall's proof of Theorem 4.1 that we may extend the result somewhat if we put some restrictions on G itself. That is, if $\phi^*(G)$ is locally residually finite for all G in some quotientand subgroup-closed class Σ , then ϕ satisfies the Schur-Baer property for all G in Σ .

THEOREM 4.3. If G satisfies the maximum or the minimum condition, or if G is an SN^* group, then e_n satisfies the Schur-Baer property for G.

Proof. Suppose G satisfies the maximum condition. Then, by [12, Theorem VI. 8. j], we have that the set of left Engel elements (of all lengths) is the Hirsch-Plotkin radical R. Since then $e_n^*(G) \subseteq R$ is locally nilpotent, it is locally residually finite. By the preceding remark, we have that e_n satisfies the Schur-Baer property for G.

Vilyacer [18] has shown that an Engel group satisfying the minimum condition is locally nilpotent. Plotkin [11] has proved that an Engel group which is also an SN^* group is locally nilpotent. Hence the remainder of the theorem follows as above.

The validity of the Schur-Baer property in general is one of several conjectures which have been proposed for the group functions ϕ and ϕ^* (see [9] and [16]). Modified solutions of two of these come from the following lemma.

LEMMA 4.4. Suppose G is in a class of groups in which the Schur-Baer property is satisfied locally for ϕ . If G is locally residually finite and ϕ is finite-valued on G, then $\phi(G)$ is finite.

Proof. This follows from the arguments used in the proofs of Proposition 1 and its two corollaries in [17].

We note in particular in these proofs that there is a finitely generated subgroup H of G such that $\phi(H) = \phi(G)$. It follows that $H/\phi^*(H)$ is finite. Since H and ϕ satisfy the Schur-Baer property, $\phi(H) = \phi(G)$ is finite.

The following two theorems are immediate from these observations.

THEOREM 4.5. If $\phi \in \{e_2, e_3\}$, G is locally residually finite, and ϕ is finite-valued on G, then $\phi(G)$ is finite.

THEOREM 4.6. If $\phi \in \{e_2, e_3\}$, ϕ is finite-valued on G, and G is finitely generated and residually finite, then $G/\phi^*(G)$ is finite.

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