ON THE ENGEL MARGIN

TOMMY KAY TEAGUE
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T. K. TEAGUE

The marginal subgroup for any outer commutator word has been characterized by R. F. Turner-Smith. This paper considers the marginal subgroup $E(G)$ of $G$ for the Engel word $e_2(x, y) = [x, y, y]$ of length two. The principal result is that an element $a$ of $G$ is in $E(G)$ if and only if $[x, y, a][a, y, x]$ is a law in $G$. The method of proof relies upon properties of Engel elements established by W. Kappe.

Among other results are the following: (a) $E(G)/Z_2(G)$ is an elementary Abelian 3-group of central automorphisms on the commutator subgroup $G_f$. (b) If $Z(G) < \gamma_3(G)$ has no elements of order 3 or if $G_f$ is Černikov complete, then $E(G) = Z_2(G)$. (c) If $[G:E(G)] = m$ is finite, then the verbal subgroup $e_2(G)$ is finite with order dividing a power of $m$.

1. Notation and assumed results. Let $\phi(x_1, \ldots, x_n)$ be any word in the variables $x_1, \ldots, x_n$. The verbal subgroup $\phi(G)$ is the subgroup of $G$ generated by all elements of the form $\phi(a_1, \ldots, a_n)$ with $a_1, \ldots, a_n$ in $G$. We say $\phi$ is a law in $G$, or that $G$ is in the variety determined by $\phi$, if $\phi(G) = 1$.

The associated marginal subgroup $\phi^*(G)$ of $G$ consists of all $a$ in $G$ such that $\phi(g_1, \ldots, g_n) = \phi(g_i, \ldots, g_i, \ldots, g_n)$ for every $g_i$ in $G$, $1 \leq i \leq n$. We also refer to $\phi^*(G)$ as the $\phi$-margin of $G$.

For $x, y, a_i$ in $G$, define $[x, y] = x^{-1}y^{-1}xy = x^{-1}x^y$, $[a_i, \ldots, a_n] = [[a_i, \ldots, a_{n-1}], a_n]$, and $[x, (n+1)y] = [[x, ny], y]$. Similarly, for subgroups $H$ and $K$ of $G$, $[H, K]$ denotes the subgroup generated by all elements of the form $[h, k]$, where $h \in H, k \in K$. We define $[H, (n+1)K] = [[H, nK], K]$. If $H_1, \ldots, H_n$ are subgroups, then $[H_1, \ldots, H_n] = [[H_1, \ldots, H_{n-1}], H_n]$.

The word $\gamma_1 = d_0 = x$ is an outer commutator word of weight one. If $\theta = \theta(x_1, \ldots, x_m), \lambda = \lambda(y_1, \ldots, y_n)$ are outer commutator words of weights $m$ and $n$ respectively, then $\phi = \phi(x_1, \ldots, x_{m+n}) = [\theta(x_1, \ldots, x_m), \lambda(x_{m+1}, \ldots, x_{m+n})]$ is an outer commutator word of weight $m + n$. We write $\phi = [\theta, \lambda]$. Particular examples are the derived (or solvable) words, defined by $d_n = [d_{n-1}, d_{n-1}]$, and the nilpotent (or lower central) words, defined by $\gamma_{n+1} = [\gamma_n, \gamma_1]$.

The following two theorems appear in [15]:

**Theorem 1.1.** For any group $G$ and word $\phi$,
(a) $\phi(G)$ is fully invariant in $G$ and $\phi^*(G)$ is characteristic in $G$.
(b) $\phi(\phi^*(G)) = 1$. 

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(c) If $K/\phi^*(G)$ is the center of $G/\phi^*(G)$, then $[K, \phi(G)] = 1$. In particular, $[\phi^*(G), \phi(G)] = 1$.

(d) If $H$ is a subgroup such that $G = H\phi^*(G)$, then $\phi^*(H) = H \cap \phi^*(G)$ and $\phi(G) = \phi(H)$.

**Theorem 1.2.** Let $\theta$ and $\lambda$ be two words in independent variables and $\phi = [\theta, \lambda]$. Then, in any group $G$,

(a) $\phi(G) = [\theta(G), \lambda(G)]$.

(b) If $U = C_G(\theta(G))$, $V = C_G(\lambda(G))$, $L/U = \gamma^*(G/U)$, and $M/V = \theta^*(G/V)$, then $\phi^*(G) = L \cap M$.

An immediate result of Theorem 1.2(b) is that $\gamma^*_{n+1}(G) = Z_n(G)$, the $n$th center of $G$. It is this theorem which makes possible a classification of marginal subgroups for all outer commutator words, since the variables in $\theta$ and $\lambda$ are independent of each other (see [16, p. 328]).

An element $x$ of $G$ is called a left (right) Engel element of $G$ if for every $y$ in $G$ there is a positive integer $n$ such that $[y, nx] = 1$ ($[x, ny] = 1$). The Engel word of length $n$ is $e_n(x, y) = [x, ny]$. We note that Theorem 1.2(b) can not be used to determine $e_n^*(G)$, since $e_{n-1}(x, y)$ and $y$ are not independent.

For $H$ a subgroup of $G$, $[G: H]$ is the index of $H$ in $G$. If $H$ is a proper (normal) subgroup of $G$, write $H < G(H \triangleleft G)$. If $G$ is isomorphic to a subgroup of a group $K$, write $G \cong K$. $C_0(H)$ is the centralizer of $H$ in $G$. For $x$ in $G$, $x^G$ denotes the subgroup generated by all conjugates of $x$ in $G$.

2. The Engel margin. In this section “Engel word” will mean “Engel word of length two”. We write $M(G) = d^*_s(G)$ and $E(G) = e^*_2(G)$ for the metabelian and Engel margins of $G$ respectively.

Recall that $[Z_n(G), \gamma_m(G)] \subseteq Z_{n-m}(G)$ for all positive integers $m$ and $n$.

**Lemma 2.1.** In any group $G$,

(a) $d^*_s(G)/C_0(d_{n-1}(G)) = d^*_s(G/C_0(d_{n-1}(G)))$. In particular, $M(G) = \{a \in G \mid [[a, x], [y, z]]$ is a law in $G\}$.

(b) $Z_{n(n+1)/2}(G) \subseteq d^*_n(G)$. In particular, $Z_n(G) \subseteq M(G)$.

**Proof.** Part (a) follows from Theorem 1.2(b) with $\theta = \lambda = d_{n-1}$. We prove (b) by induction on $n$. For $n = 1$, $Z_1(G) \subseteq d^*_1(G) = Z(G)$. For $n > 1$, let $\overline{G} = G/C_0(d_{n-1}(G))$. Then

$$\overline{d^*_n(G)} = \overline{d^*_n(\overline{G})} \supseteq Z_{n(n-1)/2}(\overline{G})$$

by part (a) and the induction hypothesis. Furthermore,
\[ Z_{n(n+1)/2}(G), n(n - 1)/2(G) \subseteq Z_{n(n+1)/2-n(n-1)/d(G)} = Z_n(G) \]

and \( [Z_n(G), d_{n-1}(G)] \subseteq [Z_n(G), \gamma_n(G)] = 1 \) so that

\[ [Z_{n(n+1)/2}(G), n(n - 1)/2(G)] \subseteq C_\gamma(d_{n-1}(G)). \]

Consequently,

\[ \overline{Z_{n(n+1)/2}(G)} \subseteq Z_{n(n-1)/2}(G) \subseteq \overline{d_{n-1}(G)} \]

and \( Z_{n(n+1)/2}(G) \subseteq d_{n}(G)C_\gamma(d_{n-1}(G)) = \overline{d_{n-1}(G)} \), as desired.

We define \( E^r_i(G) = \{ a \in G \mid [ax, y, y] = [x, y, y] \text{ for all } x, y \in G \} \) and \( L(G) = \{ a \in G \mid [a, x, x] \text{ is a law in } G \} \) to be the subgroup of right Engel elements of length two. It is not difficult to show that \( E(G) \subseteq E_i(G) \) and \( E_i(G) \) is a characteristic subgroup of \( G \).

The following properties of \( L(G) \) were established by W. Kappe in [6]:

**Lemma 2.2.** In any group \( G \), where \( a \in L(G), g, h, e G \),

(a) \( L(G) \) is a characteristic subgroup of \( G \).

(b) \([a, g, h] = [a, h, g]^{-1} \).

(c) \([a, [g, h]] = [a, g, h]^2 \).

(d) \([a, g, [h, g]] = 1 \).

(e) \( a^4 \in Z_n(G) \).

**Theorem 2.3.** In any group \( G \),

(a) \( Z_2(G) \subseteq E(G) \subseteq L(G) \).

(b) \( E^r_i(G) = \{ a \in G \mid [ax, x] \in C_\gamma(x^a) \text{ for all } x \in G \} = L(G) \).

(c) \([a, x] \in C_\gamma(x^a) \cap C_\gamma(a) \text{ for all } a \in E^r_i(G), x \in G. \) Furthermore, \( [a, x]^r = [a^r, x^r] \) for all integers \( r \) and \( s \).

(d) \( a^g \) and \( x^{\gamma(G)} \) are Abelian for all \( a \) in \( L(G) \), \( x \) in \( G \).

(e) \( E^r_i(G) \subseteq C_\gamma(x^{\gamma(G)}) \) for all \( x \) in \( G \).

**Proof.** Part (a) follows immediately from the definitions.

(b) Let \( a \in E^r_i(G) \). Then \( [ay, x, x] = [y, x, x] \) for all \( x, y \) in \( G \). This is equivalent to saying that \( 1 = [[ay, x][y, x]^{-1}, x] = [[a, x]^y \times [y, x][y, x]^{-1}, x] = [[a, x]^y, x] \) for all \( x, y \) in \( G \). Since \( x \) and \( y \) are independent, we may conclude that \( a \) is in \( E^r_i(G) \) if and only if \( 1 = [a, x, x^y] \) for all \( x, y \) in \( G \) or, equivalently, \( [a, x] \in C_\gamma(x^a) \) for all \( x \).

That \( E^r_i(G) \subseteq L(G) \) follows from \([a, x, x^y] = 1 \) by letting \( y = 1 \). Finally, let \( a \in L(G) \). We have for \( x, y \) in \( G \) that

\[ [a, x, x^y] = [a, x, x[x, y]] = [a, x, [x, y]][a, x, x[^x,y]. \]

From the definition of \( L(G) \) we must have that \([a, x, x] = 1 \). By Lemma 2.2(d) we also have that \([a, x, [x, y]] = 1 \). Hence \([a, x, x^y] = 1 \) and \( a \in E^r_i(G) \).
(c) Since \( a \) is a right Engel element, we have that \([a, x]\) is in \( C_0(a) \) by [6, Lemma 2.1]. Part (b) says that \([a, x] \in C_0(x^a)\) for all \( x \) in \( G \). The remainder of part (c) follows from [13, Theorem 3.4.4].

(d) From part (c) we see that \( a^x = a[a, x] \in C_0(a) \), since \( a \) and \([a, x]\) are in \( C_0(a) \). This implies that \( a^x \) is Abelian.

The proof that \( x^{L(G)} \) is Abelian follows similarly from the observation that \( a^x = x[x, a] \), \([x, a] \in C_0(x^a) \subseteq C_0(x) \).

(e) By part (c) we may conclude that \([a, x^y] \in C_0((x^a)^y) = C_0(x^a)\) for all \( a \) in \( E_3(G) \), \( x, y \) in \( G \).

Let \( a \in E_1(G) \). By Lemma 2.2(c), we have \([a, [x^y, x^a]] = [[a, x^y], x^a]^y = 1\). This implies that \( a \in C_0((x^a)^y) \).

**Theorem 2.4.** In any group \( G \), \( E(G) = \{a \in G \mid [x, a, y][x, y, a] = 1 \text{ for all } x, y \text{ in } G\} \).

**Proof.** Set \( E_3(G) = \{a \in G \mid [x, ay, ay] = [x, y, y] \text{ for all } x, y \text{ in } G\} \). We see then that \( E(G) = E_3(G) \cap E_4(G) \). Let \( S \) be the set described on the right in the statement of the theorem. Suppose \( a \in S \), \( x \in G \). Then \( 1 = [x, a, x][x, x, a] = [x, a, x] \). This implies that \( a \in E_1(G) = L(G) \).

Since also \( E(G) \subseteq E_1(G) \), it suffices to show that \( E(G) \cap E_1(G) = E_3(G) \cap E_4(G) = E_1(G) \cap S \). Then, for \( x, y \) in \( G \), \( a \in E_3(G) \cap E_4(G) \) if and only if

\[
[x, y, y] = [x, ay, ay] = [x, ay, y][x, ay, a]^y = [[x, y][x, a]^y, y][[x, y][x, a]^y, a]^y = [x, y, y][x, a]^y[[x, a]^y, y][x, y, a][x, a]^y[[x, a]^y, a]^y.
\]

By assumption, \([a, x] \in C_0(x^a) \). Since \( C_0(x^a) < G \), we also have that \([a, x]^y \in C_0(x^a) \). Consequently, conjugation by \([x, a]^y \) is irrelevant in the last statement above because all the commutators are in \( x^a \).

Therefore, the above is equivalent to

\[
[x, y, y] = [x, y, y][[x, a]^y, y][x, y, a][x, a]^y[[x, a]^y, a]^y
\]

or

\[
1 = [x, a, y][x, y, a][[x, a]^y, a]
\]

for all \( x, y \in G \), \( a \in E(G) \).

Now \( a \) and \([x, a]^y \) are elements of \( a^x \). By Theorem 2.3(d), \( a^x \) is Abelian. This implies that \([[[x, a]^y, a] = 1 \). Therefore, \( E(G) \) is contained in the set \( S \).

We have already shown that \( S \) is a subset of \( E_1(G) = L(G) \). Consequently, all the above arguments are reversible and we may conclude that \( S = E(G) \).
Lemma 2.5. (a) \( E(G) \cap C_0(G') = Z_3(G) \).
(b) \([x, a, y] = [a, y, x]\) for all \( x, y \) in \( G \), \( a \) in \( L(G) \).

Proof. (a) We need only verify that \( E(G) \cap C_0(G') \subseteq Z_3(G) \) by Theorem 2.3(a) and the remark before Lemma 2.1. Let \( a \in E(G) \cap C_0(G') \). By Theorem 2.4, \( 1 = [x, a, y][x, y, a] \) for all \( x, y \) in \( G \). But \( a \in C_0(G') \) implies that \( [x, y, a] = 1 \) and thus that \( [x, a, y] = 1 \) for all \( x, y \) in \( G \). Hence \( a \in Z_3(G) \).

(b) \([a, y, x] = [a, x, y]^{-1}\) by Lemma 2.2(b), \( = [[x, a]^{-1}, y]^{-1} = (([x, a, y]^{-1})^{-1})^{[a, x, y]} = [a, x, y] \) since \([a, x] \in C_0(x^a) \) by Theorem 2.3(c).

From Theorem 2.4 and Lemma 2.5(b) we have our characterization of \( E(G) \):

Theorem 2.6. For any group \( G \), \( E(G) = \{a \in G \mid [x, y, a][a, y, x] \) is a law in \( G \} \).

Corollary 2.7. For any \( a \in E(G) \), \([a, G, G]^3 = [a^3, G, G] = 1\).

Proof. Let \( x, y \in G \). By Theorem 2.6, \([x, y, a][a, y, x] = 1\). Then \([x, y, a] = [a, x, y]^{-1} = ([a, x, y]^3]^{-1}\) by Lemma 2.2(c), \( = [a, y, x]^3\) by Lemma 2.2(b). Hence \( 1 = [x, y, a][a, y, x] = [a, y, x]^3[a, x, y] = [a, y, x]^3\).

By Theorem 2.3(d) we have that \( a^3 \) is Abelian. Hence \([a, x, y]^3 = 1\) for all \( x, y \in G \) implies \([a, G, G] \) has exponent dividing three, and \([a, x, y]^3 = [a^3, x, y] = 1\).

Corollary 2.8. For any group \( G \), \( E(G) \subseteq Z_3(G) \subseteq M(G) \).

Proof. Let \( a \in E(G) \). By Lemma 2.2(e) we have that \( a^3 \in Z_3(G) \).

Since also \( a^3 \in Z_3(G) \subseteq Z_3(G) \) by Corollary 2.7, it follows that \( a \in Z_3(G) \).

We recall a theorem of F. W. Levi (see [12]): If \( e_2 \) is a law in a group \( G \), then \( G \) is nilpotent of class at most three and \( \gamma_3(G) \) has exponent dividing three. This, together with Theorem 1.1(b), yields the first statement in the following:

Theorem 2.9. \( E(G) \) is nilpotent of class no greater then three and metabelian, and \( \gamma_3(E(G)) \) has exponent dividing three. If \( C_0(G') \subseteq E(G) \), then \( M(G) = Z_3(G) \).

Proof. Suppose \( C_0(G') \subseteq E(G) \). By Lemma 2.5(a) this implies that \( C_0(G') = Z_3(G) \). From Lemma 2.1(a), \( M(G)/C_0(G') = Z(G/C_0(G')) \).

Hence \( M(G) = Z_3(G) \).
THEOREM 2.10. Let $G$ be a group, $M = M(G)$, $E_{1} = E_{1}(G) = L(G)$. Then

(a) $[G', M, E_{1}] = [G', E_{1}, M] = [M, G, G'] = 1$.

(b) $[G, M', E_{1}] = [M', E_{1}, G] = [G', M'] = 1$. In particular, $[M', E_{1}] \subseteq Z(G)$.

Proof. (a) By Lemma 2.1(a), $[M, G] \subseteq C_{G}(G') \cap G' = Z(G')$ so that $1 = [M, G, G']$. Now let $a \in E_{1}$, $m \in M$, $x \in G'$. By Lemma 2.2(c), $[[a, m], x] = [a, m, x]^{x} = 1$. This implies $[G', M, E_{1}] = 1$. Consequently $[G', E_{1}, M] = 1$ by [13, Theorem 3.4.8(i)].

(b) As in the proof of part (a), we have $M' \subseteq Z(G')$ so that $1 = [G', M']$. Let $a \in E_{1}$, $x \in M'$, $g \in G$. Then $[a, [g, x]] = [a, g, x]^{x} = 1$. Hence $[M', G, E_{1}] = 1$ and, as above, $[M', E_{1}, G] = 1$.

3. Central automorphisms on $G'$. It follows from Theorem 2.10(a) that $[M(G), G'] \subseteq Z(G')$. This implies that $M(G)/C_{G}(G')$ acts as an Abelian group of central automorphisms on $G'$. Then

$$(E_{1}(G) \cap M(G))/(E_{1}(G) \cap C_{G}(G')) \subseteq M(G)/C_{G}(G')$$

is also such a group. Denote the corresponding group of automorphisms on $G'$ by $\mathcal{A}_{2}$. Furthermore,

$$E(G)/Z_{2}(G) = (E(G) \cap M(G))/(E(G) \cap C_{G}(G')) \subseteq \mathcal{A}_{2}$$

by Lemma 2.5(a) and Corollary 2.8. Let $\mathcal{A}_{1} \subseteq \mathcal{A}_{2}$ denote the corresponding group of automorphisms. From Corollary 2.7 we see that $E(G)/Z_{2}(G)$ has exponent 3. Hence $\mathcal{A}_{1}$ is an elementary Abelian 3-group of central automorphisms on $G'$.

THEOREM 3.1. (a) If the exponent $\text{Exp}(Z(G')) = n$ is finite, then $\text{Exp}(\mathcal{A}_{2})$ divides $n$.

(b) If $G'$ is a $p$-group, $\mathcal{A} \subseteq \mathcal{A}_{2}$ is periodic, then $\mathcal{A}$ is a $p$-group.

(c) Assume $G'$ is polycyclic; that is, $G'$ has a finite ascending normal series with cyclic factors. Then $E(G)/Z_{2}(G)$ is finite.

Proof. (a) Suppose $Z(G')$ has exponent $n$. Then, for $x \in G'$, $a \in \mathcal{A}_{2}$, $1 = [x, a]^{x} = [x, a^{x}]$ by Theorem 2.3(c). Consequently, $a^{x} = 1$ and $\mathcal{A}_{2}$ has exponent dividing $n$.

(b) Now assume $\mathcal{A}$ is periodic. By Theorem 2.10(a) we may conclude that $[G', M(G), E_{1}(G)] = [G', \mathcal{A}, \mathcal{A}] = 1$. Thus $\mathcal{A}$ stabilizes the normal series $1 < [G', \mathcal{A}] < G'$ of $G'$. By [1, Corollary 5.3.3] we have that $\mathcal{A}$ is a $p$-group.

(c) Smirnov [14] has shown that a solvable group of automorphisms of a polycyclic group is polycyclic. Since then $\mathcal{A}_{1}$ is finitely generated, it must be finite.
Theorem 3.2. If \( \mathfrak{A}_2 \neq 1 \) is not torsionfree, then \( G' \) has a proper subgroup of finite index and \( Z(G') \) is not torsionfree.

**Proof.** For \( 1 \neq \alpha \in \mathfrak{A}_2 \), the homomorphism from \( G' \) into \( Z(G') \) defined by \( f_\alpha(x) = [x, \alpha] \) for each \( x \) in \( G' \) is nontrivial. We choose \( \alpha \in E_2(G) \cap M(G) \cap E_2(G) \cap C_0(G') \) such that \( [x, \alpha] = [x, \alpha] \) for all \( x \) in \( G' \). If \( \alpha \) has finite order, then there is an integer \( n \) such that \( \alpha^n \in C_0(G') \). Thus \( 1 = [x, \alpha]^n = [x, \alpha^n] \) and \( G'/\text{Ker}_a \subseteq Z(G') \) is a nontrivial direct sum of cyclic groups each of order bounded by \( n \). In particular, there are subgroups \( H \) and \( C \) of \( G' \) such that \( G'/\text{Ker}_a = H/\text{Ker}_a + C/\text{Ker}_a \) and \( C/\text{Ker}_a \) is nontrivial and finite. Consequently \( H < G' \) and \( G'/\text{Ker}_a \cong C/\text{Ker}_a \) is finite.

Let \( 1 \neq \alpha \in \mathfrak{A}_2 \), \( o(\alpha) = n < \infty \). Then there is an \( x \in G' \) such that \( 1 \neq [x, \alpha] \in Z(G') \). But \( [x, \alpha]^n = [x, \alpha^n] = 1 \) so that the order of \( [x, \alpha] \) divides \( n \).

Corollary 3.3. If \( E(G) > Z_2(G) \), then \( G' \) has a proper subgroup of finite index.

**Proof.** If \( E(G) > Z_2(G) \), then \( \mathfrak{A}_1 \) is a nontrivial torsion subgroup of \( \mathfrak{A}_2 \). Hence \( \mathfrak{A}_2 \neq 1 \) is not torsionfree and the theorem applies.

It is known that no complete, or even Černikov complete, group can have a proper subgroup of finite index (see [7, p. 234]). From this fact we derive part of the following:

Corollary 3.4. If \( G' \) is Černikov complete, or if \( Z(G) \cap \gamma_3(G) \) has no elements of order three, then \( E(G) = Z_3(G) \).

**Proof.** We shall show that \( \mathfrak{A}_1 = 1 \). By Corollary 2.8, \( E(G) \subseteq Z_3(G) \). Hence \( [G', E(G)] = [G', \mathfrak{A}_1] \subseteq Z(G) \cap \gamma_3(G) \).

Let \( a \in \mathfrak{A}_1, x \in G' \). Then, by Corollary 2.7 and Theorem 2.3(c), \( 1 = [x, a^n] = [x, a^n] \). By hypothesis, this implies that \( 1 = [x, a] \). Consequently \( a = 1 \).

Example 3.5. We now construct a group \( G \) such that \( Z_2(G) < E(G) < Z_3(G) \).

Let \( H = \langle a_1, a_2, a_3; x^3 \rangle \). Levi and van der Waerden [8] have shown that \( H \) has nilpotence class exactly three and is in the variety determined by \( e_3 \). Hence \( E(H) = H = Z_2(H) > Z_3(H) \). Let \( K \) be any group of nilpotence class at least three having no elements of order three (see for example [12, p. 198]). By Corollary 3.4, \( E(K) = Z_2(K) < Z_3(K) \subseteq K \). Letting \( G = H \times K \), we see that \( E(G) = E(H) \times E(K) = H \times Z_3(K) \). Hence \( Z_2(G) < E(G) < Z_3(G) \).
REMARK 3.6. Define $N_A(G) = \bigcap \{N_H(H) \mid H \text{ maximal Abelian subgroup of } G\}$ to be the A-Norm of $G$. Kappe [6] has shown that $a \in N_A(G)$ if and only if $[g, h] = 1$ for $g, h$ in $G$ implies that $[a, g, h] = 1$. From Theorem 2.6 it follows immediately that $E(G) \subseteq N_A(G) \subseteq E(G)$.

4. Finiteness conditions. We shall say that a word $\phi$ satisfies the Schur-Baer property if $[G; \phi^*(G)] = m$ finite implies $\phi(G)$ finite with order which divides a power of $m$ for all groups $G$.

Schur showed that $\gamma_2$ satisfies the Schur-Baer property; Baer extended this result to any outer commutator word $\phi$ (see [15]).

Recall that a group $G$ is residually finite if for every $x$ in $G$, $x \neq 1$, there is a normal subgroup $N_x$ of $G$ such that $x \notin N_x$ and $G/N_x$ is finite. A group is locally residually finite if every finitely generated subgroup is residually finite.

We shall need the following theorem. For a proof (due to P. Hall), see [15, Theorem 2].

**Theorem 4.1.** If $\phi$ generates a locally residually finite variety, then $\phi$ satisfies the Schur-Baer property.

**Theorem 4.2.** If $\phi \in \{e_2, e_3\}$, then $\phi$ satisfies the Schur-Baer property.

**Proof.** Suppose $\phi = e_2$. A group in the variety generated by $\phi$ is nilpotent by Levi’s Theorem. A finitely generated nilpotent group is residually finite by P. Hall [4]. Therefore, a finitely generated group in the variety generated by $\phi$ is residually finite and Theorem 4.1 applies.

Let $\phi = e_3$. Heineken [5] has shown that a group in the variety generated by $\phi$ is locally nilpotent. Hence a finitely generated group in this variety is also residually finite and the theorem follows as above.

Recall that a group is an $\text{SN}^*$ group if it possesses an ascending normal series with Abelian factors (see [7]). Also, the unique maximum locally nilpotent normal subgroup of a group is called its Hirsch-Plotkin radical (see [12]).

We note that in P. Hall’s proof of Theorem 4.1 that we may extend the result somewhat if we put some restrictions on $G$ itself. That is, if $\phi^*(G)$ is locally residually finite for all $G$ in some quotient-and subgroup-closed class $\Sigma$, then $\phi$ satisfies the Schur-Baer property for all $G$ in $\Sigma$.

**Theorem 4.3.** If $G$ satisfies the maximum or the minimum condition, or if $G$ is an $\text{SN}^*$ group, then $e_n$ satisfies the Schur-Baer property for $G$. 
Proof. Suppose $G$ satisfies the maximum condition. Then, by [12, Theorem VI. 8. j], we have that the set of left Engel elements (of all lengths) is the Hirsch-Plotkin radical $R$. Since then $e^*_n(G) \subseteq R$ is locally nilpotent, it is locally residually finite. By the preceding remark, we have that $e_n$ satisfies the Schur-Baer property for $G$.

Vilyacer [18] has shown that an Engel group satisfying the minimum condition is locally nilpotent. Plotkin [11] has proved that an Engel group which is also an $SN^*$ group is locally nilpotent. Hence the remainder of the theorem follows as above.

The validity of the Schur-Baer property in general is one of several conjectures which have been proposed for the group functions $\phi$ and $\phi^*$ (see [9] and [16]). Modified solutions of two of these come from the following lemma.

**Lemma 4.4.** Suppose $G$ is in a class of groups in which the Schur-Baer property is satisfied locally for $\phi$. If $G$ is locally residually finite and $\phi$ is finite-valued on $G$, then $\phi(G)$ is finite.

**Proof.** This follows from the arguments used in the proofs of Proposition 1 and its two corollaries in [17].

We note in particular in these proofs that there is a finitely generated subgroup $H$ of $G$ such that $\phi(H) = \phi(G)$. It follows that $H/\phi^*(H)$ is finite. Since $H$ and $\phi$ satisfy the Schur-Baer property, $\phi(H) = \phi(G)$ is finite.

The following two theorems are immediate from these observations.

**Theorem 4.5.** If $\phi \in \{e, e_2\}$, $G$ is locally residually finite, and $\phi$ is finite-valued on $G$, then $\phi(G)$ is finite.

**Theorem 4.6.** If $\phi \in \{e, e_3\}$, $\phi$ is finite-valued on $G$, and $G$ is finitely generated and residually finite, then $G/\phi^*(G)$ is finite.

**References**


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