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**A COUNTER EXAMPLE TO THE BLUM HANSON THEOREM
IN GENERAL SPACES**

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A COUNTER EXAMPLE TO THE BLUM HANSON THEOREM IN GENERAL SPACES

M. A. AKCOGLU, J. P. HUNEKE AND H. ROST

Let T and S be two bounded linear operators on a Banach space B . One studies the question whether weak convergence of the powers T^n to S implies convergence of the Cesaro averages $1/n \sum_{k=1}^n T^{i(k)}$ to S for all subsequences $0 \leq i(1) < i(2) < \dots$ of the integers. It is well known that this implication holds if B is the L^2 of a finite measure space and T is induced by a measure preserving transformation of that space (this is the Blum Hanson theorem) or, more generally, if B is a Hilbert space and T of norm at most 1, or if B is a L^1 space and T a positive operator of norm at most 1. In the present paper the conjecture that the above implication holds in general Banach spaces for all T with $\|T\| \leq 1$ is disproved by constructing a counterexample in a Banach space of the type $B = \mathcal{C}(X)$, X a compact Hausdorff space.

Specifically, let B and B^* be a Banach space and its adjoint, respectively, and let $T: B \rightarrow B$ and $S: B \rightarrow B$ be two linear and bounded operators. Consider the following two statements:

(i) T^n converges weakly to S ; i.e., for each $f \in B$ and $F \in B^*$, $\lim_{n \rightarrow \infty} F(T^n f) = F(Sf)$.

(ii) Let $i(n)$ be a sequence of integers so that $0 \leq i(n) < i(n+1)$ for each $n \geq 1$. Then, $1/n \sum_{k=1}^n T^{i(k)}$ converges strongly to S ; i.e., if $f \in B$, then $\lim \|1/n \sum_{k=1}^n T^{i(k)} f - Sf\| = 0$.

It is easy to see that (ii) always implies (i). The Blum Hanson theorem [2] states that if B is the L_2 space of a finite measure space and if T is induced by a measure preserving transformation of this measure space, then (i) also implies (ii). Later it was shown that the equivalence of (i) and (ii) is true if T is a contraction (i.e., if $\|T\| \leq 1$) and B is a Hilbert space [1], [3] or the L_1 space of a σ -finite measure space [1]. It is then natural to ask if these two conditions (i) and (ii) are always equivalent. In this note, we give a counterexample to show that in general (i) does not imply (ii), even if T is a contraction.

2. Reducing the question to a topological one. Let X be a compact Hausdorff space and let $\mathcal{C} = \mathcal{C}(X)$ be the Banach space of all real valued continuous functions on X , with the usual, supremum norm. If $\tau: X \rightarrow X$ is a continuous transformation, then there is an induced linear contraction $T: \mathcal{C} \rightarrow \mathcal{C}$, defined as $(Tf)(x) = f(\tau x)$ for each $f \in \mathcal{C}$ and $x \in X$. Note that $T^n f$ converges weakly in \mathcal{C} if and only if $f(\tau^n x)$ converges for each $x \in X$, as a sequence of real numbers.

Hence, if there is a point $x_0 \in X$ so that $\lim_{n \rightarrow \infty} \tau^n x = x_0$ for every $x \in X$, then T^n converges weakly to $S: \mathcal{E} \rightarrow \mathcal{E}$, defined as $(Sf)(x) = f(x_0)$ for each $x \in X$ and $f \in \mathcal{E}$.

Now assume that τ is such a transformation and also that there is a compact $K \subset X$, not containing the point x_0 and satisfying the following condition:

(A) For each integer $N \geq 0$ there is a point $x = x(N)$ in X so that K contains more than N terms of the sequence $\tau^n x$, $n = 0, 1, 2, \dots$.

Before we give an example for such an X and τ in the next section, here we note that in this case (i) does not imply (ii). Let $f \in \mathcal{E}$ be a nonnegative function so that $f(x_0) = 0$ and $f(x) \geq 1$ for all $x \in K$. Hence, $T^n f$ converges weakly to zero. Now define a sequence $i(n)$ of integers as follows. Let $i(1) = 0$. For each $r \geq 1$, if the first 2^{r-1} terms are determined then the next 2^{r-1} terms [i.e., the terms $i(2^{r-1} + 1), \dots, i(2^r)$] are chosen as follows. With the notations of Condition (A), let $x_r = x(i(2^{r-1}) + 2^{r-1})$ and let the following conditions be satisfied: $\tau^i(2^{r-1} + s)_{x_r} \in K$ for each $s = 1, 2, \dots, 2^{r-1}$ and $i(2^{r-1}) < i(2^{r-1} + 1) < i(2^r)$. Then,

$$\frac{1}{2^r} \sum_{k=1}^{2^r} f(\tau^{i(k)} x_r) \geq \frac{1}{2}$$

for each $r \geq 1$. Hence,

$$\frac{1}{n} \sum_{k=1}^n T^{i(k)} f$$

does not converge strongly to zero.

3. The topological example. We are now going to give an example of a compact Hausdorff space X and a continuous transformation $\tau: X \rightarrow X$ so that all the assumptions of the second section are satisfied.

Let R be the real line with the usual topology and let $C = [0, 1) = \{x \mid 0 \leq x < 1\}$ be the unit interval with its circle topology. Let $\varphi: C \rightarrow C$ be a homeomorphism that is linear in $[0, 1/2)$ and in $[1/2, 1)$ and satisfies $\varphi 0 = 0$, $\varphi 1/2 = 3/4$. Note that if $0 < x < 1$ then $\varphi^n x \rightarrow 1$ in R . Hence, $\varphi^n x \rightarrow 0$ in C , for each $x \in C$. Also, let $\alpha: C \rightarrow C$ be a continuous function that is linear in $[1/4, 1/2)$, vanishes identically on $[1/2, 1)$ and is equal to $-A/\log x$ at every $x \in (0, 1/4)$. Here, A is a positive constant so that $\max_{x \in C} \alpha x = +A/\log 4$ is less than $1/4$.

Now let $X = C^2$ be the two-dimensional torus with its usual topology. The points of X are denoted as (x, y) , where $x, y \in C$. Let a mapping $\tau: X \rightarrow X$ be defined as $\tau(x, y) = (\{\varphi x + \alpha y\} \bmod 1, \varphi y)$. It is then clear that τ is continuous.

LEMMA. If $(x, y) \in X$, then $\lim_{n \rightarrow \infty} \tau^n(x, y) = (0, 0)$.

Proof. Let $\tau^n(x, y) = (x_n, y_n)$. Hence, $x_0 = x, y_0 = y$ and if $n \geq 1$, then $y_n = \varphi^n y, x_n = \xi_n \pmod 1$, where $\xi_n = \varphi x_{n-1} + \alpha y_{n-1}$. If $y_0 = 0$, then $y_n = 0$ for all $n \geq 0$ and $x_n = \varphi x_{n-1} = \varphi^n x_0 \rightarrow 0$ in C . If $y_0 > 0$, then there is an integer $m \geq 0$ so that $y_n = \varphi^n y_0 > 1/2$ for all $n \geq m$. This means that $\alpha y_n = 0$ and $x_n = \varphi^{n-m} x_m$ for all $n \geq m$. Hence, $(x_n, y_n) = (\varphi^{n-m} x_m, \varphi^n y_0)$ converges to $(0, 0)$ for all $(x, y) \in X$.

LEMMA. If $K = \{(x, y) \mid (x, y) \in X, 1/8 \leq x \leq 7/8\}$, then K satisfies Condition A of § 2.

Proof. With the notations of the previous proof, let

$$\delta_n = \delta_n(x, y) = \xi_n - x_{n-1} = \varphi x_{n-1} - x_{n-1} + \alpha y_{n-1}.$$

Then,

$$x_n = \left[x_0 + \sum_{k=1}^n \delta_k \right] \pmod 1.$$

Now note that if

$$\sum_{k=n_1}^{n_2} \delta_k \geq 1$$

then there is an integer n , so that $n_1 \leq n \leq n_2$ and that $(x_n, y_n) \in K$. In fact, for each $k \geq 1, 0 \leq \delta_k \leq \max_{x \in C} (\varphi x - x) + \max_{y \in C} \alpha y \leq 1/2$, and hence, if

$$\sum_{k=n_1}^{n_2} \delta_k \geq 1$$

then,

$$\left[x_0 + \sum_{k=1}^n \delta_k \right] \pmod 1$$

is between $1/8$ and $7/8$ for some $n, n_1 \leq n \leq n_2$. Therefore, to prove the present lemma, it is enough to show that given any number N , there is a point $(x, y) \in X$ so that

$$\sum_{n=1}^{\infty} \delta_n(x, y) \geq N.$$

Let $0 < y < 1/4$ be given and let $M = M(y)$ be the largest integer in the set $\{n \mid n \geq 0, \varphi^n y < 1/4\}$. Let $z = \varphi^M y$. Hence, $z < 1/4$, but $\varphi z = (3/2)z \geq 1/4$. Therefore, if $0 \leq n \leq M$, then $y_n = \varphi^n y = (3/2)^n y = (3/2)^{n-M} z \geq (3/2)^{n-M} 1/6$, and $\alpha y_n = -A/\log y_n \geq A/((M-n) \log 3/2 + \log 6)$. This means that

$$\sum_{k=1}^M \delta_n \geq \sum_{n=1}^M \alpha y_n \geq A \sum_{n=0}^{M-1} \frac{1}{n \log 3/2 + \log 6}.$$

But it is clear that there are points $y \in (0, 1/4)$ for which $M = M(y)$ is arbitrarily large, hence, for which $\sum_{n=1}^{\infty} \delta_n$ is also arbitrarily large.

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Mustafa Agah Akcoglu, John Philip Huneke and Hermann Rost, <i>A counter example to the Blum Hanson theorem in general spaces</i>	305
Huzihiro Araki, <i>Some properties of modular conjugation operator of von Neumann algebras and a non-commutative Radon-Nikodym theorem with a chain rule</i>	309
E. F. Beckenbach, Fook H. Eng and Richard Edward Tafel, <i>Global properties of rational and logarithmico-rational minimal surfaces</i>	355
David W. Boyd, <i>A new class of infinite sphere packings</i>	383
K. G. Choo, <i>Whitehead Groups of twisted free associative algebras</i>	399
Charles Kam-Tai Chui and Milton N. Parnes, <i>Limit sets of power series outside the circles of convergence</i>	403
Allan Clark and John Harwood Ewing, <i>The realization of polynomial algebras as cohomology rings</i>	425
Dennis Garbanati, <i>Classes of circulants over the p-adic and rational integers</i>	435
Arjun K. Gupta, <i>On a "square" functional equation</i>	449
David James Hallenbeck and Thomas Harold MacGregor, <i>Subordination and extreme-point theory</i>	455
Douglas Harris, <i>The local compactness of vX</i>	469
William Emery Haver, <i>Monotone mappings of a two-disk onto itself which fix the disk's boundary can be canonically approximated by homeomorphisms</i>	477
Norman Peter Herzberg, <i>On a problem of Hurwitz</i>	485
Chin-Shui Hsu, <i>A class of Abelian groups closed under direct limits and subgroups formation</i>	495
Bjarni Jónsson and Thomas Paul Whaley, <i>Congruence relations and multiplicity types of algebras</i>	505
Lowell Duane Loveland, <i>Vertically countable spheres and their wild sets</i>	521
Nimrod Megiddo, <i>Kernels of compound games with simple components</i>	531
Russell L. Merris, <i>An identity for matrix functions</i>	557
E. O. Milton, <i>Fourier transforms of odd and even tempered distributions</i>	563
Dix Hayes Pettey, <i>One-one-mappings onto locally connected generalized continua</i>	573
Mark Bernard Ramras, <i>Orders with finite global dimension</i>	583
Doron Ravdin, <i>Various types of local homogeneity</i>	589
George Michael Reed, <i>On metrizable complete Moore spaces</i>	595
Charles Small, <i>Normal bases for quadratic extensions</i>	601
Philip C. Tonne, <i>Polynomials and Hausdorff matrices</i>	613
Robert Earl Weber, <i>The range of a derivation and ideals</i>	617