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For a cyclic and separating vector Ψ of a von Neumann algebra R, the corresponding modular conjugation operator J_Ψ is characterized by the property that it is an antiunitary involution satisfying $J_\Psi \Psi = \Psi$, $J_\Psi R J_\Psi = R'$ and $(\Psi, Q j_\Psi(Q)\Psi) \geqq 0$ for all $Q \in R$ where $j_\Psi(Q) = J_\Psi Q J_\Psi$.

The strong closure V_Ψ of the vectors $Q j_\Psi(Q)\Psi$ is shown to be a J_Ψ-invariant pointed closed convex cone which algebraically span the Hilbert space H. Any J_Ψ-invariant $\Phi \in H$ has a unique decomposition $\Phi = \Phi_1 - \Phi_2$ such that $\Phi_j \in V_\Psi$ and $s^R(\Phi_1) \perp s^R(\Phi_2)$.

There exists a unique bijective homeomorphism σ_Ψ from the set of all normal linear functionals on R onto V_Ψ such that the expectation functional by the vector $\sigma_\Psi(\rho)$ is ρ. It satisfies

$$\begin{aligned} \| \sigma_\Psi(\rho_1) - \sigma_\Psi(\rho_2) \|^2 &\leqq \| \rho_1 - \rho_2 \| \\ &\leqq \{\| \sigma_\Psi(\rho_1) + \rho_\Psi(\rho_2) \|\} \| \sigma_\Psi(\rho_1) - \sigma_\Psi(\rho_2) \| . \end{aligned}$$

Any two σ_Ψ and $\sigma_{\Psi'}$ are related by a unitary u' in R' by $u'\sigma_\Psi(\rho) = \sigma_{\Psi'}(\rho)$ for all ρ.

The relation $l\rho_1 \geqq \rho_2$ holds if and only if there exists $A(\rho_2/\rho_1) \in R$ such that $A(\rho_2/\rho_1)\sigma_\Psi(\rho_1) = \sigma_\Psi(\rho_2)$. The smallest l is given by $\| A(\rho_2/\rho_1) \|$. It satisfies the chain rule $A(\rho_3/\rho_2)A(\rho_2/\rho_1) = A(\rho_3/\rho_1)$. It coincides with the positive square root of the measure theoretical Radon-Nikodym derivative if R is commutative.

As an application, it is shown that product of any two modular conjugation $j_\Psi j_\Phi$ is an inner automorphism of R.

For a product state $\otimes \rho_j$ of a C^* algebra generated by finite W^* tensor products $\{\otimes_{j \in I} R_j\} \otimes \{\otimes_{j \in I} 1_j\}$ of von Neumman algebras R_j, it is shown that $\otimes \rho_j$ and $\otimes \rho'_j$ are equivalent if and only if $\Sigma \| \sigma_\Psi(\rho_j) - \sigma_\Psi(\rho'_j) \|^2 < \infty$ where $\| \sigma_\Psi(\rho) - \sigma_\Psi(\rho') \|$ is independent of Ψ.

It is shown that there exists a unitary representation $U_\Psi(g)$ of the group of all $*$-automorphisms of R such that $U_\Psi(g) x U_\Psi(g)^* = g(x)$ for all $x \in R$ and $U_\Psi(g)\sigma_\Psi(g^*\rho) = \sigma_\Psi)\rho)$ for all normal positive linear functionals ρ.

1. Introduction. In the Tomita-Takesaki theory of modular automorphisms [9], two operators Δ_Ψ and J_Ψ are associated with each cyclic and separating vector Ψ of a von Neumann algebra R on a Hilbert space H.

Δ_Ψ is a positive selfadjoint operator such that

$$\Delta_\Psi \Psi = \Psi \, , \tag{1.1}$$

$$\tau_\Psi(t)Q \equiv (\Delta_\Psi)^{it} Q (\Delta_\Psi)^{-it} \in R \tag{1.2}$$

for every $Q \in R$ and real t. It is called a modular operator and the automorphisms $\tau_\Psi(t)$ of R is called modular automorphisms.

$J = J_\Psi$ is an antiunitary involution, namely

$$(Jx, Jy) = (y, x) \tag{1.3}$$

$$J^2 = 1 \, . \tag{1.4}$$

It satisfies

$$J\Psi = \Psi \, , \tag{1.5}$$

$$JRJ = R' \, . \tag{1.6}$$

We shall call J_Ψ a modular conjugation operator.

Δ_Ψ and J_Ψ are defined through the polar decomposition

$$\bar{S} = J_\Psi \Delta_\Psi^{1/2} \tag{1.7}$$

of the closure of an antilinear operator S, which is defined on its domain $R\Psi$ by

$$SQ\Psi = Q^*\Psi \, , \qquad Q \in R \, . \tag{1.8}$$

An important property is

$$J_\Psi \Delta_\Psi J_\Psi = \Delta_\Psi^{-1} \, . \tag{1.9}$$

Our investigation centers around the following property of $J = J_\Psi$ observed in [2]. For any $Q \in R$, $Q \geqq 0$, $Q \neq 0$, the following strict inequality holds:

$$(\Psi, Qj_\Psi(Q)\Psi) > 0 \tag{1.10}$$

where

$$j_\Psi(Q) \equiv J_\Psi Q J_\Psi \in R' \, . \tag{1.11}$$

The validity of (1.10) comes from the property $\Delta_\Psi > 0$ and the following identity obtained from (1.5), (1.7), and (1.8):

$$(\Psi, Qj_\Psi(Q)\Psi) = (Q^*\Psi, \Delta_\Psi^{1/2} Q^*\Psi) \, . \tag{1.12}$$

Our first result is the characterization of the modular conjugation J_Ψ for a given Ψ by (1.3), (1.4), (1.5), (1.6), and (1.10). It should be remarked that (1.3), (1.4), (1.5), and (1.6) without (1.10) are not sufficient to characterize J_Ψ. If (1.5) is dropped, then there exists a unitary u in the center such that $J = J_{u\Psi}$.

Our second result is concerned with the strong closure of the set of all vectors $Qj(Q)\Psi$, $Q \in R$. It is shown to be a pointed closed convex cone which algebraically span H and is selfdual in the sense that any $\Phi \in H$ satisfying

$$(\Phi, x) \geqq 0 \tag{1.13}$$

for all $x \in V_\Psi$ must be in V_Ψ. Any $\Phi \in V_\Psi$ is shown to have a unique decomposition $\Phi = \Phi_1 - \Phi_2$, satisfying $\Phi_1 \in V_\Psi$, $\Phi_2 \in V_\Psi$ and $s^R(\Phi_1) \perp s^R(\Phi_2)$.

Our third result is concerned with a possibility of having some $\Phi \in V_\Psi$ for a given normal positive linear functional ρ such that $\omega_\Phi = \rho$ where ω_Φ denotes the expectation functional on R by the vector Φ. This turns out to be possible for all ρ in a unique and nice manner. It is shown that there exists one and only one element in V_Ψ—denoted as $\sigma_\Psi\rho$—for any given normal positive linear functional ρ on R, such that the expectation functional $\omega_{\sigma_\Psi\rho}$ by the vector $\sigma_\Psi\rho \in V_\Psi$ is ρ. The mapping σ_Ψ is bicontinuous due to the following inequality:

$$\begin{aligned}||\sigma_\Psi(\rho_1) - \sigma_\Psi(\rho_2)||^2 &\leqq ||\rho_1 - \rho_2|| \\ &\leqq \{||\sigma_\Psi(\rho_1) + \sigma_\Psi(\rho_2)||\}\, ||\sigma_\Psi(\rho_1) - \sigma_\Psi(\rho_2)|| \,.\end{aligned}$$

Any two σ_Ψ and $\sigma_{\Psi'}$ are equivalent up to a unitary equivalence, namely there exists a unitary $u' \in R'$ satisfying

$$u'\sigma_\Psi(\rho) = \sigma_{\Psi'}(\rho)$$

for all ρ.

The fourth result is concerned with the Radon-Nikodym derivative satisfying a chain rule. The relation $l\rho_1 \geqq \rho_2$ for two normal positive linear functional ρ_1 and ρ_2 holds if and only if there exists $A(\rho_2/\rho_1) \in R$ such that $A(\rho_2/\rho_1)\sigma_\Psi(\rho_1) = \sigma_\Psi(\rho_2)$. It satisfies the chain rule

$$A(\rho_3/\rho_2)A(\rho_2/\rho_1) = A(\rho_3/\rho_1)\,.$$

If R is commutative, $A(\rho_2/\rho_1)$ is the positive square root of the measure theoretical Radon-Nikodym derivative. For a general R, $A(\rho_2/\rho_1)$ is different from the noncommutative Radon-Nikodym derivative found by Sakai [8].

As a corollary to our investigation, we find that product of any two modular conjugation $j_\Psi j_\Phi$ is an inner $*$ automorphism of R.

Another application is made in connection with an infinite tensor product of von Neumann algebras R_j. We define

$$d'(\rho_1, \rho_2) = ||\sigma_\Psi(\rho_1) - \sigma_\Psi(\rho_2)||$$

which is independent of the choice of cyclic and separating vector Ψ. For normal states ρ_j and ρ'_j of each R_j, we consider product states $\otimes \rho_j$ and $\otimes \rho'_j$ on the C^* algebra A generated (as an inductive limit) by finite W^* tensor products $\{\bigotimes_{j \in I} R_j\} \equiv R(I)$ where I is any finite index set. The representations of A canonically associated with $\otimes \rho_j$ and $\otimes \rho'_j$ are quasi-equivalent if and only if

$$\Sigma d'(\rho_j, \rho'_j)^2 < \infty$$

and the central supports of ρ_j and ρ'_j are the same. The distance d' is in general larger than Bures distance [5]. They coincides if ρ_1 and ρ_2 commute.

As a further application, we show that there exists a unitary representation $U_\Psi(g)$ of the group of all $*$-automorphisms of R such that $U_\Psi(g)xU_\Psi(g)^* = g(x)$ for all $x \in R$ and $U_\Psi(g)\sigma_\Psi(g^*\rho) = \sigma_\Psi(\rho)$ for all normal positive linear functionals ρ.

We also give a simple proof of the continuity of the modular automorphism $\tau_\rho(t)x$ in ρ for a fixed $x \in R$ and bounded t.

2. A characterization of the modular conjugation operator.

THEOREM 1. *Let Ψ be a cyclic and separating vector for a von Neumann algebra R on H. An operator J is the modular conjugation for Ψ if and only if the following* 5 *conditions are fulfilled.*

(i) $(Jx, Jy) = (y, x)$ *for all* $x, y \in H$.

(ii) $J^2 = 1$.

(iii) $JRJ = R'$.

(iv) $J\Psi = \Psi$.

(v) $(\Psi, Qj(Q)\Psi) \geqq 0$ *for all* $Q \in R$ *where* $j(Q) \equiv JQJ$. *The equality in* (v) *holds if and only if* $Q = 0$.

Proof. It is known [9] that the modular conjugation J_Ψ for the vector Ψ satisfies (i), (ii), (iii), and (iv). (v) with the strict inequality for $Q \neq 0$ is already proved in § 1.

We now prove that J satisfying the 5 conditions must by J_Ψ. From (i), it follows that J is antilinear. From (ii), it follows that J is bijective. Hence J is antiunitary.

Let T be defined on $R\Psi$ by

$$TQ\Psi = JQ^*\Psi \,, \qquad Q \in R \,. \tag{2.1}$$

Since Ψ is separating for R, $Q_1\Psi = Q_2\Psi$ implies $Q_1 = Q_2$ and hence $JQ_1^*\Psi = JQ_2^*\Psi$. Therefore, T is well-defined and is linear. Since Ψ is cyclic for R, T has a dense domain. By (iv) and (v),

$$(Q\Psi, TQ\Psi) = (\Psi, Q^*j(Q^*)\Psi) \geqq 0 \,, \quad Q \in R \,. \tag{2.2}$$

Thus T is positive on its domain and hence is symmetric.

By (1.8) and (2.1), we have

$$T = JS . \tag{2.3}$$

Since J preserves norm, we have $\bar{T} = J\bar{S}$ and

$$D(\bar{T}) = D(\bar{S}) = D(\Delta_{\Psi}^{1/2}) . \tag{2.4}$$

Define

$$u \equiv JJ_{\Psi} . \tag{2.5}$$

Both J and J_{Ψ} are antiunitary. Hence u is unitary. We have

$$\bar{T} = u\Delta_{\Psi}^{1/2} , \tag{2.6}$$

where (1.7) is used. We shall now show that $\bar{T}$ is selfadjoint. Then (2.2) implies that $\bar{T}$ is positive and hence (2.6) implies $\bar{T} = \Delta_{\Psi}^{1/2}$ and $u = 1$, which proves $J = J_{\Psi}$ by (2.5).

From (2.3), we have[1]

$$T^* = S^*J . \tag{2.7}$$

It is known [9] that $R'\Psi$ is a core of S^*. (Namely, the closure of restriction of S^* to $R'\Psi$ is S^*.) By (iii), $JR\Psi = R'\Psi$. Hence RT is a core of T^*. Since $R\Psi$ is the domain of T and $T^* \supset T$, we have $T^* = \bar{T}$.

The condition (iv) of Theorem 1 is not essential as is seen in the next result.

THEOREM 2. *Let Ψ be cyclic and separating for R in H. An operator J satisfies conditions* (i), (ii), (iii), *and* (v) *of Theorem* 1 *if and only if there exists a unitary u in the center of R such that*

$$J = J_{u\Psi}(= uJ_{\Psi}u^*) . \tag{2.8}$$

The condition (2.8) *is equivalent to JJ_{Ψ} being in $R \cap R'$.*

For the proof we need preliminary lemmas.

LEMMA 1. *The weakly closed linear hull of the set of all operators $Qj(Q)$, $Q \in R$ is $\{R \cup R'\}''$.*

Proof. For arbitrary $Q_1 \in R$ and $Q_2 \in R'$, we have

$$Q_1Q_2 = 4^{-1}\sum_{n=0}^{3} e^{in\pi/2}X_nj(X_n) ,$$

$$X_n = Q_1 + e^{in\pi/2}j(Q_2) \in R ,$$

[1] This part of proof has been simplified by a suggestion of Dr. G. Elliott.

where $j(Q_2) \equiv JQ_2J \in R$, $j(X_n) \equiv JX_nJ$.

LEMMA 2. *Let W be a von Neumann algebra on H such that W' is commutative. If $\Psi = \Psi_+ + \Psi_-$ is a cyclic vector for W in H, and*

$$(\Psi_+, Q\Psi_-) + (\Psi_-, Q\Psi_+) = 0 \tag{2.9}$$

for all $Q \in W$, then there exists a selfadjoint operator A such that its spectral projections are in the center W' of W and

$$s^{W'}(\Psi_+)\Psi_- = iAs^{W'}(\Psi_-)\Psi_+ \tag{2.10}$$

where $s^{W'}(\Psi_\pm)$ are projections onto the closures of $W\Psi_\pm$.

Proof. $s^{W'}(\Psi_\pm)$ belong to W' which is commutative and hence is the center of W. Let

$$E = s^{W'}(\Psi_+)s^{W'}(\Psi_-) .$$

Then

$$E\Psi_\mp = s^{W'}(\Psi_\pm)\Psi_\mp . \tag{2.11}$$

We define A to be 0 on $(1-E)H$. If $E=0$, (2.10) is trivially satisfied. Hence we consider the case $E \neq 0$.

We are going to define a selfadjoint operator $A_1 = AE$ on EH satisfying

$$E\Psi_- = iA_1E\Psi_+ \tag{2.12}$$

which implies (2.10) in view of (2.11).

Since $WE\Psi_\pm = EW\Psi_\pm$ are dense in $Es^{W'}(\Psi_\pm)H = EH$, $E\Psi_\pm$ are both cyclic for WE on EH. Define an operator A_2 by

$$A_2QE\Psi_+ = -iQE\Psi_- , \qquad Q \in W \tag{2.13}$$

on a dense subset $WE\Psi_+$ of EH.

If $QE\Psi_+ = 0$, then (2.9), where Q is replaced by EQ_1^*QE, implies

$$\begin{aligned}(Q_1E\Psi_+, QE\Psi_-) &= (\Psi_+, EQ_1^*QE\Psi_-)\\ &= -(\Psi_-, EQ_1^*QE\Psi_+)\\ &= 0\end{aligned}$$

for all $Q_1 \in W$. Therefore $QE\Psi_- = 0$. Hence $QE\Psi_+ = Q'E\Psi_+$ for $Q, Q' \in W$ implies $QE\Psi_- = Q'E\Psi_-$, which shows that A_2 is well-defined. A_2 is obviously linear.

From (2.9), we have for $Q_1, Q_2 \in W$

$$\begin{aligned}(Q_1E\Psi_+, A_2Q_2E\Psi_+) &= (\Psi_+, -iEQ_1^*Q_2E\Psi_-) \\ &= (-i\Psi_-, EQ_1^*Q_2E\Psi_+) \\ &= (A_2Q_1E\Psi_+, Q_2E\Psi_+) .\end{aligned}$$

Therefore A_2 is symmetric. A_2 obviously commutes with $Q \in W$ on its domain.

Since Ψ is cyclic for W, $WE\Psi = EW\Psi$ is dense in EH. Hence $E\Psi_+ + E\Psi_- = E\Psi$ is cyclic for EW on EH. It is therefore separating for the commutant of EW on EH, which is EW'.

From (2.9), we have

$$(E\Psi_+ - E\Psi_-, Q(E\Psi_+ - E\Psi_-)) = (E\Psi_+ + E\Psi_-, Q(E\Psi, +E\Psi_-)) .$$

Hence $\| Q(E\Psi_+ - E\Psi_-) \|^2 = 0$ implies $\| QE\Psi \|^2 = 0$ for any $Q \in W$. As we have seen, $E\Psi$ is separating for EW' and hence $E\Psi_+ - E\Psi_-$ is also separating for EW'. It is therefore cyclic on EH for the commutant of EW' on EH which is EW.

Since

$$(A_2 + i)QE\Psi_+ = iQ(E\Psi_+ - E\Psi_-) ,$$

$$(A_2 - i)QE\Psi_+ = -iQ(E\Psi_+ + E\Psi_-) ,$$

for all $Q \in W$, $A_2 + i$ and $A_2 - i$ have both dense ranges in EH by cyclicity of $E\Psi_+ - E\Psi_-$ and $E\Psi_+ + E\Psi_-$ for EW. Therefore, the closure $A_1 = \overline{A_2}$ of A_2 is selfadjoint. By (2.13) with $Q = 1$, we have (2.12).

REMARK. The assumption that Ψ is cyclic for W can be omitted. Let $e = s^{W'}(\Psi)$. Then $(1-e)\Psi_+ = -(1-e)\Psi_-$. Substituting $Q = (1-e)$ into (2.9), we obtain

$$\| (1 - e)\Psi_+ \|^2 = \| (1 - e)\Psi_- \|^2 = 0 .$$

Hence we may restrict our attention to eW on eH with Ψ, Ψ_+, Ψ_- all in eH and apply proof of Lemma 2.

LEMMA 3. *If* $Q \in R \cap R'$, *then*

$$J_\Psi Q J_\Psi = Q^* \tag{2.14}$$

where J_Ψ *is the modular conjugation operator for a cyclic and separating vector* Ψ *of* R.

Proof. It is known ([1], [9]) that the center of R is elementwise invariant under any *KMS* automorphisms. Hence $Q \in R \cap R'$ commutes with Δ_Ψ. We have

$$
\begin{aligned}
(J_\Psi Q J_\Psi)\Psi = J_\Psi Q\Psi &= \Delta_\Psi^{1/2} Q^* \Psi \\
&= Q^* \Delta_\Psi^{1/2}\Psi = Q^*\Psi \ .
\end{aligned}
$$

By (iii) of Theorem 1, $J_\Psi(R \cap R')J_\Psi = R \cap R'$. Since Ψ is separating for $R \supset R \cap R'$, we have (2.14).

Proof of Theorem 2. Assume that J satisfies (i), (ii), (iii), and (v) of Theorem 1. From (i) and (ii), J is an antiunitary operator. Set

$$\Psi_\pm = 2^{-1}(\Psi \pm J\Psi) \ . \tag{2.15}$$

We have

$$J\Psi_\pm = \pm\Psi_\pm \ , \tag{2.16}$$

$$\Psi = \Psi_+ + \Psi_- \ . \tag{2.17}$$

By (2.16), we have for $Q \in R$

$$
\begin{aligned}
(\Psi_\pm, Qj(Q)\Psi_\pm) &= (J\Psi_\pm, Qj(Q)J\Psi_\pm) \\
&= (J\Psi_\pm, JQj(Q)\Psi_\pm) \\
&= \overline{(\Psi_\pm, Qj(Q)\Psi_\pm)}
\end{aligned}
$$

where the second equality is due to $Qj(Q) = j(Q)Q$ and the last equality is due to (i). Similarly,

$$(\Psi_\pm, Qj(Q)\Psi_\mp) = -\overline{(\Psi_\pm, Qj(Q)\Psi_\mp)} \ .$$

Hence

$$i \operatorname{Im}(\Psi, Qj(Q)\Psi) = (\Psi_+, Qj(Q)\Psi_-) + (\Psi_-, Qj(Q)\Psi_+) \ .$$

By (v), this must vanish. By Lemma 1, the weakly closed linear hull of $Qj(Q)$, $Q \in R$ is $(R \cup R')''$. Setting $W = (R \cup R')''$, the premises of Lemma 2 are satisfied. Note that $W' = R \cap R'$ is the center of R and is commutative.

Hence there exists a selfadjoint operator A affiliated with $R \cap R'$ such that (2.10) is satisfied. We define a unitary operator u in $R \cap R'$ by

$$
\begin{aligned}
u = {} & s^{W'}(\Psi_+)(1 - s^{W'}(\Psi_-)) \\
& + (1 - iA)(1 + A^2)^{-1/2} s^{W'}(\Psi_+) s^{W'}(\Psi_-) \\
& + i s^{W'}(\Psi_-)(1 - s^{W'}(\Psi_+)) \ .
\end{aligned} \tag{2.18}
$$

Because Ψ is cyclic for R, it is cyclic for W. Hence $s^{W'}(\Psi_+) \vee s^{W'}(\Psi_-) \geqq s^{W'}(\Psi) = 1$. Thus

$$(1 - s^{W'}(\Psi_-))(1 - s^{W'}(\Psi_+)) = 0$$

and u is unitary.

From (2.10) and (2.18), we have

$$
u\Psi = (1 - s^{W'}(\Psi_-))\Psi_+ + (1 + A^2)^{1/2} s^{W'}(\Psi_-)\Psi_+ + i(1 - s^{W'}(\Psi_+))\Psi_- . \tag{2.19}
$$

Since $JWJ = W$, both $W\Psi_+$ and $W\Psi_-$ are invariant under J. Therefore $s^{W'}(\Psi_\pm)$ both commute with J. We shall next prove that A commutes with J.

As we have seen, $E = s^{W'}(\Psi_+)s^{W'}(\Psi_-)$ commutes with J. From (2.16) and $JWJ = W$, the domain $WE\Psi_+$ of A_2 is invariant under J and A_2 commutes with J. Hence the closure A_1 of A_2 commutes with J, because J preserves norm. From the uniqueness of the spectral projections and

$$
\int \lambda dE_\lambda = A_1 = JA_1J = \int \lambda d(JE_\lambda J) ,
$$

we have $E_\lambda = JE_\lambda J$ for all spectral projections E_λ of A_1. Hence J commutes with $(1 + A^2)^{1/2}$.

From (2.19) and (2.16), we have

$$
Ju\Psi = u\Psi .
$$

Since u is in the center of R, it commutes with $Qj(Q)$, $Q \in R$. Since u is unitary, we have

$$
(u\Psi, Qj(Q)u\Psi) = (\Psi, Qj(Q)\Psi) \geqq 0 .
$$

By Theorem 1,

$$
J = J_{u\Psi} .
$$

Since the unitary mapping $H \to uH = H$, $\Psi \to u\Psi$, $R \to uRu^* = R$ brings S_Ψ to $uS_\Psi u^* = S_{u\Psi}$, we have

$$
uJ_\Psi u^* = J_{u\Psi} .
$$

Hence we have (2.8).

By Lemma 3, we have

$$
JJ_\Psi = uJ_\Psi u^* J_\Psi = u^2
$$

which is a unitary operator in the center of R.

Conversely, let w be a unitary operator in $R \cap R'$ and $JJ_\Psi = w$. Then $J = wJ_\Psi$ satisfies (i), (ii), (iii), and (v) of Theorem 1, where (ii) is due to Lemma 3:

$$
(wJ_\Psi)^2 = wJ_\Psi wJ_\Psi = ww^* = 1 .
$$

The following example shows the case where (i), (ii), (iii), and (iv) are satisfied but $J \neq J_\Psi$. The center in this example is trivial and $J \neq uJ_\Psi u^*$ for any unitary u in the center.

EXAMPLE. Let H_n be n dimensional Hilbert space and $R = B(H_2) \otimes 1$ be the algebra of 2×2 matrices on $H_4 = H_2 \otimes H_2$. Let e_1, e_2 be an orthonormal basis of H_2 and $e_{ij} = e_i \otimes e_j \in H_4$. Let $\Psi = 2^{-1/2}(e_{11} + e_{22})$, $\Phi = 2^{-1/2}(e_{12} + e_{21})$. Then $J_\Psi e_{ij} = e_{ji}$ while $J_\Phi e_{ij} = e_{ij}$ for $i \neq j$ and $J_\Phi e_{ii} = e_{jj}$ for $i \neq j$. Hence $J_\Psi \neq J_\Phi$. However, $J = J_\Phi$ satisfies (i), (ii), and (iii) because it is a modular conjugation operator for Φ and satisfies (iv).

REMARK. The condition (iii) is used only in the proof of the essential selfadjointness in Theorem 1. If R is a finite matrix algebra then (i), (ii), (iv), and (v) are sufficient to prove $J = J_\Psi$. Whether (iii) is necessary for more general case is an open question.

3. Technical lemmas concerning $\Delta_\Psi^z Q \Delta_\Psi^{-z}$. We denote by $\mathfrak{A}_\Psi$ the set of all operators Q such that there exists a family of bounded linear operators $\tau_\Psi(z)Q$ depending on a complex parameter z, which is holomorphic in z for all z and satisfies

$$\tau_\Psi(t)Q = \Delta_\Psi^{it} Q \Delta_\Psi^{-it} \tag{3.1}$$

for real t.

For real z, we have

$$\tau_\Psi(z)Q\Delta_\Psi^{iz}\Phi = \Delta_\Psi^{iz}Q\Phi\ , \quad \Phi \in D(\Delta_\Psi^{iz})\ . \tag{3.2}$$

If Φ is an entire vector of $\log \Delta_\Psi$, then the left hand side is an entire function of z and hence $Q\Phi$ must be an entire vector of $\log \Delta_\Psi$ and (3.2) holds for all z. Since vectors, on which $\log \Delta_\Psi$ is bounded, are entire vector of $\log \Delta_\Psi$ and form a dense set of analytic vectors for Δ_Ψ^α for any real α, (3.2) holds for any z and $\Phi \in D(\Delta_\Psi^{iz})$ by Nelson's theorem.

If Q_1 and Q_2 are in $\mathfrak{A}_\Psi$, then $(\tau_\Psi(z)Q_1)\tau_\Psi(z)Q_2$ is an entire function of z and satisfies (3.1) for $Q = Q_1Q_2$. Hence $Q_1Q_2 \in \mathfrak{A}_\Psi$ and

$$\tau_\Psi(z)(Q_1Q_2) = \{\tau_\Psi(z)Q_1\}\tau_\Psi(z)Q_2\ . \tag{3.3}$$

Similarly, $Q \in \mathfrak{A}_\Psi$ implies $Q^* \in \mathfrak{A}_\Psi$ and

$$\tau_\Psi(z)(Q^*) = (\tau_\Psi(\bar{z})Q)^*\ . \tag{3.4}$$

We define

$$\mathfrak{A}_{\Psi 1} = \mathfrak{A}_\Psi \cap R\ , \quad D_{\Psi 1} = \mathfrak{A}_{1\Psi}\Psi\ , \tag{3.5}$$

$$\mathfrak{A}_{\Psi 2} = \mathfrak{A}_\Psi \cap R'\ , \quad D_{\Psi 2} = \mathfrak{A}_{\Psi 2}\Psi\ . \tag{3.6}$$

If $Q \in \mathfrak{A}_{\Psi 1}$, then $[\tau_\Psi(z)Q, Q_1] = 0$ for any $Q_1 \in R'$ and real z, hence for all z by an analytic continuation. Therefore $\tau_\Psi(z)Q \in \mathfrak{A}_{\Psi 1}$. Similarly, if $Q \in \mathfrak{A}_{\Psi 2}$, then $\tau_\Psi(z)Q \in \mathfrak{A}_{\Psi 2}$ for all z.

For any L^1 function f, we define

$$Q(f) = \int \Delta_\Psi^{it} Q \Delta_\Psi^{-it} f(t) \mathrm{d}t \ . \tag{3.7}$$

It is bounded $\left(\|Q(f)\| \leqq \|Q\| \int |f(t)|\,\mathrm{d}t\right)$, $Q(f) \in R$ if $Q \in R$ and $Q(f) \in R'$ if $Q \in R'$. If $\tilde{f}$ is a C^∞ function such that $e^{\alpha\lambda}\tilde{f}(\lambda)$ is bounded for any real α, and

$$f(t) = (2\pi)^{-1} \int e^{-i\lambda t} \tilde{f}(\lambda) \mathrm{d}\lambda \ , \tag{3.8}$$

then $Q(f) \in \mathfrak{A}_\Psi$ and

$$\tau_\Psi(z) Q(f) = Q(f_z) \ , \tag{3.9}$$

$$f_z(t) = (2\pi)^{-1} \int e^{-i\lambda(t-z)} \tilde{f}(\lambda) \mathrm{d}\lambda \ . \tag{3.10}$$

We shall use the following specific function later:

$$f_\beta^G(t) = (\beta\pi)^{-1/2} \exp\{-t^2/\beta\} \ , \quad \beta > 0 \ . \tag{3.11}$$

It has the property that $Q(f_\beta^G)$ is in the weak closure of convex hull of $\Delta_\Psi^{it} Q \Delta_\Psi^{-it}$ and

$$\lim_{\beta \to 0} Q(f_\beta^G) = Q \ . \tag{3.12}$$

If $\tilde{f}$ has a compact support, then $Q(f)\Psi$ is an analytic vector of Δ_Ψ^α for any real α. Since

$$Q(f)\Psi = \tilde{f}(\log \Delta_\Psi) Q\Psi$$

and $R\Psi$ is dense, such vectors $Q(f)\Psi$ are dense and hence $D_{\Psi 1}$ is a core of Δ_Ψ^z for arbitrary z. Similarly, $D_{\Psi 2}$ is also a core of Δ_Ψ^z for arbitrary z.

LEMMA 4. *Let $Y = \int \lambda \mathrm{d}p_\lambda$ be a positive selfadjoint operator and D be a core of Y. Then D is a core of Y^α for $0 \leqq \alpha \leqq 1$.*

Proof. Any vector in the domain of Y is in the domain of Y^α, $0 \leqq \alpha \leqq 1$. Then

$$\begin{aligned} \| Y^\alpha x \|^2 &= \| p_1 Y^\alpha x \| + \| (1 - p_1) Y^\alpha x \|^2 \\ &\leqq \| p_1 x \|^2 + \| (1 - p_1) Y x \|^2 \\ &\leqq \| x \|^2 + \| Y x \|^2 \ . \end{aligned} \tag{3.13}$$

If $x_n \in D$, $x_n \to x \in D(Y)$ and $Yx_n \to Yx$, then $Y^\alpha x_n$ is Cauchy by (3.13) and hence $x \in D((Y^\alpha \mid D)^-)$. Since $D(Y)$ is a core of Y^α, $0 \leqq \alpha \leqq 1$, D is also a core of Y^α.

LEMMA 5. *For $Q \in R$, the following two conditions are equivalent.*

$$Q\Psi \in D(\Delta_\Psi^{(1/2)+\alpha}) . \tag{3.14}$$

$$Q^*\Psi \in D(\Delta_\Psi^{-\alpha}) . \tag{3.15}$$

If these conditions are satisfied for an $\alpha > 0$, then there exists a family of closable operators $\hat{\tau}_\Psi(z)Q$ for $\operatorname{Im} z \in [-\alpha, 0]$ *with a common domain $D_{\Psi 2}$ such that*

(1) *$\hat{\tau}_\Psi(z)Q$ is affiliated with R,*

(2) *$\hat{\tau}_\Psi(z)Qx$ is continuous in z for* $\operatorname{Im} z \in [-\alpha, 0]$ *and analytic in z for $z \in [-\alpha, 0)$ if $x \in D_{\Psi 2}$,*

(3) *$\hat{\tau}_\Psi(z)Qx = \Delta_\Psi^{iz} Q \Delta_\Psi^{-iz} x$, $x \in D_{\Psi 2}$,*

(4) *$(\hat{\tau}_\Psi(z)Q)^*x = \Delta_\Psi^{i\bar{z}} Q^* \Delta_\Psi^{-i\bar{z}} x$, $x \in D_{\Psi 2}$.*

Proof. Due to $J_\Psi \Delta_\Psi^\alpha = \Delta_\Psi^{-\alpha} J_\Psi$, we have

$$D(\Delta_\Psi^{-\alpha}) = J_\Psi D(\Delta_\Psi^\alpha) . \tag{3.16}$$

Hence (3.15) is equivalent to

$$\Delta_\Psi^{1/2} Q\Psi = J_\Psi Q^*\Psi \in D(\Delta_\Psi^\alpha)$$

which is equivalent to (3.14).

Assume that Q satisfies (3.14) and (3.15). Define an operator A_z on $D_{\Psi 2}$ by

$$A_z Q'\Psi = Q'\Delta_\Psi^{iz} Q\Psi , \quad Q' \in \mathfrak{A}_{\Psi 2} , \tag{3.17}$$

where $\operatorname{Im} z \in [-\alpha, 0]$. By (3.14), $Q\Psi$ is in the domain of Δ_Ψ^{iz} for $\operatorname{Im} z \in [-\alpha, 0]$. Since Ψ is separating for $R' \supset \mathfrak{A}_{\Psi 2}$, A_z is well-defined and linear.

To show that A_z is closable, we show that its adjoint has a dense domain. For Q_1' and Q_2' in $\mathfrak{A}_{\Psi 2}$, we have

$$\begin{aligned}
(Q_1'\Psi, A_z Q_2'\Psi) &= (Q_2'^* Q_1'\Psi, \Delta_\Psi^{iz} Q\Psi) \\
&= (\Delta_\Psi^{-1/2}\{\tau(-\bar{z} - i/2)(Q_2'^* Q_1')\}\Psi, Q\Psi) \\
&= (J_\Psi \Delta_\Psi^{-iz-1/2} Q_1'^* Q_2'\Psi, J_\Psi \Delta_\Psi^{1/2} Q^*\Psi) \\
&= (\Delta_\Psi^{1/2} Q^*\Psi, \Delta_\Psi^{-iz-1/2} Q_1'^* Q_2'\Psi) \\
&= (Q_1' \Delta_\Psi^{i\bar{z}} Q^*\Psi, Q_2'\Psi)
\end{aligned} \tag{3.18}$$

where $Q^*\Psi$ is in the domain of $\Delta_\Psi^{i\bar{z}}$ by (3.15). This proves that $D(A_z^*)$ contains a dense set $D_{\Psi 2}$ and A_z is closable. We denote $A_z = \hat{\tau}_\Psi(z)Q$.

(1) By (3.17), we have

$$Q'_1 A_z Q'_2 \Psi = Q'_1 Q'_2 \Delta_\Psi^{iz} Q \Psi = A_z Q'_1 Q'_2 \Psi$$

for any Q'_1 and Q'_2 in $\mathfrak{A}_{\Psi 2}$. Hence A_z commutes with $Q'_1 \in \mathfrak{A}_{\Psi 2}$ and is affiliated with $(\mathfrak{A}_{\Psi 2})' = R$.

(2) By (3.17), we have

$$(\hat{\tau}_\Psi(z)Q)Q'\Psi = Q'\Delta_\Psi^{iz} Q\Psi$$

which has the stated continuity and analyticity due to (3.14).

(3) This follows from the following computation:

$$\begin{aligned}\Delta_\Psi^{iz} Q \Delta_\Psi^{-iz} Q'\Psi &= \Delta_\Psi^{iz} Q\{\tau_\Psi(-z)Q'\}\Psi \\ &= \Delta_\Psi^{iz}\{\tau_\Psi(-z)Q'\}Q\Psi \\ &= Q'\Delta_\Psi^{iz} Q\Psi = A_z Q'\Psi .\end{aligned}$$

(4) This follows from the following computation where (3.18) is used.

$$\begin{aligned}(Q'_1\Psi, (\hat{\tau}_\Psi(z)Q)Q'_2\Psi) &= (Q'_1\Delta_\Psi^{i\bar{z}} Q^*\Psi, Q'_2\Psi) \\ &= (\Delta_\Psi^{i\bar{z}}\{\tau_\Psi(-\bar{z})Q'_1\}Q^*\Psi, Q'_2\Psi) \\ &= (\Delta_\Psi^{i\bar{z}} Q^*\{\tau_\Psi(-\bar{z})Q'_1\}\Psi, Q'_2\Psi) \\ &= (\Delta_\Psi^{i\bar{z}} Q^*\Delta_\Psi^{-i\bar{z}} Q'_1\Psi, Q'_2\Psi) .\end{aligned}$$

COROLLARY. *For $Q \in R$, the following two conditions are equivalent.*

(3.19) $$Q\Psi \in D(\Delta_\Psi^{-\alpha}) .$$

(3.20) $$Q^*\Psi \in D(\Delta_\Psi^{(1/2)+\alpha}) .$$

If these conditions are satisfied for an $\alpha > 0$, then there exists a family of closable operators $\hat{\tau}_\Psi(z)Q$ for $\operatorname{Im} z \in [0, \alpha]$ with a common domain $D_{\Psi 2}$ such that

(1) *$\hat{\tau}(z)Q$ is affiliated with R,*

(2) *$\hat{\tau}_\Psi(z)Qx$ is continuous in z for $\operatorname{Im} z \in [0, \alpha]$ and analytic in z for $\operatorname{Im} z \in (0, \alpha)$ if $x \in D_{\Psi 2}$,*

(3) *$\hat{\tau}_\Psi(z)Qx = \Delta_\Psi^{iz} Q \Delta_\Psi^{-iz} x$, $x \in D_{\Psi 2}$,*

(4) *$(\hat{\tau}_\Psi(z)Q)^*x = \Delta_\Psi^{i\bar{z}} Q^* \Delta_\Psi^{-i\bar{z}} x$, $x \in D_{\Psi 2}$.*

Proof. Interchange roles of Q and Q^* in Lemma 5 and denote the restriction of $\{\hat{\tau}_\Psi(\bar{z})(Q^*)\}^*$ to $D_{\Psi 2}$ by $\hat{\tau}_\Psi(z)Q$. The only change is in the analyticity at the boundary $\operatorname{Im} z = \alpha$.

LEMMA 6. *Assume that $Q \in R$ and*

(3.21) $$\Delta^\alpha Q\Psi = Q_1\Psi$$

for some $Q_1 \in R$ *and a real* $\alpha \neq 0$. *Then there exists a family of operators* $\tau_\Psi(z)Q \in R$ *for* $\mathrm{Im}\, z$ *between* 0 *and* $-\alpha$ (*i.e., in* $[0, -\alpha]$ *if* $\alpha < 0$ *and* $[-\alpha, 0]$ *if* $\alpha > 0$) *such that*

(1) $\tau_\Psi(z)Q$ *is strongly continuous in z for* $\mathrm{Im}\, z \in [0, -\alpha]$ *or* $[-\alpha, 0]$ *and analytic in z for* $\mathrm{Im}\, z \in (0, -\alpha)$ *or* $(-\alpha, 0)$.

(2) $\tau_\Psi(z)Qx = \Delta_\Psi^{iz} Q \Delta_\Psi^{-iz} x$, $x \in D(\Delta_\Psi^{-iz})$.

(3) $(\tau_\Psi(z)Q)^* x = \Delta_\Psi^{i\bar{z}} Q^* \Delta_\Psi^{-i\bar{z}} x$, $x \in D(\Delta_\Psi^{i\bar{z}})$.

(4) $\| \tau_\Psi(z)Q \| \leqq \max \{\| Q \|, \| Q_1 \|\}$.

(5) $\tau_\Psi(0)Q = Q$, $\tau_\Psi(-i\alpha)Q = Q_1$.

Proof. First assume $\alpha > 0$. Since $Q_1\Psi \in D(\Delta_\Psi^{1/2})$ for any $Q_1 \in R$, (3.21) implies (3.14). Consider

$$f(z) \equiv (x, \hat{\tau}_\Psi(z)Qy)$$

for $x, y \in D_{\Psi 2}$. If $x = Q_1'\Psi$, $y = Q_2'\Psi$, then

$$\begin{aligned} |f(z)| &= |(Q_2'^* Q_1'\Psi, \Delta_\Psi^{iz} Q\Psi)| \\ &\leqq \| Q_2'^* Q_1'\Psi \| \, \| \Delta_\Psi^{-\mathrm{Im}\, z} Q\Psi \| \\ &\leqq \| Q_2'^* Q_1'\Psi \| \{\| \Delta_\Psi^{\alpha} Q\Psi \|^2 + \| Q\Psi \|^2\}^{1/2} \end{aligned}$$

for $\mathrm{Im}\, z \in [-\alpha, 0]$ due to (3.13). Since $f(z)$ is continuous for $\mathrm{Im}\, z \in [-\alpha, 0]$ and is holomorphic for $\mathrm{Im}\, z \in (-\alpha, 0)$, the three line theorem is applicable.

On the boundary $\mathrm{Im}\, z = 0$, we have

$$|f(t)| \leqq \| x \| \, \| y \| \, \| Q \| , \quad t \text{ real.}$$

For $z = s - i\alpha$, we have

$$\begin{aligned} \{\hat{\tau}_\Psi(z)Q\}Q'\Psi &= Q'\Delta_\Psi^{iz} Q\Psi \\ &= Q'\Delta_\Psi^{is} Q_1\Psi = \{\Delta_\Psi^{is} Q_1 \Delta_\Psi^{-is}\}Q'\Psi . \end{aligned}$$

Hence

$$|f(s - i\alpha)| \leqq \| x \| \, \| y \| \, \| Q_1 \| , \quad s \text{ real.}$$

Therefore,

$$|f(z)| \leqq \| x \| \, \| y \| \max \{\| Q_1 \|, \| Q \|\} .$$

This implies that $\hat{\tau}(z)Q$, $\mathrm{Im}\, z \in [-\alpha, 0]$ is bounded. We denote its closure by $\tau(z)Q$. It satisfies (4) due to the above estimate. (5) follows from definition. From (1) of Lemma 5, $\tau(z)Q \in R$. Since $D_{\Psi 2}$ is a core of Δ_Ψ^{iz} for any z, we have (2) and (3) from (3) and (4) of Lemma 5.

(1) holds on a dense set $D_{\Psi 2}$ by (2) of Lemma 5. Due to the uniform boundedness (4), the continuity statement holds on any vector. Then analyticity statement also holds on any vector by Cauchy integral theorem.

The proof for the case $\alpha < 0$ is the same as the case $\alpha > 0$.

4. The cone V_Ψ^α. Let V_Ψ^α be the weak closure of the set of vectors

$$\{\Delta_\Psi^\alpha Q\Psi;\ Q \in R,\ Q \geqq 0\} \tag{4.1}$$

where $\alpha \in [0, 1/2]$. V_Ψ^0 is $\mathscr{P}^\#$ of Takesaki [9]. Since $\Delta_\Psi^{1/2} Q\Psi = J_\Psi Q\Psi = j_\Psi(Q)\Psi$ for $Q \in R$, $Q \geqq 0$, $V_\Psi^{1/2}$ is $\mathscr{P}^b$ of Takesaki.

THEOREM 3.

(1) *V_Ψ^α is a pointed weakly closed convex cone invariant under Δ_Ψ^{it}.*

(2) *$\Phi \in V_\Psi^\alpha$ is in the domain of $\Delta_\Psi^{1/2-2\alpha}$ and*

$$J_\Psi \Phi = \Delta_\Psi^{1/2-2\alpha}\Phi\ . \tag{4.2}$$

(3) *$\Delta_\Psi^\alpha V_\Psi^0$ is a dense subset of V_Ψ^α.*

(4) *$J_\Psi V_\Psi^\alpha = V_\Psi^{1/2-\alpha}$.*

(5) *The dual of V_Ψ^α is $V_\Psi^{1/2-\alpha}$.*

(6) *$V_\Psi^\alpha = \Delta_\Psi^{\alpha-1/4}\{V_\Psi^{1/4} \cap D(\Delta_\Psi^{\alpha-1/4})\}$.*

(7) *If $Q \in R$ and $Q\Psi \in V_\Psi^\alpha$, then $\Delta_\Psi^{iz} Q \Delta_\Psi^{-iz}$ is bounded by $\|Q\|$ for* $\operatorname{Im} z \in [0, 2\alpha]$ *and satisfies*

$$(\Delta_\Psi^{-2\alpha} Q \Delta_\Psi^{2\alpha})^- = Q^*\ , \tag{4.3}$$

$$(\Delta_\Psi^{-\alpha} Q \Delta_\Psi^{\alpha})^- \geqq 0\ , \tag{4.4}$$

where the bar indicates the closure.

Conversely, if $\Delta_\Psi^{-\alpha} Q \Delta_\Psi^\alpha$ is a positive bounded operator with a dense domain affiliated with R, then $Q\Psi \in V_\Psi^\alpha$.

(8) *If $\Phi \in V_\Psi^\alpha$, $\alpha \leqq 1/4$ and $\omega_\Phi \leqq l\omega_\Psi$ for some $l > 0$, then there exists $Q \in R$ such that*

$$\Phi = Q\Psi\ , \qquad \|Q\| \leqq l^{1/2}\ , \tag{4.5}$$

$(\Delta_\Psi^{iz} Q \Delta_\Psi^{-iz})^-$ is bounded by $l^{1/2}$ for $\operatorname{Im} z \in [2\alpha - 1/2, 1/2]$.

(9) *If $Q\Psi \in V_\Psi^\alpha$, $Q \in R$, then $(\|Q\| - Q)\Psi \in V_\Psi^\alpha$.*

Proof. V_Ψ^α is obviously a weakly closed convex cone. Since

$$\Delta_\Psi^{it}(\Delta_\Psi^\alpha Q\Psi) = \Delta_\Psi^\alpha Q_t \Psi\ , \quad Q_t \equiv \Delta_\Psi^{it} Q \Delta_\Psi^{-it}\ ,$$

and $Q_t \in R$, $Q_t \geqq 0$, V_Ψ^α is invariant under Δ_Ψ^{it}.

We shall prove that V_Ψ^α is pointed after (6).

(2) If $Q \in R$, $Q \geqq 0$, we have

$$\begin{aligned} J_\Psi(\Delta_\Psi^\alpha Q\Psi) &= \Delta_\Psi^{-\alpha} J_\Psi Q\Psi = \Delta_\Psi^{1/2-\alpha} Q\Psi \\ &= \Delta_\Psi^{(1/2-2\alpha)}(\Delta_\Psi^\alpha Q\Psi)\ . \end{aligned}$$

Hence $\Delta_{\Psi}^{\alpha}Q\Psi$ satisfies (4.2).

Since J_{Ψ} is bounded and $\Delta_{\Psi}^{(1/2-2\alpha)}$ is closed, (4.2) holds for any Φ in the strong closure of the set (4.1). Since the set (4.1) is convex, its strong and weak closures coincide.

(3) Since (4.1) is convex, V_{Ψ}^{α} is the strong closure of (4.1). If $\Phi \in V_{\Psi}^{0}$, there exists $Q_n \in R$, $Q_n \geqq 0$ satisfying $\lim Q_n\Psi = \Phi$. By (3.13),

$$\begin{aligned} \| \Delta_{\Psi}^{\alpha}(Q_n\Psi - \Phi) \|^2 &\leqq \| \Delta_{\Psi}^{1/2}(Q_n\Psi - \Phi) \|^2 + \| Q_n\Psi - \Phi \|^2 \\ &= \| J_{\Psi}(Q_n\Psi - \Phi) \|^2 + \| Q_n\Psi - \Phi \|^2 \rightarrow 0 \,. \end{aligned}$$

This proves $\Delta_{\Psi}^{\alpha}\mathscr{P}^{\natural} \subset V_{\Psi}^{\alpha}$. By definition, $\Delta_{\Psi}^{\alpha}\mathscr{P}^{\natural}$ contains a dense subset of V_{Ψ}^{α}.

(4) This follows from $J_{\Psi}^2 = 1$ and

$$J_{\Psi}\Delta_{\Psi}^{\alpha}Q\Psi = \Delta_{\Psi}^{1/2-\alpha}Q\Psi$$

for $Q \in R$, $Q \geqq 0$.

(5) Let $Q_1, Q_2 \in R$, $Q_1 \geqq 0$, $Q_2 \geqq 0$. Then

$$\begin{aligned} (\Delta_{\Psi}^{\alpha}Q_1\Psi, \Delta_{\Psi}^{1/2-\alpha}Q_2\Psi) &= (Q_1\Psi, \Delta_{\Psi}^{1/2}Q_2\Psi) \\ &= (\Psi, Q_1 j_{\Psi}(Q_2)\Psi) \geqq 0 \end{aligned}$$

due to $Q_1 \geqq 0$, $j_{\Psi}(Q_2) \geqq 0$ and $[Q_1, j_{\Psi}(Q_2)] = 0$. Hence

$$(V_{\Psi}^{\alpha})' \supset V_{\Psi}^{1/2-\alpha} \tag{4.6}$$

where $(V_{\Psi}^{\alpha})'$ denotes the set of all Φ such that $(\Phi, x) \geqq 0$ for every $x \in V_{\Psi}^{\alpha}$.

Next let $\Phi \in (\Delta_{\Psi}^{\alpha}\mathscr{P}^{\natural})'$. Let f_{β}^{G} be given by (3.11) and let

$$\Phi_{\beta} \equiv \int \Delta_{\Psi}^{it}\Phi f_{\beta}^{G}(t)\mathrm{d}t \,. \tag{4.7}$$

Since $\Delta_{\Psi}^{\alpha}\mathscr{P}^{\natural}$ is invariant under Δ_{Ψ}^{it}, we have $\Phi_{\beta} \in (\Delta_{\Psi}^{\alpha}\mathscr{P}^{\natural})'$. Furthermore,

$$\Delta_{\Psi}^{iz}\Phi_{\beta} = \int \Delta_{\Psi}^{it}\Phi f_{\beta}^{G}(t - z)\mathrm{d}t \tag{4.8}$$

for real z and the right hand side has an analytic continuation to all z. Hence Φ_{β} is an entire vector of $\log \Delta_{\Psi}$ and is in domain of Δ_{Ψ}^{iz} for arbitrary z. Hence

$$\Delta_{\Psi}^{\alpha}\Phi_{\beta} \in (\mathscr{P}^{\natural})' = \mathscr{P}^{b} = \Delta_{\Psi}^{1/2}V_{\Psi}^{0}$$

where the last equality is due to (2) and (4), for example, and the first equality is due to [9]. Hence $\Phi_{\beta} \in \Delta^{1/2-\alpha}\mathscr{P}^{\natural}$. By (3), $\Phi_{\beta} \in V^{1/2-\alpha}$. Since $\Phi = \lim_{\beta\to 0}\Phi_{\beta}$, we have $\Phi \in V_{\Psi}^{1/2-\alpha}$.

By (3), we now have

$$(V_{\Psi}^{\alpha})' \subset (\Delta_{\Psi}^{\alpha}\mathscr{P}^{\natural})' \subset V_{\Psi}^{1/2-\alpha} \,. \tag{4.9}$$

By (4.6) and (4.9) we have (5).

(6) First consider the case $\alpha < 1/4$.

For $\Phi \in V_\Psi^\alpha$, there exists $Q_n \in R$, $Q_n \geqq 0$ such that $\Phi = \lim \Delta_\Psi^\alpha Q_n \Psi$. We use (3.13), in which we replace x by $\Delta_\Psi^\alpha(Q_n - Q_m)\Psi$, Y by $\Delta_\Psi^{(1/2-2\alpha)}$ and α by 1/2. We have

$$\begin{aligned} &\| \Delta_\Psi^{(1/4-\alpha)}\{\Delta_\Psi^\alpha(Q_n - Q_m)\Psi\} \|^2 \\ &\leqq \| \Delta_\Psi^\alpha(Q_n - Q_m)\Psi \|^2 + \| \Delta_\Psi^{1/2-\alpha}(Q_n - Q_m)\Psi \|^2 \\ &= \| \Delta_\Psi^\alpha(Q_n - Q_m)\Psi \|^2 + \| J_\Psi \Delta_\Psi^\alpha(Q_n - Q_m)\Psi \|^2 . \end{aligned}$$

Hence $\Delta_\Psi^{1/4}Q_n\Psi$ is Cauchy and has a strong limit $\Delta_\Psi^{1/4-\alpha}\Phi$, which must be in $V_\Psi^{1/4}$ by definition. Hence

$$V_\Psi^\alpha \subset \Delta_\Psi^{\alpha-1/4}\{V_\Psi^{1/4} \cap D(\Delta_\Psi^{\alpha-1/4})\} .$$

Let $x \in V_\Psi^{1/4} \cap D(\Delta_\Psi^{\alpha-1/4})$ and $y \in V_\Psi^0$. Then

$$(\Delta_\Psi^{1/2-\alpha}y, \Delta_\Psi^{\alpha-1/4}x) = (\Delta_\Psi^{1/4}y, x) \geqq 0$$

due to $\Delta_\Psi^{1/4}y \in V_\Psi^{1/4} \subset (V_\Psi^{1/4})'$. By (3),

$$(V_\Psi^{1/2-\alpha})' \supset \Delta_\Psi^{\alpha-1/4}\{V_\Psi^{1/4} \cap D(\Delta_\Psi^{\alpha-1/4})\} .$$

By (5), $(V_\Psi^{1/2-\alpha})' = V_\Psi^\alpha$ and hence we have (6).

The case $\alpha > 1/4$ follows from the case $\alpha < 1/4$ by (4).

(1) Let $\Phi \in V_\Psi^\alpha$ and $-\Phi \in V_\Psi^\alpha$. By (5), $\Phi \perp V_\Psi^{1/2-\alpha}$. The linear span of $V_\Psi^{1/2-\alpha}$ contains $\Delta_\Psi^{1/2-\alpha}\mathfrak{A}_{\Psi 1}\Psi = \mathfrak{A}_{\Psi 1}\Psi$, which is dense. Hence $\Phi = 0$ and V_Ψ^α is pointed.

(7) If $Q\Psi \in V_\Psi^\alpha$, then $Q\Psi \in D(\Delta_\Psi^{1/2-2\alpha})$ and

$$J_\Psi Q\Psi = \Delta_\Psi^{1/2}Q^*\Psi = \Delta_\Psi^{1/2-2\alpha}Q\Psi$$

due to (4.2). Hence $Q\Psi \in D(\Delta_\Psi^{-2\alpha})$ and

$$\Delta_\Psi^{-2\alpha}Q\Psi = Q^*\Psi .$$

By Lemma 6, we obtain the first half of (7) except for (4.4).

By (3) and (4), $V_\Psi^{1/2-\alpha} \supset \Delta_\Psi^{-\alpha}\mathscr{P}^b$. By (5),

$$0 \leqq (\Delta_\Psi^{-\alpha}x, Q\Psi) = (x, \tau_\Psi(i\alpha)Q\Psi)$$

for all $x \in \mathscr{P}^b$. Hence $\tau_\Psi(i\alpha)Q \geqq 0$ which shows (4.4).

Let $Q_1 = (\Delta_\Psi^{-\alpha}Q\Delta_\Psi^\alpha)^-$. Then $Q\Delta_\Psi^\alpha\Phi = \Delta_\Psi^\alpha Q_1\Phi$ holds for a dense set of vectors Φ. Hence $\Delta_\Psi^\alpha Q^*\Psi = Q_1^*\Psi$, which implies $\Delta_\Psi^{1/2-\alpha}Q\Psi = J_\Psi\Delta_\Psi^\alpha Q^*\Psi = \Delta_\Psi^{1/2}Q_1\Psi$. Therefore $Q\Psi = \Delta_\Psi^\alpha Q_1\Psi$. Since $Q_1 \geqq 0$ by (4.4), $Q\Psi \in V_\Psi^\alpha$.

(8) If $\omega_\Phi \leqq l\omega_\Psi$, there exists $Q' \in R'$ such that $\omega_\Phi = \omega_{Q'\Psi}$, and $\| Q' \| \leqq l^{1/2}$. Then there exists a partial isometry u' in R' such that

$$\Phi = u'Q'\Psi .$$

By (4.2) we have

$$\Delta_{\Psi}^{1/2-2\alpha}\Phi = J_{\Psi}\Phi = j_{\Psi}(u'Q')\Psi .$$

By (4), $J_{\Psi}\Phi \in V_{\Psi}^{1/2-\alpha}$ and hence by (7), $Q_1 = j_{\Psi}(u'Q') \in R$ has bounded $\tau_{\Psi}(z)Q_1$ for $\operatorname{Im} z \in [0, 1-2\alpha]$. Setting $Q = \tau_{\Psi}(i/2 - 2i\alpha)Q_1$, we have $\Phi = Q\Psi$, $Q \in R$ and $\|Q\| \leqq \|Q_1\| \leqq l^{1/2}$. $(\Delta_{\Psi}^{iz} Q \Delta_{\Psi}^{-iz})^{-} = \tau_{\Psi}(z')Q_1$ with $z' = z + (1/2 - 2\alpha)i$ and hence is bounded by $l^{1/2}$ for $\operatorname{Im} z \in [2\alpha - 1/2, 1/2]$ and is positive for $\operatorname{Im} z = \alpha$.

(9) If $Q\Psi \in V_{\Psi}^{\alpha}$, $Q \in R$, then $\Delta_{\Psi}^{\alpha} Q \Delta_{\Psi}^{-\alpha}$ is bounded by $\|Q\|$, symmetric and affiliated with R due to (7). Hence

$$\Delta_{\Psi}^{\alpha}(\|Q\| - Q)\Delta_{\Psi}^{-\alpha} = \|Q\| - \Delta_{\Psi}^{\alpha} Q \Delta_{\Psi}^{-\alpha}$$

is bounded, positive and affiliated with R. By the last half of (7), $(\|Q\| - Q)\Psi \in V_{\Psi}^{\alpha}$.

5. The cone V_{Ψ}. We denote $V_{\Psi} = V_{\Psi}^{1/4}$ due to an importance of $V_{\Psi}^{1/4}$.

THEOREM 4. *Let Ψ be a cyclic and separating vector for R on H.*

(1) *V_{Ψ} is a pointed closed selfdual convex cone.*

(2) *V_{Ψ} satisfies*

(5.1) $$\Delta_{\Psi}^{it} V_{\Psi} = V_{\Psi}, \qquad -\infty < t < \infty .$$

(5.2) $$J_{\Psi} x = x , \qquad x \in V_{\Psi} .$$

(5.3) $$Q j_{\Psi}(Q) V_{\Psi} \subset V_{\Psi} , \qquad Q \in R .$$

(5.4) $$(x, Q j_{\Psi}(Q) y) \geqq 0 , \quad x, y \in V_{\Psi} , \quad Q \in R .$$

(3) *V_{Ψ} is the strong closure of the set of*

(5.5) $$Q j_{\Psi}(Q)\Psi , \qquad Q \in R .$$

(4) *If $\Phi \in V$ and Φ is separating or cyclic for R, then Φ is separating and cyclic for R and $V_{\Phi} = V_{\Psi}$.*

(5) *If Φ is a cyclic and separating vector for R, then $\Phi \in V_{\Psi}$ if and only if $J_{\Phi} = J_{\Psi}$ and*

(5.6) $$(\Phi, z\Psi) \geqq 0$$

for all $z \in R \cap R'$, $z \geqq 0$.

(6) *Any $\Phi \in H$ has a unique decomposition*

(5.7) $$\Phi = \Phi_1 - \Phi_2 + i(\Phi_3 - \Phi_4)$$

such that $\Phi_i \in V_{\Psi}$, $i = 1, 2, 3, 4$, and

(5.8) $$\Phi_1 \perp \Phi_2 , \qquad \Phi_3 \perp \Phi_4 .$$

(7) *If* $\Phi_1 \in V_\Psi$, $\Phi_2 \in V_\Psi$ *and* $\Phi_1 \perp \Phi_2$, *then*

$$s^R(\Phi_1) \perp s^R(\Phi_2) , \quad s^{R'}(\Phi_1) \perp s^{R'}(\Phi_2) , \tag{5.9}$$

where $s^R(\Phi)$ *and* $s^{R'}(\Phi)$ *denote projections onto closures of* $R'\Phi$ *and* $R\Phi$, *respectively.*

(8) *If* $\Phi_1 \in V_\Psi$ *and* $\Phi_2 \in V_\Psi$, *then*

$$\| \omega^R_{\Phi_1} - \omega^R_{\Phi_2} \| \geqq \| \Phi_1 - \Phi_2 \|^2 \tag{5.10}$$

where ω^R_Φ *is the expectation functional on* R *by a vector* Φ.

Proof. (1), (5.1) and (5.2) follows from Theorem 3. Because

$$\{Qj_\Psi(Q)\}\{Q_1 j_\Psi(Q_1)\} = (QQ_1)j_\Psi(QQ_1) ,$$

(5.3) follows from (3). (5.4) then follows by $V'_\Psi = V_\Psi$.

(3) Let $Q(f^G_\beta)$ be given by (3.7) and (3.11) for $Q \in R$. Then

$$\begin{aligned} Q(f^G_\beta)j_\Psi(Q(f^G_\beta))\Psi &= Q(f^G_\beta)\Delta_\Psi^{1/2}Q(f^G_\beta)^*\Psi \\ &= \Delta_\Psi^{1/4}Q_1Q_1^*\Psi \in V_\Psi \end{aligned}$$

where

$$Q_1 = \tau_\Psi(i/4)Q(f^G_\beta) \in R .$$

Hence

$$Qj_\Psi(Q)\Psi = \lim_{\beta\to 0} Q(f^G_\beta)j_\Psi(Q(f^G_\beta))\Psi \in V_\Psi .$$

On the other hand, if we set

$$Q_{2\beta} \equiv \tau_\Psi(-i/4)\{Q^{1/2}(f^G_\beta)\} , \quad Q \in R , \quad Q \geqq 0 ,$$

then

$$Q_{2\beta}j_\Psi(Q_{2\beta})\Psi = Q_{2\beta}\Delta_\Psi^{1/2}Q_{2\beta}^*\Psi = \Delta_\Psi^{1/4}Q^{1/2}(f^G_\beta)^2\Psi .$$

We have

$$\begin{aligned} &\lim_{\beta\to 0} Q^{1/2}(f^G_\beta)^2\Psi = (Q^{1/2})^2\Psi = Q\Psi , \\ &\qquad \| \Delta_\Psi^{1/4}\{Q^{1/2}(f^G_\beta)^2\Psi - Q\Psi\} \|^2 \\ &\qquad \leqq \| \Delta_\Psi^{1/2}\{Q^{1/2}(f^G_\beta)^2\Psi - Q\Psi\} \|^2 + \| Q^{1/2}(f^G_\beta)^2\Psi - Q\Psi \|^2 \\ &\qquad = 2 \| Q^{1/2}(f^G_\beta)^2\Psi - Q\Psi \|^2 \to 0 . \end{aligned}$$

Hence $\Delta_\Psi^{1/4}\mathscr{P}^\sharp$ is in the strong closure of the set (5.5) and we have (3).

(4) If $R'\Phi$ or $R\Phi$ is dense, then $R\Phi = J_\Psi R'J_\Psi\Phi = J_\Psi R'\Phi$ or $R'\Phi = J_\Psi RJ_\Psi\Phi = J_\Psi R\Phi$ is dense. Hence if Φ in V_Ψ is separating or cyclic, then Φ is cyclic and separating. If $\Phi \in V_\Psi$, then J_Ψ satisfies

$$J_\Psi \Phi = \Phi , \qquad (\Phi, Qj_\Psi(Q)\Phi) \geqq 0$$

due to (5.2) and (5.4). Hence $J_\Phi = J_\Psi$ by Theorem 1. Since V_Φ is the strong closure of $Qj_\Phi(Q)\Phi$, we have $V_\Phi \subset V_\Psi$ due to (5.3) and $J_\Psi = J_\Phi$. Since V_Φ and V_Ψ are selfdual, we have $V_\Phi = V'_\Phi \supset V'_\Psi = V_\Psi$ and hence $V_\Phi = V_\Psi$.

(5) If $\Phi \in V_\Psi$, then $J_\Phi = J_\Psi$ as we have seen and (5.6) holds because $z = z^{1/2} j_\Psi(z^{1/2})$ due to Lemma 3. Conversely, assume $J_\Phi = J_\Psi$. By (6) and (7), which we shall prove below, we have

$$\Phi = \Phi_1 - \Phi_2 , \quad \Phi_1 \in V_\Psi , \quad \Phi_2 \in V_\Psi , \tag{5.11}$$

$$s^R(\Phi_1) \perp s^R(\Phi_2) . \tag{5.12}$$

Assume that $(\Phi_1, Qj_\Psi(Q)\Phi_2) > 0$ for some $Q \in R$. Let $Q_1 = s^R(\Phi_1)Qs^R(\Phi_2)$. We then have by (5.12)

$$\begin{aligned}&(\Phi, Q_1 j_\Psi(Q_1)\Phi)\\ &\quad = -(\Phi_1, Q_1 j_\Psi(Q_1)Q_2) = -(\Phi_1, Qj_\Psi(Q)\Phi_2) < 0 ,\end{aligned}$$

where we have used $s^R(\Phi_k)\Phi_k = \Phi_k$, $j_\Psi\{s^R(\Phi_k)\} = s^{R'}(\Phi_k)$ (because of $J_\Psi R'\Phi_k = j_\Psi(R')J_\Psi\Phi_k = R\Phi_k$) and $s^{R'}(\Phi_k)\Phi_k = \Phi_k$, in the second equality. This contradicts with $J_\Psi = J_\Phi$ and (5.4) for the cone V_Φ. Hence

$$(\Phi_1, Qj_\Psi(Q)\Phi_2) = 0 \tag{5.13}$$

due to (5.4) and (5.11).

From (5.13), we have

$$s^{W'}(\Phi_1) \perp s^{W'}(\Phi_2)$$

where W is the von Neumann algebra generated by $Qj_\Psi(Q)$. By Lemma 1, $W' = R \cap R'$. Hence $z \equiv s^{W'}(\Phi_2) \in R \cap R'$ and

$$(\Psi, z\Phi) = -(\Psi, \Phi_2) \geqq 0$$

by (5.6). Since $\Phi_2 \in V_\Psi$, we have $(\Psi, \Phi_2) \geqq 0$ by $V'_\Psi = V_\Psi$ and hence $(\Psi, \Phi_2) = 0$. We shall see that this implies $\Phi_2 = 0$ in the proof of (7) and hence $\Phi = \Phi_1 \in V_\Psi$.

(6) Let $\Phi \in H$. Define

$$\Phi_r = 2^{-1}(\Phi + J_\Psi\Phi) , \quad \Phi_i = (2i)^{-1}(\Phi - J\Phi_\Psi) . \tag{5.14}$$

Then

$$\Phi = \Phi_r + i\Phi_i , \quad J_\Psi\Phi_r = \Phi_r , \quad J_\Psi\Phi_i = \Phi_i . \tag{5.15}$$

Conversely, if (5.15) is satisfied, Φ_r and Φ_i are uniquely given by (5.14).

We now show that any $\Phi \in H$ satisfying $J_\Psi\Phi = \Phi$ has a unique decomposition

$$\Phi = \Phi_1 - \Phi_2 \,, \quad \Phi_1 \in V_\Psi \,, \quad \Phi_2 \in V_\Psi \,, \quad \Phi_1 \perp \Phi_2 \,. \tag{5.16}$$

Let

$$d = \inf \{ \| \Phi - \Phi' \| ; \Phi' \in V_\Psi \} \tag{5.17}$$

$$\lim_n \| \Phi'_n - \Phi \| = d \,, \qquad \Phi'_n \in V_\Psi \,. \tag{5.18}$$

Since (5.18) implies that the sequence Φ'_n is uniformly bounded, there exists a weakly converging subsequence $\Phi'_{n(k)}$:

$$w - \lim_k \Phi'_{n(k)} = \Phi_1 \,.$$

Then

$$\| \Phi - \Phi_1 \|^2 = \| \Phi_1 \|^2 + d^2 - \lim \| \Phi'_{n(k)} \|^2 \,.$$

By (5.17) and $\| \Phi_1 \|^2 \leqq \lim \| \Phi'_{n(k)} \|^2$, we have

$$\| \Phi - \Phi_1 \|^2 = d^2 \,. \tag{5.19}$$

Let $\Phi_2 = \Phi_1 - \Phi$ and $x \in V_\Psi$. Then $\Phi_1 + \lambda x \in V_\Psi$ for $\lambda \geqq 0$. We have from (5.17) and (5.19)

$$\begin{aligned} \| \Phi_2 \|^2 = d^2 &\leqq \| \Phi - (\Phi_1 + \lambda x) \|^2 \\ &= \| \Phi_2 \|^2 + \lambda \{ 2(\Phi_2, x) + \| x \|^2 \lambda \} \end{aligned}$$

where (Φ_2, x) is real due to $J_\Psi \Phi_2 = \Phi_2$ and $J_\Psi x = x$. We then have

$$(\Phi_2, x) \geqq 0$$

which implies $\Phi_2 \in V'_\Psi = V_\Psi$.

Since Φ_1 and Φ_2 are in V_Ψ, $(\Phi_1, \Phi_2) \geqq 0$. For $\lambda > 0$,

$$d^2 \leqq \| \Phi - (1 - \lambda)\Phi_1 \|^2 = \| \Phi_2 \|^2 - \lambda (2(\Phi_1, \Phi_2) - \lambda \| \Phi_1 \|^2)$$

which implies $(\Phi_1, \Phi_2) = 0$.

To prove the uniqueness of the decomposition (5.16), let $\Phi = \Phi_1 - \Phi_2 = \Phi'_1 - \Phi'_2$ be two such decompositions. For any vectors x_1, x_2, x_3, we have

$$G(x_1, x_2, x_3) \equiv \det ((x_i, x_j)) (= \mathrm{dex}\, X^* X) \geqq 0 \,. \tag{5.20}$$

Since (Φ_k, Φ'_l) are all real, we have

$$\begin{aligned} 0 &\leqq G(\Phi, \Phi'_1, -\Phi_2) \\ &= (\| \Phi_1 \|^2 - \| \Phi'_1 \|^2) \| \Phi'_1 \|^2 \| \Phi_2 \|^2 \\ &\quad - (\Phi'_1, \Phi_2)^2 \| \Phi \|^2 - 2 \| \Phi'_1 \|^2 \| \Phi_2 \|^2 (\Phi'_1, \Phi_2) \,, \end{aligned} \tag{5.21}$$

$$
\begin{aligned}
& 0 \leqq G(\Phi, \Phi_1, -\Phi_2') \\
(5.22) \qquad & = (\|\Phi_1'\|^2 - \|\Phi_1\|^2)\|\Phi_1\|^2\|\Phi_2'\|^2 \\
& \quad - (\Phi_1, \Phi_2')^2\|\Phi\|^2 - 2\|\Phi_1\|^2\|\Phi_2'\|^2(\Phi_1, \Phi_2') .
\end{aligned}
$$

Since $(\Phi_k, \Phi_l') \geqq 0$ by $V_\Psi' = V_\Psi$, either all terms in (5.21) are negative or all terms in (5.22) are negative. In the first case, all terms in (5.21) vanish and we have the following three alternatives:

Case (i). $\Phi_1' = 0, \Phi = -\Phi_2'$. Then

$$\|\Phi_1\|^2 = (\Phi_1, \Phi) = -(\Phi_1, \Phi_2') \leqq 0$$

and hence $\Phi_1 = 0 = \Phi_1'$ and $\Phi_2 = -\Phi = \Phi_2'$.

Case (ii). $\Phi_2 = 0, \Phi = \Phi_1$. Then

$$\|\Phi_2'\|^2 = -(\Phi_2', \Phi) = -(\Phi_2', \Phi_1) \leqq 0$$

and hence $\Phi_2' = 0 = \Phi_2, \Phi_1' = \Phi = \Phi_1$.

Case (iii). $(\Phi_1', \Phi_2) = 0$ and $\|\Phi_1\|^2 = \|\Phi_1'\|^2$. Then

$$\|\Phi_1\|^2 = \|\Phi_1'\|^2 = (\Phi_1', \Phi) = (\Phi_1', \Phi_1)$$

which implies $\|\Phi_1 - \Phi_1'\|^2 = 0$. Hence $\Phi_1 = \Phi_1', \Phi_2 = \Phi_2'$. If all terms in (5.22) vanish, we have the same argument.

(7) First we prove that any nonzero $\Phi \in V_\Psi$ is never orthogonal to Ψ. By (3), there exists $Q_n \in R$ such that

$$\Phi = \lim_n Q_n j_\Psi(Q_n)\Psi .$$

Assume that $(\Psi, \Phi) = 0$. Then

$$
\begin{aligned}
0 &= \lim_n (\Psi, Q_n j_\Psi(Q_n)\Psi) \\
&= \lim_n \|\Delta_\Psi^{1/4} Q_n \Psi\|^2 .
\end{aligned}
$$

Let $x = Qj(Q)\Psi, Q' \in R, Q = Q'(f_\beta^G)$. Then

$$
\begin{aligned}
(x, \Phi) &= \lim (x, Q_n j_\Psi(Q_n)\Psi) \\
&= \lim (j_\Psi(Q_n^* Q)\Psi, Q^* Q_n \Psi) \\
&= \lim \|\Delta_\Psi^{1/4} Q^* Q_n \Psi\|^2 \\
&= \lim \|\{\tau_\Psi(i/4)Q\}^* \Delta_\Psi^{1/4} Q_n \Psi\|^2 \\
&= 0 .
\end{aligned}
$$

By (3.12) and Lemma 1 (or (3) and (6)), such x is total in H and hence $\Phi = 0$.

Since $V_{\Phi_1} = V_\Psi$ for any separating Φ_1 in V_Ψ, we have

$$(5.23) \qquad (\Phi_1, \Phi_2) > 0$$

if $\Phi_1 \in V_\Psi, \Phi_2 \in V_\Psi, \Phi_1$ is separating for R and $\Phi_2 \neq 0$.

We now assume that $\Phi_1 \in V_\Psi$, $\Phi_2 \in V_\Psi$ and $\Phi_1 \perp \Phi_2$. Let s and s' denote $s^R(\Phi_1)$ and $s^{R'}(\Phi_1)$, respectively. Since $J_\Psi R'\Phi_1 = j_\Psi(R')\Phi_1 = R\Phi_1$, we have $j_\Psi(s) = s'$. Hence J_Ψ commutes with ss'.

Consider the space $\hat{H} = ss'H$ and a von Neumann algebra $\hat{R} = sRss'$ on $\hat{H}$. Φ_1 is in $\hat{H}$ and is cyclic and separating for $\hat{R}$ by definition of s and s'. Since J_Ψ commutes with ss', the restriction of J_Ψ to $\hat{H}$ is the modular conjugation operator $\hat{J}_{\Phi_1}$ for Φ_1 on $\hat{H}$ due to Theorem 1. We also have

$$ss'Qj_\Psi(Q)\Phi_1 = ss'Qj_\Psi(Q)ss'\Phi_1 = \hat{Q}j_{\Phi_1}(\hat{Q})\Phi_1$$

where $\hat{Q} = sQs$. Hence $ss'V_\Psi = \hat{V}_{\Phi_1}$.

Let $\hat{\Phi}_2 = ss'\Phi_2$. $\hat{\Phi}_2 \in \hat{V}_{\Phi_1}$ because $\Phi_2 \in V_\Psi$. We also have

$$(\hat{\Phi}_2, \Phi_1) = (\Phi_2, \Phi_1) = 0 \ .$$

By (5.23), we have $\hat{\Phi}_2 = 0$.

Denoting $\varphi = (1-s)(1-s')\Phi_2$, $\varphi_1 = s(1-s')\Phi_2$, and $\varphi_2 = (1-s)s'\Phi_2$, we have

$$\Phi_2 = \varphi + \varphi_1 + \varphi_2 \ .$$

Since $J_\Psi\Phi_2 = \Phi_2$, and $j_\Psi(s) = s'$, we have $J_\Psi\varphi_1 = \varphi_2$. We now prove $\varphi_1 = \varphi_2 = 0$.

Assume $\varphi_1 \neq 0$ and let $s_k = s^R(\varphi_k)$, $s'_k = s^{R'}(\varphi_k)$, $k = 1, 2$. Then $j_\Psi(s_1) = s'_2$, $j_\Psi(s_2) = s'_1$, $s_1 \leqq s$, $s_2 \leqq 1 - s$. Let $c(E)$ denote the central support of $E \in (R \cup R')''$. Then $j_\Psi(c(E)) = c(E)^* = c(E)$ by Lemma 3. Hence $c(j_\Psi(E)) = c(E)$. Setting $E = s_1s'_1$, we have $c(s_1s'_1) = c(s_2s'_2)$. Since $s_1s'_1\varphi_1 = \varphi_1 \neq 0$, $c(s_1s'_1) \neq 0$. We have $c(s_1) \geqq c(s_1s'_1) = c(s_2s'_2)$ and $c(s_2) \geqq c(s_2s'_2)$. Therefore, there exists a partial isometry $u \in R$ such that $u^*u \leqq s_1$, $uu^* \leqq s_2$, $c(uu^*) = c(u^*u) = c(s_2s'_2)$.

Since s_1 is the support of φ_1, $u^*u\varphi_1 \neq 0$. Then $s'' \equiv s^{R'}(u^*u\varphi_1) \leqq s'_1$ is nonzero and $c(s'') \leqq c(s_2s'_2) \leqq c(s'_2)$. Hence there exists a partial isometry $v \in R'$ such that $v^*v \leqq s''$, $vv^* \leqq s'_2$, $v \neq 0$. Again $v^*vu^*u\varphi_1 \neq 0$.

Since

$$uv\varphi_1 \in uH \subseteqq s_2H \ , \quad uv\varphi_1 \in vH \subseteqq s'_2H \ ,$$

there exists $A \in s_2Rs_2$ such that

$$\mathrm{Re}\,(uv\varphi_1, A\varphi_2) > 0 \ . \tag{5.24}$$

Let $Q = A^*u - j_\Psi(v)$. A^*u vanishes on $(1-s)H$ and its range is in $(1-s)H$. $j_\Psi(v)$ vanishes on sH and its range is in sH. v vanishes on $s'H$ and its range is in $s'H$. $j_\Psi(A^*u)$ vanishes on $(1-s')H$ and its range is in $(1-s')H$. Therefore,

$$
\begin{aligned}
0 &\leqq (\Phi_2, Qj_\Psi(Q)\Phi_2) \\
&= -(\varphi_1, j_\Psi(A^*u)j_\Psi(v)\varphi_2) - (\varphi_2, A^*uv\varphi_1) \\
&= -2\,\mathrm{Re}\,(\varphi_2, A^*uv\varphi_1)
\end{aligned}
$$

where we have used $J_\Psi\varphi_1 = \varphi_2$, $\varphi_1 = J_\Psi\varphi_2$. This contradicts with (5.24). Therefore $\varphi_1 = \varphi_2 = 0$ and $\Phi_2 = \varphi$.

We now have

$$s^R(\Phi_2) = s^R(\varphi) \leqq 1 - s\ ,$$

$$s^{R'}(\Phi_2) = s^{R'}(\varphi) \leqq 1 - s'\ .$$

Hence (5.9) is satisfied.

(8) For $\Phi_1 \in V_\Psi$ and $\Phi_2 \in V_\Psi$, we have a decomposition

$$\Phi_1 - \Phi_2 = \Phi_+ - \Phi_-$$

satisfying $\Phi_\pm \in V_\Psi$, $\Phi_+ \perp \Phi_-$, due to (6). By (7), we have $s^R(\Phi_+) \perp s^R(\Phi_-)$.

Let $E \equiv s^R(\Phi_+) - s^R(\Phi_-)$. Then $\|E\| \leqq 1$. We have

$$
\begin{aligned}
\|\omega_{\Phi_1} - \omega_{\Phi_2}\| &\geqq \|\omega_{\Phi_1}(E) - \omega_{\Phi_2}(E)\| \\
&= 2^{-1}|(\Phi_1 - \Phi_2, E(\Phi_1 + \Phi_2)) + (\Phi_1 + \Phi_2, E(\Phi_1 - \Phi_2))| \\
&= |\mathrm{Re}\,(\Phi_+ + \Phi_-, \Phi_1 + \Phi_2)| \\
&= (\Phi_+ + \Phi_-, \Phi_1 + \Phi_2) \\
&\geqq (\Phi_+ - \Phi_-, \Phi_1 - \Phi_2) = \|\Phi_1 - \Phi_2\|^2\ ,
\end{aligned}
$$

where we have used $(\Phi_1, \Phi_-) \geqq 0$ and $(\Phi_2, \Phi_+) \geqq 0$ due to $\Phi_1, \Phi_2, \Phi_-, \Phi_+ \in V_\Psi$.

6. Some Radon-Nikodym theorems.

THEOREM 5. *Let μ be a normal positive linear functional on a von Neumann algebra R with a cyclic and separating vector Ψ such that $\mu \leqq \omega_\Psi$. Then there exists $h_\alpha \in R$, $\|h_\alpha\| \leqq 1$, $h_\alpha \geqq 0$ for each $\alpha \in [0,1]$ such that*

$$2\mu(Q) = (\Delta_\Psi^{\alpha/2}Q^*\Psi, \Delta_\Psi^{\alpha/2}h_\alpha\Psi) + (\Delta_\Psi^{\alpha/2}h_\alpha\Psi, \Delta_\Psi^{\alpha/2}Q\Psi)\ . \tag{6.1}$$

Proof. Let $h \in R$, $h^* = h$ and

$$f_h^\alpha(Q) = \{(\Delta_\Psi^{\alpha/2}Q^*\Psi, \Delta_\Psi^{\alpha/2}h\Psi) + (\Delta_\Psi^{\alpha/2}h\Psi, \Delta_\Psi^{\alpha/2}Q\Psi)\}/2\ . \tag{6.2}$$

If $\alpha \leqq 1/2$, then

$$f_h^\alpha(Q) = (1/2)\{(\Psi, Q\Delta_\Psi^\alpha h\Psi) + (\Delta_\Psi^\alpha h\Psi, Q\Psi)\}\ . \tag{6.3}$$

If $\alpha \geqq 1/2$, then

$$
\begin{aligned}
f_h^\alpha(Q) &= (1/2)\{(J_\Psi\Delta_\Psi^{\alpha/2}h\Psi, J_\Psi\Delta_\Psi^{\alpha/2}Q^*\Psi) \\
&\quad + (J_\Psi\Delta_\Psi^{\alpha/2}Q\Psi, J_\Psi\Delta_\Psi^{\alpha/2}h\Psi)\} = f_h^{1-\alpha}(Q)\ .
\end{aligned} \tag{6.4}
$$

Hence f_h^α is a normal linear functional on R. If $Q^* = Q$, then

$$f_h^\alpha(Q) = \operatorname{Re}(\Delta_\Psi^{\alpha/2}Q^*\Psi, \Delta_\Psi^{\alpha/2}h\Psi) \tag{6.5}$$

and hence f_h^α is selfadjoint. Since

$$f_h^\alpha(Q) = \{(\Delta_\Psi^\alpha Q^*\Psi, h\Psi) + (h\Psi, \Delta_\Psi^\alpha Q\Psi)\}/2$$

for $\alpha \leqq 1/2$ and $f_h^\alpha(Q) = f_h^{1-\alpha}(Q)$, f_h^α is weakly continuous in h.

Let F be the set of f_h^α, $h \in R$, $h^* = h$, $1 \geqq h \geqq 0$. Then as an image of a compact, convex set under continuous real linear map, F is weakly compact and convex. F contains 0. Let F^0 be the polar of F, namely the set of $Q \in R$, $Q^* = Q$ and $f(Q) \leqq 1$ for all $f \in F$. Then $(F^0)^0 = F$, where $(F^0)^0$ is the set of all normal linear selfadjoint functionals f satisfying $f(Q) \leqq 1$ for all $Q \in F^0$.

For each real $\alpha \in [0, 1]$, consider

$$m_h^\alpha(Q) = \sup_t \operatorname{Re} f_h(\alpha + it) ,$$
$$f_h(\alpha + it) = (\Delta_\Psi^{(\alpha-it)/2}Q^*\Psi, \Delta_\Psi^{(\alpha+it)/2}h\Psi) .$$

$f_h(z)$ is obviously an analytic function of z for $\operatorname{Re} z \in (0, 1)$. It is continuous for $\operatorname{Re} z \in [0, 1]$. Furthermore,

$$\begin{aligned} |f_h(\alpha + it)| &\leqq \| \Delta_\Psi^{(\alpha-it)/2}Q^*\Psi \| \, \| \Delta_\Psi^{(\alpha+it)/2}h\Psi \| \\ &\leqq \{\| J_\Psi Q\Psi \|^2 + \| Q^*\Psi \|^2\}^{1/2}\{\| J_\Psi h\Psi \|^2 + \| h\Psi \|^2\}^{1/2} . \end{aligned}$$

By the three line theorem,

$$\sup_t \operatorname{Re} f_h(\alpha + it) = \log \sup_t | e^{f_h(\alpha+it)} |$$

is a convex function of α. Hence

$$g^\alpha(Q) = \sup_h \{m_h^\alpha(Q); h \in R, h^* = h, 1 \geqq h \geqq 0\}$$

is also a convex function of α.

Since $f_h(\alpha + it) = f_{h'}(\alpha)$, $h' = \Delta_\Psi^{it}h\Delta_\Psi^{-it}$, we have for $Q^* = Q$

$$g^\alpha(Q) = \sup_h \{f_h^\alpha(Q); h \in R, h^* = h, 1 \geqq h \geqq 0\} .$$

By (6.4) we have

$$g^\alpha(Q) = g^{1-\alpha}(Q) .$$

Due to convexity,

$$g^\alpha(Q) \geqq g^{1/2}(Q) . \tag{6.6}$$

We have

$$\begin{aligned} f_h^{1/2}(Q) &= (\Psi, j_\Psi(h)Q\Psi) \\ &= \omega_\Phi(Q) , \end{aligned}$$

$$\Phi = j_\Psi(h)^{1/2}\Psi .$$

The set of such ω_Φ for $h \in R$, $h^* = h$, $1 \geqq h \geqq 0$ is exactly the set of all normal positive linear functionals μ of R satisfying $\mu \leqq \omega_\Psi$. Hence $g^{1/2}(Q) \geqq \mu(Q)$ and by (6.6)

$$g^\alpha(Q) \geqq \mu(Q)$$

for any $Q^* = Q$, $Q \in R$, $\alpha \in [0, 1]$. Hence $\mu \in (F^0)^0 = F$.

REMARK. $h_{1-\alpha} = h_\alpha$. h_α is unique. (If $\mu = 0$, set $Q = h_\alpha$.)

COROLLARY. *If $\Phi \in V_\Psi^\alpha$, $l\Psi - \Phi \in V_\Psi^\alpha$ and $\alpha \leqq 1/4$, then there exists $h \in R$ such that $0 \leqq h \leqq l$ and*

$$2\Phi = h\Psi + \Delta_\Psi^{2\alpha} h\Psi . \tag{6.7}$$

Such h is unique. If $\Phi \in V_\Psi^\alpha$, $l\Psi - \Phi \in V_\Psi^\alpha$ and $\alpha \geqq 1/4$, then there exists $h' \in R'$ such that $0 \leqq h' \leqq l$ and

$$2\Phi = h'\Psi + \Delta_\Psi^{2\alpha-1} h'\Psi . \tag{6.8}$$

Such h' is unique.

Proof. Let $\alpha \leqq 1/4$, $\beta = 1/2 - \alpha$ and

$$\mu(Q) \equiv (\Phi, \Delta_\Psi^\beta Q\Psi)/l , \qquad Q \in R .$$

Since $\Delta_\Psi^\beta Q\Psi \in V_\Psi^\beta = (V_\Psi^\alpha)'$ for $Q \geqq 0$, we have $\mu \geqq 0$. By $l\Psi - \Phi \in V_\Psi^\alpha$, we also have $\mu \leqq \omega_\Psi$. By applying Theorem 5 to μ and setting $h = lh_\beta$, we have

$$2l\mu(Q) = (h\Psi, \Delta_\Psi^\beta Q\Psi) + (\Delta_\Psi^\beta Q^*\Psi, h\Psi) .$$

Since

$$\begin{aligned}(\Delta_\Psi^\beta Q^*\Psi, h\Psi) &= (J_\Psi h\Psi, J_\Psi \Delta_\Psi^\beta Q^*\Psi)\\ &= (\Delta_\Psi^{1/2} h\Psi, \Delta_\Psi^{1/2-\beta} Q\Psi)\\ &= (\Delta_\Psi^{2\alpha} h\Psi, \Delta_\Psi^\beta Q\Psi) ,\end{aligned}$$

we have (6.7).

If h_1 and h_2 yield the same Φ, then we have for $h = h_1 - h_2$

$$0 = (h\Psi + \Delta_\Psi^{2\alpha} h\Psi, h\Psi) = \| h\Psi \|^2 + \| \Delta_\Psi^\alpha h\Psi \|^2 .$$

Hence $h\Psi = 0$ and $h_1 = h_2$, which proves the uniqueness of h.

If $\alpha \geqq 1/4$, then we interchange the role of R and R'. Then Δ_Ψ^{-1} replaces Δ_Ψ and $1/2 - \alpha$ replaces α. We then obtain the latter half of corollary.

REMARK. If $\alpha = 1/4$, then $\Delta_\Psi^{2\alpha} h\Psi = J_\Psi h\Psi$, $\Delta_\Psi^{2\alpha-1} h'\Psi = J_\Psi h'\Psi$ and hence $h' = j_\Psi(h)$.

THEOREM 6. *For any normal state μ of a von Neumann algebra R with a cyclic and separating vector Ψ, there exists $\Phi \in V_\Psi$ such that $\omega_\Phi = \mu$.*

We first prove a technical lemma.

LEMMA 7. *Let Ψ be a cyclic and separating vector for R and S be an operator in R with a bounded inverse $S^{-1} \in R$ such that $S\Psi \in V_\Psi$. If $\Delta_\Psi^{1/2} Q\Psi = Q_1\Psi$ for some $Q \in R$ and $Q_1 \in R$, then*

$$\Delta_{S\Psi}^{1/2} Q(S\Psi) = Q_2(S\Psi) , \qquad Q_2 = SQ_1S^{-1} . \tag{6.9}$$

Proof. By using $J_\Psi = J_{S\Psi}$ due to $S\Psi \in V_\Psi$, we have

$$\begin{aligned} \Delta_{S\Psi}^{1/2} QS\Psi &= J_{S\Psi} Q^* S\Psi = j_{S\Psi}(Q^*)S\Psi \\ &= Sj_{S\Psi}(Q^*)\Psi = Sj_\Psi(Q^*)\Psi \\ &= SJ_\Psi Q^*\Psi = S\Delta_\Psi^{1/2} Q\Psi \\ &= SQ_1\Psi = SQ_1S^{-1}(S\Psi) . \end{aligned}$$

Proof of Theorem 6.

Step (i). Let $0 < \delta \leqq 2^{-4}$. We prove that if Ψ_1 is cyclic and separating vector belonging to V_Ψ, $t_1 \in R$, $t_1' \in R$ and

$$\Phi_1 = \Psi_1 + t_1\Psi_1 , \tag{6.10}$$

$$\|t_1\| \leqq \delta , \qquad \|t_1'\| \leqq \delta , \tag{6.11}$$

$$\Delta_{\Psi_1}^{1/2} t_1{}^*\Psi_1 = t_1'\Psi_1 , \tag{6.12}$$

then there exists $\Phi \in V_\Psi$ such that

$$\omega_\Phi = \omega_{\Phi_1} . \tag{6.13}$$

We first note that by Theorem 4 (4) and (5), $J_{\Psi_1} = J_\Psi$ and $V_\Psi = V_{\Psi_1}$. Let

$$t_{1\pm} \equiv (1/2)\{t_1 \pm t_1'\} . \tag{6.14}$$

Then

$$J_{\Psi_1} t_{1\pm}\Psi_1 = \pm t_{1\pm}\Psi_1 .$$

By Theorem 4 (6) and (7), there exists $\Psi_{11} \in V_{\Psi_1}$ and $\Psi_{12} \in V_{\Psi_1}$ such that

$$-it_{1-}\Psi_1 = \Psi_{11} - \Psi_{12} ,$$

$$s^R(\Psi_{11}) \perp s^R(\Psi_{12}) .$$

Let

$$t_{11} \equiv -is^R(\Psi_{11})t_{1-} , \qquad t_{12} \equiv is^R(\Psi_{12})t_{1-} .$$

Then

$$t_{11}\Psi_1 = \Psi_{11} \in V_{\Psi_1} , \qquad t_{12}\Psi_1 = \Psi_{12} \in V_{\Psi_1} ,$$

$$\| t_{11} \| \leqq \delta , \qquad \| t_{12} \| \leqq \delta .$$

By Theorem 3 (9), $(\delta - t_{11})\Psi_1 \in V_{\Psi_1}$, $(\delta - t_{12})\Psi_1 \in V_{\Psi_1}$. Hence by corollary to Theorem 5, there exists $h_1 \in R$ and $h_2 \in R$ such that

$$0 \leqq h_1 \leqq \delta , \qquad 0 \leqq h_2 \leqq \delta ,$$

$$t_{11}\Psi_1 = (h_1\Psi_1 + J_{\Psi_1}h_1\Psi_1)/2 ,$$

$$t_{12}\Psi_1 = (h_2\Psi_1 + J_{\Psi_1}h_2\Psi_1)/2 .$$

From $J_{\Psi_1}h_k\Psi_1 = \Delta_{\Psi_1}^{1/2}h_k^*\Psi_1 = \Delta_{\Psi_1}^{1/2}h_k\Psi_1$, we obtain

$$h_1' \equiv \tau_{\Psi_1}(-i/2)h_1 = 2t_{11} - h_1 ,$$

$$h_2' \equiv \tau_{\Psi_1}(-i/2)h_2 = 2t_{12} - h_2 .$$

Thus

$$\| h_1' - h_2' \| \leqq 2 \| t_{1-} \| + \| h_1 - h_2 \| \leqq 3\delta .$$

We set

$$\Phi_2 \equiv u'\Phi_1 , \qquad u' \equiv \exp\{-ij_{\Psi_1}(h_1 - h_2)\} ,$$

$$\Psi_2 \equiv S_1\Psi_1 , \qquad S_1 \equiv 1 + t_1 - i(h_1' - h_2') ,$$

$$t_2' \equiv (1 + t_1)(-1 + i(h_1' - h_2') + \exp\{-i(h_1' - h_2')\}) - it_1(h_1' - h_2') ,$$

$$t_2 \equiv t_2'S_1^{-1} .$$

Since u' commutes with t_1 and $u'\Psi_1 = \exp\{-i(h_1' - h_2')\}\Psi_1$ due to $j_{\Psi_1}(h_1 - h_2)\Psi_1 = (h_1' - h_2')\Psi_1$, we obtain

$$\Phi_2 = \Psi_2 + t_2\Psi_2 ,$$

$$\omega_{\Phi_2} = \omega_{\Phi_1} .$$

We have

$$S_1 = 1 + t_{1+} + (i/2)\{(h_1 - h_2) - (h_1' - h_2')\} .$$

Hence $\tau_{\Psi_1}(-i/2)S_1^* = S_1$ and $(\tau_{\Psi_1}(i/4)S_1)$ is symmetric. Furthermore,

$$(6.15) \qquad \| t_{1+} + (i/2)\{(h_1 - h_2) - (h'_1 - h'_2)\} \| \leqq 3\delta < 1 .$$

Hence $\| \tau_{\Psi_1}(-i/4)(S_1 - 1)^* \| \leqq 3\delta$ and $\tau_{\Psi_1}(i/4)S_1 \geqq 0$. Therefore $\Psi_2 \in V_{\Psi_1} = V_{\Psi}$.

Since S_1 is invertible, Ψ_2 is again cyclic and separating.

We have

$$\| t'_2 \| \leqq (1 + \delta)(e^{3\delta} - 1 - 3\delta) + 3\delta^2 ,$$

$$\| S_1^{-1} \| \leqq (1 - 3\delta)^{-1} .$$

Hence

$$\| t_2 \| \leqq a_1 \delta^2$$

with

$$\begin{aligned} a_1 &\equiv (1 - 3\delta)^{-1}\{3 + (1 + \delta)(e^{3\delta} - 1 - 3\delta)/\delta^2\} \\ &\leqq (1 - 3\delta)^{-1}(3 + (9/2)(1 + \delta)e^{3\delta}) < 16 \end{aligned}$$

for $\delta \leqq 2^{-4}$. Hence

$$\| t_2 \| \leqq a\delta$$

with $a = a_1 2^{-4} < 1$.

By Lemma 7,

$$\tau_{\Psi_2}(-i/2)(t_2{}^*) = S_1\{\tau_{\Psi_1}(-i/2)(S_1^{*-1}t_2'{}^*)\}S_1^{-1} .$$

Since $\tau_{\Psi_1}(-i/2)S_1^* = S_1$,

$$\begin{aligned} \tau_{\Psi_2}(-i/2)(t_2^*) &= \{\tau_{\Psi_1}(-i/2)(t_2'{}^*)\}S_1^{-1} \\ &= \{(-1 - i(h_1 - h_2) + \exp\{i(h_1 - h_2)\})(1 + \tau_{\Psi_1}(-i/2)(t_1^*)) \\ &\quad + i(h_1 - h_2)\tau_{\Psi_1}(-i/2)(t_1^*)\}S_1^{-1} . \end{aligned}$$

Therefore,

$$\begin{aligned} \| \tau_{\Psi_2}(-i/2)(t_2^*) \| &\leqq \{(1 + \delta)(e^{\delta} - 1 - \delta) + \delta^2\}(1 - 3\delta)^{-1} \\ &\leqq a\delta . \end{aligned}$$

From (6.15), we also have

$$\begin{aligned} \| \Psi_1 - \Psi_2 \| &\leqq \| 1 - S_1 \| \, \| \Psi_1 \| \\ &\leqq \| 1 - S_1 \| \, \| (1 + t_1)^{-1} \| \, \omega_{\Phi_1}(1)^{1/2} \\ &\leqq 3\delta(1 - \delta)^{-1}\omega_{\Phi_1}(1)^{1/2} \\ &\leqq 4\delta\omega_{\Phi_1}(1)^{1/2} . \end{aligned}$$

We can now repeat the process and obtain a sequence of vectors Φ_n, Ψ_n and operators $t_n \in R$ such that Ψ_n is cyclic and separating, $\Psi_n \in V_{\Psi}$,

$$\Phi_n = \Psi_n + t_n\Psi_n ,$$

$$\| t_n \| \leqq a^{n-1}\delta , \qquad \| \tau_{\Psi_n}(-i/2)(t_n^*) \| \leqq a^{n-1}\delta ,$$

$$\omega_{\Phi_n} = \omega_{\Phi_1} ,$$

$$\| \Psi_n - \Psi_{n-1} \| \leqq 4a^{n-2}\delta\omega_{\Phi_1}(1)^{1/2} .$$

Ψ_n is a Cauchy sequence and has a limit

$$\Phi = \lim \Psi_n \in V_\Psi .$$

Since $\lim \| t_n\Psi_n \| = 0$, we have

$$\Phi = \lim \Phi_n ,$$

$$\omega_\Phi = \lim \omega_{\Phi_n} = \omega_{\Phi_1} .$$

Step (ii). We prove that if $t^* = t \in R$ and $\tau_\Psi(z)t \in R$ for $\operatorname{Im} z \in [-1, 1]$, then there exists $\Phi \in V_\Psi$ such that $\omega_\Phi = \omega_{(\exp t)\Psi}$.

Let $x(\lambda) \equiv (\exp \lambda t)\Psi$, $0 \leqq \lambda \leqq 1$. It is cyclic and separating because Ψ is cyclic and separating and $e^{\lambda t}$ is invertible. We have

$$\begin{aligned} J_{x(\lambda)}\Delta_{x(\lambda)}^{1/2}tx(\lambda) &= tx(\lambda) = e^{\lambda t}t\Psi \\ &= e^{\lambda t}J_\Psi\{\tau_\Psi(-i/2)t\}\Psi = t'e^{\lambda t}\Psi \end{aligned}$$

where $t' \equiv j_\Psi\{\tau_\Psi(-i/2)t\} \in R'$. Then

$$\begin{aligned} \Delta_{x(\lambda)}^{1/2}J_{x(\lambda)}t'x(\lambda) &= t'^*x(\lambda) = e^{\lambda t}t'^*\Psi \\ &= e^{\lambda t}J_\Psi\Delta_\Psi^{-1/2}t'\Psi = e^{\lambda t}\Delta_\Psi t\Psi = t''x(\lambda) \end{aligned}$$

where $t'' = e^{\lambda t}\{\tau_\Psi(-i)t\}e^{-\lambda t}$. Combining two computations, we have

$$\Delta_{x(\lambda)}tx(\lambda) = t''x(\lambda) .$$

By Lemma 6, $\tau_{x(\lambda)}(z)t \in R$ for $\operatorname{Im} z \in [-1, 0]$. Since $(\tau_{x(\lambda)}(\bar{z})t)^*$ is holomorphic for $\operatorname{Im} z \in (0, 1)$ and coincides with $\tau_{x(\lambda)}(z)t$ at $\operatorname{Im} z = 0$, it is an analytic continuation of $\tau_{x(\lambda)}(z)t$. We have $\tau_{x(\lambda)}(z) \in R$ for $\operatorname{Im} z \in [-1, 1]$ and $\| \tau_{x(\lambda)}(z)t \| \leqq \| t'' \|$. We note that $\| t \| = \| \tau_{x(\lambda)}(0)t \| \leqq \| t'' \|$.

For $y \in D_{x(\lambda)}$ we have convergence of

$$\sum_{n=0}^{\infty} (n!)^{-1}(\lambda't)^n\Delta_{x(\lambda)}^{-iz}y = e^{\lambda't}\Delta_{x(\lambda)}^{-iz}y ,$$

and

$$\sum_{n=0}^{\infty} (n!)^{-1}\Delta_{x(\lambda)}^{iz}(\lambda't)^n\Delta_{x(\lambda)}^{-iz}y = \exp\{\lambda'\tau_{x(\lambda)}(z)t\}y$$

for $\operatorname{Im} z \in [-1, 1]$. Hence

$$\Delta_{x(\lambda)}^{iz}e^{\lambda't}\Delta_{x(\lambda)}^{-iz}y = \exp\{\lambda'\tau_{x(\lambda)}(z)t\}y .$$

In particular, for $\lambda' > 0$,

$$\| \tau_{x(\lambda)}(-i/2)e^{\lambda' t} - 1 \| \leqq e^{\lambda' \|t''\|} - 1 .$$

Let N be a natural number satisfying

$$N \geqq 2^4 C e^C , \quad C = e^{2\|t\|} \| \tau_{\Psi}(-i)t \| \geqq \| t'' \| .$$

Let $\lambda_n = n/N$. We have

$$\begin{aligned} \| e^{\lambda_1 t} - 1 \| &\leqq e^{\lambda_1 \|t\|} - 1 \\ &\leqq e^{\lambda_1 \|t''\|} - 1 \leqq \lambda_1 \| t'' \| e^{\lambda_1 \|t''\|} \\ &\leqq 2^{-4} . \end{aligned}$$

Similarly, for $0 \leqq \lambda \leqq 1$,

$$\| \tau_{x(\lambda)}(-i/2)e^{\lambda_1 t} - 1 \| \leqq 2^{-4} .$$

In other words, $t''' \equiv e^{\lambda_1 t} - 1$ satisfies $\| t''' \| \leqq 2^{-4}$ and

$$\| \tau_{x(\lambda)}(-i/2)t''' \| \leqq 2^{-4}$$

for $0 \leqq \lambda \leqq 1$, and $e^{\lambda_1 t} = 1 + t'''$.

Let $y(n) = \exp(t/N)\Phi(n-1)$, where $\Phi(0) \equiv \Psi$ and $\Phi(n)$ is to be determined inductively such that $\Phi(n) \in V_{\Psi}$, $\Phi(n)$ is cyclic and separating, $\omega_{\Phi(n)} = \omega_{x(\lambda_n)}$ and $n \leqq N$. $\Phi(0) \equiv \Psi$ obviously satisfies requirements for $\Phi(n)$, $n = 0$.

If $\omega_{\Phi(n-1)} = \omega_{x(\lambda_{n-1})}$, then $\omega_{y(n)} = \omega_{\exp(tN)x(\lambda_{n-1})} = \omega_{x(\lambda_n)}$. Since $y(n) = (1 + t''')\Phi(n-1)$, we can apply Step (i) if $\Phi(n-1) \in V_{\Psi}$ and $\Phi(n-1)$ is cyclic and separating. There exists $\Phi(n) \in V_{\Psi}$ such that $\omega_{\Phi(n)} = \omega_{y(n)} = \omega_{x(\lambda_n)}$. Since $x(\lambda)$ is separating, $s^R(\omega_{\Phi(n)}) = 1$. Hence $s^{R'}(\Phi(n)) = j_{\Psi}\{s^R(\Phi(n))\} = 1$ due to $\Phi(n) \in V_{\Psi}$. Thus, by induction, we have desired $\Phi(n)$, $n \leqq N$. In particular, $\Phi(N) \in V_{\Psi}$ satisfies $\omega_{\Phi(N)} = \omega_{(\exp t)\Psi}$.

Step (iii). Let S_{Ψ} be the set of all ω_x, $x \in V_{\Psi}$. S_{Ψ} is a norm closed subset of R_*^+ by (5.10). We prove that any $\rho \in R_*^+$ is in S_{Ψ}.

Since Ψ is cyclic and separating, there exists a positive selfadjoint operator A_2 affiliated with R such that Ψ is in the domain of A_2 and $\rho = \omega_{A_2\Psi}$[3]. Let $A_2 = \int \lambda dE_\lambda$, $A_2^L = A_2(E_L - E_{1/L}) + \{1 - E_L + (1/\mathrm{L})E_{1/L})$, $t = (\log A_2^L)(f_\beta^G)$, $\rho_{L\beta} = \omega_{(\exp t)\Psi}$. Then t is a selfadjoint element of $\mathfrak{A}_{\Psi 1}$. By Step (ii), $\rho_{L\beta} \in S_{\Psi}$. Since $\lim_{L\to+\infty} \lim_{\beta\to+0} \| \rho_{L\beta} - \rho \| = 0$, we have $\rho \in S_{\Psi}$.

7. Representation of R_*^+ by V_{Ψ}. We denote the set of all normal positive linear functionals on R by R_*^+ and the set of all normal states on R by R_{*1}^+. As before ω_x denotes the expectation functional by a vector x.

THEOREM 7. *Assume that R and R_α have cyclic and separating vectors Ψ and Ψ_α, respectively.*

(1) *The mapping σ_Ψ from $\omega_x \in R_*^+$ to $\sigma_\Psi(\omega_x) \equiv x \in V_\Psi$ is a bijective homeomorphism from R_*^+ onto V_Ψ relative to the norm topologies.*

(2) *If $\rho = \sum_n \rho_n$, $\rho \in R_*^+$, $\rho_n \in R_*^+$ and $s(\rho_n)$ are mutually orthogonal, then $\sigma_\Psi \rho = \sum \sigma_\Psi \rho_n$.*

(3) *If $R = \bigoplus_n R_n$, $\Psi = \bigoplus \Psi_n$, then $\sigma_\Psi(\bigoplus \rho_n) = \bigoplus \rho_{\Psi_n}(\rho_n)$ for any $\rho_n \in (R_n)_*^+$, $\bigoplus \rho_n \in R_*^+$.*

(4) *If $R = \bigotimes(R_\alpha, \Psi_\alpha)$ on $H = \bigotimes(H_\alpha, \Psi_\alpha)$ (the incomplete infinite tensor product containing $\Psi \equiv \bigotimes \Psi_\alpha$), then $\sigma_\Psi(\bigotimes \rho_\alpha) = \bigotimes \sigma_{\Psi_\alpha}(\rho_\alpha)$ if $\rho_\alpha \in (R_\alpha)_{*1}^+$ and $\bigotimes \sigma_\Psi(\rho_\alpha) \in \bigotimes(H_\alpha, \Psi_\alpha)$. The last condition is equivalent to existence of $\rho \in R_*^+$ such that*

$$\rho(Q \otimes (\bigotimes_{\alpha \notin J} 1_\alpha)) = (\bigotimes_{\alpha \in J} \rho_\alpha)(Q) , \qquad Q \in \bigotimes_{\alpha \in J} R_\alpha$$

for every finite index set J. (Symbolically $\bigotimes \rho_\alpha \in R_*^+$.)

(5) *For any $\Phi \in H$, there exists a unique $|\Phi|_\Psi \in V_\Psi$ and a partial isometry $u' \in R'$ such that*

$$\Phi = u' \, |\Phi|_\Psi , \tag{7.1}$$

$$u'u'^* = s^{R'}(\Phi) , \qquad u'^*u' = s^{R'}(|\Phi|_\Psi) . \tag{7.2}$$

There also exist a unique $|\Phi|'_\Psi \in V_\Psi$ and a partial isometry $u \in R$ such that

$$\Phi = u \, |\Phi|'_\Psi , \tag{7.3}$$

$$uu^* = s^R(\Phi) , \qquad u^*u = s^R(|\Phi|'_\Psi) . \tag{7.4}$$

They are related by

$$u = j_\Psi(u')^* , \qquad |\Phi|'_\Psi = u' j_\Psi(u') \, |\Phi|_\Psi . \tag{7.5}$$

(6) *If Φ is any cyclic and separating vector for R, there exists a unitary $w \in R'$ such that*

$$\sigma_\Psi(\rho) = w \sigma_\Phi(\rho) \tag{7.6}$$

for all $\rho \in R_*^+$.

Proof. (1) follows from Theorem 6, (5.10) and

$$\begin{aligned} |\omega_x(Q) - \omega_y(Q)| &= |(x + y, Q(x - y)) + (x - y, Q(x + y))|/2 \\ &\leqq \|x + y\| \, \|x - y\| \, \|Q\| , \end{aligned}$$

which implies

$$\|\omega_x - \omega_y\| \leqq \|x + y\| \, \|x - y\| . \tag{7.7}$$

(2) By (1), there exists $\Phi_n \in V_\Psi$ such that $\omega_{\Phi_n} = \rho_n$. Since $s(\rho_n)$ are mutually orthogonal, $s^R(\Phi_n) = s(\rho_n)$ are mutually orthogonal and

$$\sum \|\Phi_n\|^2 = \sum \rho_n(1) = \rho(1) < \infty .$$

Hence we have convergence of

$$\Phi = \sum \Phi_n .$$

Since $\Phi_n \in V_\Psi$, $s^{R'}(\Phi_n) = j_\Psi(s^R(\Phi_n))$ are also mutually orthogonal. Hence

$$\begin{aligned}(\Phi_n, Q\Phi_m) &= (\Phi_n, Qs^{R'}(\Phi_m)\Phi_m) \\ &= (s^{R'}(\Phi_m)\Phi_n, Q\Phi_m) = 0\end{aligned}$$

for $Q \in R$ and $m \neq n$. Therefore,

$$(\Phi, Q\Phi) = \sum (\Phi_n, Q\Phi_n) = \sum \rho_n(Q) = \rho(Q) .$$

Hence $\Phi = \sigma_\Psi \rho = \sum \sigma_\Psi \rho_n$.

(3) This follows from (2).

(4) If $\Psi = \bigotimes \Psi_\alpha$, then $J_\Psi = \bigotimes J_{\Psi_\alpha}$ and $\Delta_\Psi = \bigotimes \Delta_{\Psi_\alpha}$ which is seen as follows: Let $J = \bigotimes J_{\Psi_\alpha}$, $\Delta^{it} = \bigotimes \Delta_{\Psi_\alpha}^{it}$. Then $J\Delta^{1/2}Q\Psi = Q^*\Psi$ if $Q = \bigotimes Q_\alpha$ and $Q_\alpha = 1$ except for a finite number of α. Since such Q is * strongly total in R, $J\Delta^{1/2}Q\Psi = Q^*\Psi$ for any $Q \in R$ and hence $J\Delta^{1/2} \supset J_\Psi \Delta_\Psi^{1/2}$. J satisfies (i)-(iv) of Theorem 1. It also satisfies (v) due to $JQ^*\Psi = \Delta^{1/2}Q\Psi$ and $\Delta \geqq 0$. Hence $J = J_\Psi$. Hence $\Delta = \Delta_\Psi$.

If $\bigotimes \sigma_{\Psi_\alpha}(\rho_\alpha) \in \bigotimes (H_\alpha, \Psi_\alpha)$ and ρ_α are faithful, then

$$J_{\otimes \sigma_{\Psi_\alpha}(\rho_\alpha)} = \bigotimes J_{\sigma_{\Psi_\alpha}(\rho_\alpha)} = \bigotimes J_{\Psi_\alpha} = J_{\otimes \Psi_\alpha} .$$

Let Z_α be the center of R_α. Then $\{\bigotimes (R_\alpha, \Psi_\alpha)\}' = \bigotimes (R'_\alpha, \Psi_\alpha)$ and hence the center Z of $\bigotimes (R_\alpha, \Psi_\alpha)$ is given by $\bigotimes (Z_\alpha, \Psi_\alpha)$. If z_α is a projection in Z_α and $z_\alpha = 1$ except for a finite number of α, then $z = \bigotimes z_\alpha \in Z$ satisfies

$$(\Psi, z\{\bigotimes \sigma_{\Psi_\alpha}(\rho_\alpha)\}) = \prod (\Psi_\alpha, z\sigma_{\Psi_\alpha}(\rho_\alpha)) \geqq 0 .$$

Z_α and Z can be viewed as $L^\infty(\Xi_\alpha, \mu_\alpha)$ and $L^\infty(\prod \Xi_\alpha, \bigotimes \mu_\alpha)$ where projections are characteristic functions. Hence any projection in Z can be weakly approximated by a finite sum of projections $z = \bigotimes z_\alpha$. This implies

$$(\Psi, z\{\bigotimes \sigma_{\Psi_\alpha}(\rho_\alpha)\}) \geqq 0$$

for all projections in Z and hence for all $z \in Z$, $z \geqq 0$.

By Theorem 4 (5), we have $\bigotimes \sigma_{\Psi_\alpha}(\rho_\alpha) \in V_{\otimes \Psi_\alpha}$. The same conclusion holds for nonfaithful ρ_α, by taking a limit of faithful $\rho_\alpha + \lambda_\alpha \omega_{\Psi_\alpha}$, $\lambda_\alpha \geqq 0$ as $\sum \lambda_\alpha \to 0$. ($\bigotimes \sigma_\Psi(\rho_\alpha) \in \bigotimes (H_\alpha, \Psi_\alpha)$ implies $\sigma_\Psi(\rho_\alpha) = \Psi_\alpha$ except for a countable number of α.) We also have

$$\bigotimes \rho_\alpha \equiv \omega_{\otimes\sigma_{\Psi_\alpha}(\rho_\alpha)} \in R_*^+ .$$

Hence

$$\sigma_\Psi(\bigotimes \rho_\alpha) = \bigotimes \sigma_{\Psi_\alpha}(\rho_\alpha) .$$

Next assume $\bigotimes \rho_\alpha \in R_*^+$. Without loss of generality we may assume $\|\Psi_\alpha\| = 1$. Let $R(I) = \bigotimes_{\alpha\in I} R_\alpha$, $\Psi(I) = \bigotimes_{\alpha\in I}\Psi_\alpha$, $\Phi_\alpha = \sigma_{\Psi_\alpha}(\rho_\alpha)$, $\Phi = \sigma_\Psi(\bigotimes \rho_\alpha)$, $\rho_0(I) = \omega_{\Psi(I)}$ for an arbitrary index set I and $\rho(J) = \bigotimes_{\alpha\in J}\rho_\alpha$, $\Phi(J) = \bigotimes_{\alpha\in J}\Phi_\alpha$ for a finite index set J. J^C denotes the complement of J in the index set. $\rho_\alpha \in R_{*1}^+$ implies

$$\|\Phi_\alpha\| = \|\Phi(J)\| = \|\Phi\| = 1 .$$

Since $\Psi(J^C)\otimes z$ is total when J runs over finite index sets and z runs over $\bigotimes_{\alpha\in J} H_\alpha$, there exists a finite index set J and a $z \in \bigotimes_{\alpha\in J} H_\alpha$ such that $(\Phi, \Psi(J^C)\otimes z) \neq 0$, $\|z\| = 1$. Then for any $K \subset J^C$, we have

$$\|\rho(K) - \rho_0(K)\| = \|\omega_\Phi^{R(K)} - \omega_{\Psi(J^C)\otimes z}^{R(K)}\| < 2 .$$

(If $(x, y) \neq 0$, then (7.7) implies $\|\omega_x - \omega_{y'}\|^2 \leqq (\|x\|^2 + \|y'\|^2)^2 - 4(x, y')^2$ for $y' = e^{i\theta}y$ where θ is a real number such that $(x, y') > 0$. Hence $\|\omega_x - \omega_y\| < \|x\|^2 + \|y\|^2$.)

By the first part of the proof of (4), we have $\sigma_{\Psi(K)}(\rho(K)) = \bigotimes_{\alpha\in K}\sigma_{\Psi_\alpha}(\rho_\alpha) = \Phi(K)$ for a finite index set K where the condition $\bigotimes_{\alpha\in K}\sigma_{\Psi_\alpha}(\rho_\alpha) \in \bigotimes_{\alpha\in K} H_\alpha$ is trivially satisfied. By (5.10)

$$\|\Psi(K) - \Phi(K)\|^2 \leqq \|\rho(K) - \rho_0(K)\|$$

and hence

$$(\Psi(K), \Phi(K)) \geqq 2^{-1}(2 - \|\rho(K) - \rho_0(K)\|) \equiv \delta > 0 ,$$

where we have used $(\Psi(K), \Phi(K)) \geqq 0$ due to $\Phi(K) \in V_{\Psi(K)}$. Since $\|\Psi_\alpha\| = \|\Phi_\alpha\| = 1$, we have $1 \geqq (\Psi_\alpha, \Phi_\alpha) > 0$ and hence

$$1 \geqq \prod_{\alpha\in K}(\Psi_\alpha, \Phi_\alpha) \geqq \delta > 0$$

for any finite index set $K \subset J^C$. Hence

$$\sum_\alpha |1 - (\Psi_\alpha, \Phi_\alpha)| < \infty$$

which implies $\bigotimes \Phi_\alpha \in \bigotimes (H_\alpha, \Psi_\alpha)$.

Therefore, $\bigotimes \rho_\alpha \in R_*^+$ implies $\bigotimes \sigma_{\Psi_\alpha}(\rho_\alpha) \in \bigotimes (H_\alpha, \Psi_\alpha)$.

(5) For any $\Phi \in H$, there exists a unique $|\Phi|_\Psi \in V_\Psi$ satisfying $\omega_\Phi = \omega_{|\Phi|_\Psi}$ by (1). Then there exists a unique partial isometry $u' \in R'$ satisfying (7.1) and (7.2).

Next set

$$|\Phi|'_\Psi \equiv u' j_\Psi(u') |\Phi|_\Psi .$$

Then $|\Phi|'_\Psi = j_\Psi(u')\Phi$. Since $s^R(|\Phi|_\Psi) = j_\Psi(s^{R'}(|\Phi|_\Psi)) = j_\Psi(u'^*u')$, we have

$$\Phi = u'|\Phi|_\Psi = j_\Psi(u')^*|\Phi|'_\Psi .$$

We also have

$$\begin{aligned} j_\Psi(u')j_\Psi(u')^* &= j_\Psi(s^{R'}(\Phi)) = j_\Psi\{s(\omega_\Phi^{R'})\} = j_\Psi\{s^{R'}(\omega^{R'}_{|\Phi|'_\Psi})\} \\ &= j_\Psi\{s^{R'}(|\Phi|'_\Psi)\} = s^R(|\Phi|'_\Psi) \end{aligned}$$

where the last equality is due to $|\Phi|'_\Psi \in V_\Psi$ and $\omega_x^{R'}$ denotes the expectation functional on R' by a vector x.

Thus (7.5) satisfies (7.3) and (7.4).

To see the uniqueness of $|\Phi|'_\Psi$ and u, we note $\omega_\Phi^{R'} = \omega^{R'}_{|\Phi|'_\Psi}$. If we interchange the role of R and R' in the definition of V_Ψ, we obtain the same set V_Ψ. Hence by (1), a vector $x \in V_\Psi$ satisfying $\omega_x^{R'} = \rho$ for any given $\rho \in (R')^+_*$ is unique. Hence the uniqueness of $|\Phi|'_\Psi$. The unitary operator $u \in R$ satisfying (7.3) and (7.4) is unique because $uQ|\Phi|'_\Psi = Q\Phi$ for $Q \in R'$ determines u on $s^R(|\Phi|'_\Psi)$.

(6) Since Φ is separating $s^R(\sigma_\Psi\omega_\Phi) = s(\omega_\Phi) = 1$. Hence $s^{R'}(\sigma_\Psi\omega_\Phi) = j_\Psi\{s^R(\sigma_\Psi\omega_\Phi)\} = 1$ and $\sigma_\Psi\omega_\Phi$ is cyclic and separating. By Corollary 2 of § 4, $J_{\sigma_\Psi\omega_\Phi} = J_\Psi$ and $V_{\sigma_\Psi\omega_\Phi} = V_\Psi$.

Since $\omega_\Phi = \omega_{\sigma_\Psi\omega_\Phi}$, there exists a partial isometry $w \in R'$ such that $\sigma_\Psi\omega_\Phi = w\Phi$. Since both Φ and $\sigma_\Psi\omega_\Phi$ are cyclic, w is unitary.

Since $w \in R'$, we have for $S = J_{w\Phi}\Delta_{w\Phi}^{1/2}$ and $S_\Phi = J_\Phi\Delta_\Phi^{1/2}$,

$$SwQ\Phi = SQw\Phi = Q^*w\Phi = wQ^*\Phi = wS_\Phi Q\Phi , \quad Q \in R .$$

Hence $S = wS_\Phi w^*$ and $J_\Psi = J_{w\Phi} = wJ_\Phi w^*$. Hence

$$(w\sigma_\Phi\rho, Qj_\Psi(Q)w\Phi) = (\sigma_\Phi\rho, Qj_\Phi(Q)\Phi) \geqq 0 .$$

By Theorem 4 (1) and (4),

$$w\sigma_\Phi\rho \in V_{w\Phi} = V_\Psi .$$

By the uniqueness in (1), $w\sigma_\Phi\rho = \sigma_\Psi\rho$.

8. Applications of σ_Ψ.

The following theorems are examples of applications of Theorem 7.

THEOREM 8. *Let Ψ and Φ be cyclic and separating vectors for R. Then the $*$ automorphism*

$$Q \in R \rightarrow j_\Psi\{j_\Phi(Q)\} \in R$$

of R is inner.[1]

[1] The author is informed by Professor Takesaki that Dr. Connes has a simple proof of this.

Proof. By the proof of Theorem 7 (6),

$$J_{\Psi} = wJ_{\Phi}w^*$$

for a unitary $w \in R'$. Setting $u = j_{\Phi}(w^*)$, we have

$$j_{\Psi}\{j_{\Phi}(Q)\} = uQu^*$$

where u is unitary and $u \in R$.

THEOREM 9. *Let $\mathfrak{A}$ be the C^* algebra inductive limit of finite W^* tensor products $\{\bigotimes_{\alpha\in J} R_\alpha\} \equiv R(J)$, where J is any finite subset of given index set $\{\alpha\}$. Let $\rho_\alpha, \rho'_\alpha \in (R_\alpha)^+_{*1}$. Assume that central supports of ρ_α and ρ'_α are the same. The representations of $\mathfrak{A}$ canonically associated with $\bigotimes \rho_\alpha$ and $\bigotimes \rho'_\alpha$ are quasi-equivalent, if and only if $\sum d'(\rho_\alpha, \rho'_\alpha)^2 < \infty$, where*

$$d'(\rho_\alpha, \rho'_\alpha) \equiv \| \sigma_{\Psi_\alpha}(\rho_\alpha) - \sigma_{\Psi_\alpha}(\rho'_\alpha) \| \tag{8.1}$$

does not depend on Ψ_α.

Proof. By Theorem 7 (6), $d'(\rho_1, \rho_2)$ does not depend on Ψ.

First assume $\sum d'(\rho_\alpha, \rho'_\alpha)^2 < \infty$. Then there exists a countable index set I such that $d'(\rho_\alpha, \rho'_\alpha) = 0$ for $\alpha \notin I$. Then $\rho_\alpha = \rho'_\alpha$ for $\alpha \notin I$.

By assumption

$$\sum_\alpha |1 - (\sigma_{\Psi_\alpha}\rho_\alpha, \sigma_{\Psi_\alpha}\rho'_\alpha)| < \infty \ .$$

Hence $\Phi \equiv \bigotimes_\alpha \sigma_{\Psi_\alpha}\rho_\alpha$ and $\Phi' \equiv \bigotimes_\alpha \sigma_{\Psi_\alpha}\rho'_\alpha$ belong to the same incomplete infinite tensor product $H = \bigotimes (H_\alpha, \sigma_{\Psi_\alpha}\rho_\alpha)$. The C^* algebra $\mathfrak{A}$ has a natural representation π on H and $\bigotimes \rho_\alpha = \omega_\Phi$, $\bigotimes \rho'_\alpha = \omega_{\Phi'}$. Let E_α be the central support of ρ_α, which is the same as the central support of ρ'_α. Then $(R_\alpha \cup R'_\alpha)\sigma_{\Psi_\alpha}\rho_\alpha = E_\alpha H_\alpha$. Since $(\bigotimes R_\alpha)' = \bigotimes R'_\alpha$ in an incomplete infinite tensor product, the central support E of $\bigotimes \sigma_{\Psi_\alpha}\rho_\alpha$ satisfies $EH = \lim_{J\uparrow} (\bigotimes_{\alpha\notin J} \sigma_{\Psi_\alpha}\rho_\alpha) \otimes (\bigotimes_{\alpha\in J} E_\alpha H_\alpha)$. By the same calculation the central support of $\bigotimes \sigma_{\Psi_\alpha}\rho'_\alpha$ coincides with E. Hence $\bigotimes \rho_\alpha$ and $\bigotimes \rho'_\alpha$ produce quasi-equivalent representations of $\mathfrak{A}$.

Next assume that representations of $\mathfrak{A}$ associated with $\bigotimes \rho_\alpha$ and $\bigotimes \rho'_\alpha$ are quasi-equivalent. Let $H_\alpha, \pi_\alpha, \Phi_\alpha$ be canonically associated with ρ_α. We have $\omega_\Phi = \bigotimes \rho_\alpha$ for $\Phi = \bigotimes \Phi_\alpha$.

By assumption of quasi-equivalence, there exists $x_n \in \bigotimes (H_\alpha, \Phi_\alpha)$, $x_1 \neq 0$ such that $\bigotimes \rho'_\alpha = \sum_n \omega_{x_n}$. Since $(\bigotimes_{\alpha\notin J} \Phi_\alpha) \otimes z$ is total when J runs over all finite index sets and z runs over $\bigotimes_{\alpha\in J} H_\alpha$, there exists a finite index set J and $z \in \bigotimes_{\alpha\in J} H_\alpha$ such that $(x_1, (\bigotimes_{\alpha\notin J} \Phi_\alpha) \otimes z) \neq 0$. Denote $\rho' = \bigotimes \rho'_\alpha$ and $\rho'' = \omega_{(\otimes_{\alpha\notin J}\Phi_\alpha)\otimes z}$. Then $\| \rho' - \rho'' \| < 2$.

Let $\rho_K = \bigotimes_{\alpha\in K} \rho_\alpha$, $\rho'_K = \bigotimes_{\alpha\in K} \rho'_\alpha$. Restrictions of ρ'' and ρ' to $\bigotimes_{\alpha\in K} R_\alpha$ is ρ_K and ρ'_K for any finite index set K in J^C. By Theorem 7

(4) and (3.10), we have

$$\prod_{\alpha \in K} (\sigma_{\Psi_\alpha}\rho_\alpha, \sigma_{\Psi_\alpha}\rho'_\alpha) = \{2 - \| \sigma_{\Psi(K)}\rho_K - \sigma_{\Psi(K)}\rho'_K \|^2\}/2$$
$$\geqq \{2 - \| \rho'' - \rho' \|\}/2 > 0$$

where $\Psi(K) = \bigotimes_{\alpha \in K} \Psi_\alpha$. Since $0 \leqq (\sigma_{\Psi_\alpha}\rho_\alpha, \sigma_{\Psi_\alpha}\rho'_\alpha) \leqq 1$, we have

$$2 \sum | 1 - (\sigma_{\Psi_\alpha}\rho_\alpha, \sigma_{\Psi_\alpha}\rho'_\alpha) | = \sum \| \sigma_{\Psi_\alpha}\rho_\alpha - \sigma_{\Psi_\alpha}\rho'_\alpha \|^2 < \infty .$$

REMARK 1. The distance $d'(\rho, \rho')$ satisfies

$$d'(\rho, \rho') \geqq d(\rho, \rho') \tag{8.2}$$

where $d(\rho, \rho')$ is the Bures distance [5]. Since $\sum d(\rho_\alpha, \rho'_\alpha)^2 < \infty$ is another necessary and sufficient condition for quasi-equivalence, it must be equivalent to $\sum d'(\rho_\alpha, \rho'_\alpha)^2 < \infty$. Hence there must be a constant $\lambda > 1$, such that

$$\lambda d(\rho, \rho') \geqq d'(\rho, \rho') . \tag{8.3}$$

REMARK 2. If R is semifinite, φ if a σ-finite faithful normal trace on R, H is the Hilbert space of Hilbert-Schmidt operator affiliated with R, Hilbert-Schmidt relative to φ, and R is left multiplication, then an example of V_Ψ is the set of vector corresponding to positive Hilbert-Schmidt operators. The inequality (5.10) correspond to the inequality $\| \sigma - \rho \|_{\mathrm{tr}} \geqq \| \sigma^{1/2} - \rho^{1/2} \|^2_{\mathrm{H.S}}$ [7].

THEOREM 10 [6]. *$\tau_\rho(t)x \to \tau_\psi(t)x$ strongly as $\| \rho - \psi \| \to 0$ where ρ and ψ are faithful positive linear functionals of R, both $x \in R$ and ψ are fixed.*

Proof. Let $\xi_\rho = \sigma_\Psi(\rho)$ and $\xi_\psi = \sigma_\Psi(\psi)$ for some cyclic and separating Ψ. Then for $x \in R$,

$$\| \Delta^{1/2}_{\xi_\psi} x\xi_\psi - \Delta^{1/2}_{\xi_\rho} x\xi_\rho \| = \| J_\Psi \Delta^{1/2}_{\xi_\psi} x\xi_\psi - J_\Psi \Delta^{1/2}_{\xi_\rho} x\xi_\rho \|$$
$$= \| x^*(\xi_\psi - \xi_\rho) \| \leqq \| x \| \, \| \psi - \rho \|^{1/2}$$

where we have used Theorem 4 (5) and (8). Hence

$$\| (\Delta^{1/2}_{\xi_\psi} + 1)x\xi_\psi - (\Delta^{1/2}_{\xi_\rho} + 1)x\xi_\rho \| \leqq 2 \| x \| \, \| \psi - \rho \|^{1/2} .$$

Since $\| (\Delta^{1/2}_{\xi_\psi} + 1)^{-1} \| \leqq 1$, we have

$$\| \{(\Delta^{1/2}_{\xi_\rho} + 1)^{-1} - (\Delta^{1/2}_{\xi_\psi} + 1)^{-1}\}(\Delta^{1/2}_{\xi_\psi} + 1)x\xi_\psi \|$$
$$= \| (\Delta^{1/2}_{\xi_\rho} + 1)^{-1}\{(\Delta^{1/2}_{\xi_\psi} + 1)x\xi_\psi - (\Delta^{1/2}_{\xi_\rho} + 1)x\xi_\rho\} + x(\xi_\rho - \xi_\psi) \|$$
$$\leqq 3 \| x \| \, \| \psi - \rho \|^{1/2} .$$

Since $\Delta^{1/2}_{\xi_\psi}$ is essentially self-adjoint on $R\xi_\psi$, $(\Delta^{1/2}_{\xi_\psi} + 1)R\xi_\psi$ is dense.

Hence by uniform boundedness $\|(\Delta_{\xi_\rho}^{1/2}+1)^{-1}\|\leqq 1$,

$$(\Delta_{\xi_\rho}^{1/2}+1)^{-1}\longrightarrow(\Delta_{\xi_\psi}^{1/2}+1)^{-1}$$

strongly as $\|\xi_\rho-\xi_\psi\|\to 0$. Let $f_t((u+1)^{-1})=u^{2it}$. f_t is a family of continuous functions on $(0, 1)$, equicontinuous on compact subsets of $(0, 1)$ for bounded t and uniformly bounded. Hence by [4]

$$\Delta_{\xi_\rho}^{it}\longrightarrow\Delta_{\xi_\psi}^{it} \text{ strongly as } \|\rho-\psi\|\to 0$$

uniformly in t in a compact set. This implies $\tau_\rho(t)x\longrightarrow\tau_\psi(t)x$ strongly as $\|\rho-\psi\|\to 0$, uniformly in t in a compact set.

REMARK 3. A similar application yields an alternative proof of Theorem 3 of [6]:

In Theorem 3 of [6], let

$$\varphi_1(x)=(1-\lambda)^{-1}\{\lambda\varphi(uxu^*)+(1-\lambda)\varphi(uu^*xuu^*)\}\ .$$

Then $\varphi_1\geqq 0$, φ_1 is faithful if φ is faithful and

$$\begin{aligned}\|\varphi_1(1)-1\|&=\lambda(1-\lambda)^{-1}\varphi(uu^*)-\varphi(u^*u)\\&\leqq(1-\lambda)^{-1}\varepsilon\ .\end{aligned}$$

We also have

$$\begin{aligned}\|\varphi_1(x)-\varphi(x)\|\leqq&(1-\lambda)^{-1}|\lambda\varphi(uxu^*)-(1-\lambda)\varphi(xu^*u)|\\&+\lambda^{-1}|\lambda\varphi(uu^*xuu^*)-(1-\lambda)\varphi(u^*xuu^*u)|\\&+\lambda^{-1}|(1-\lambda)\varphi(u^*xu)-\lambda\varphi(xuu^*)|\\\leqq&(2-\lambda)(1-\lambda)^{-1}\lambda^{-1}\|x\|\varepsilon\ .\end{aligned}$$

Hence

$$\|\varphi_1-\varphi\|\leqq(2-\lambda)(1-\lambda)^{-1}\lambda^{-1}\varepsilon\ .$$

It is easily seen that $\lambda\varphi_1(xu^*)=(1-\lambda)\varphi(u^*x)$ and hence

$$(\Delta_{\varphi_1}^{1/2}-\lambda^{1/2}(1-\lambda)^{-1/2})u^*\xi_{\varphi_1}=0\ .$$

Since $\|u^*\xi_{\varphi_1}\|^2=\varphi(uu^*)\geqq 1-\lambda-\varepsilon$, we have

$$\|(\Delta_\varphi^{1/2}-\lambda^{1/2}(1-\lambda)^{-1/2})u^*\xi_\varphi\|\leqq(1+\lambda^{1/2}(1-\lambda)^{-1/2})\|\varphi-\varphi_1\|^{1/2}\ .$$

This proves Theorem 3 of [6].

Let $\mathrm{Aut}(R)$ denote the set of all $*$-automorphisms of R. Each $g\in\mathrm{Aut}(R)$ induces an adjoint mapping on R_*^+:

$$(g^*\varphi)(x)=\varphi(g(x))\ .$$

THEOREM 11. *There exists a unitary representation $U_\Psi(g)$ of*

Aut (R) *such that*

$$U_\Psi(g)xU_\Psi(g)^* = g(x)\,, \qquad x \in R\,, \tag{8.4}$$

$$U_\Psi(g)\sigma_\Psi(g^*\rho) = \sigma_\Psi(\rho)\,, \qquad \rho \in R_*^+\,. \tag{8.5}$$

Each $U_\Psi(g)$, $g \in \mathrm{Aut}\,(R)$, *commutes with* J_Ψ. *For two cyclic and separating vectors* Ψ *and* Φ, U_Ψ *and* U_Φ *are unitarily equivalent through a unitary operator* $u' \in R'$:

$$u'U_\Psi(g) = U_\Phi(g)u'\,. \tag{8.6}$$

Proof. Let $\xi(g) = \sigma_\Psi(g^*\omega_\Psi)$ where ω_Ψ is the expectation functional by Ψ. We define

$$U_0(g)x\Psi = g(x)\xi(g^{-1})\,, \qquad x \in R\,. \tag{8.7}$$

We have

$$(g(x)\xi(g^{-1}),\, g(y)\xi(g^{-1})) = (g^{-1})^*\omega_\Psi(g(x^*y)) = (x\Psi,\, y\Psi)\,.$$

Hence $U_0(g)$ is well-defined and its closure $U_\Psi(g)$ is isometric. Since $g^*\omega_\Psi$ is faithful because Ψ is separating and g is an automorphism, $\sigma_\Psi(g^*\omega_\Psi) = \xi(g)$ is separating. Since $\xi(g) \in V_\Psi$, it is cyclic if it is separating. Hence $U_\Psi(g)$ is unitary.

From the definition (8.7), $U_0(g)x = g(x)U_0(g)$ and hence (8.4) holds.

Let $S_1 = J_\Psi\Delta_\Psi^{1/2}$ and $S_2 = J_\xi\Delta_\xi^{1/2}$ for $\xi = \xi(g^{-1})$. We have

$$\begin{aligned} U(g)S_1x\Psi &= U(g)x^*\Psi = g(x^*)\xi \\ &= S_2g(x)\xi = S_2U(g)x\Psi \end{aligned}$$

for $x \in R$. Since $R\Psi$ is a core of S_1 and $R\xi$ is a core of S_2, we have $U(g)S_1U(g)^* = S_2$. By the uniqueness of polar decomposition, we have $U(g)J_\Psi U(g)^* = J_\xi$. Since $\xi(g^{-1}) \in V_\Psi$, we have $J_\xi = J_\Psi$. Hence $U(g)$ commutes with J_Ψ.

Let $x \in R$ and $\psi \in V_\Psi$. Then

$$\begin{aligned} (U(g)\psi,\, xj_\xi(x)\xi) &= (U(g)\psi,\, \{U(g)yU(g)^*\}J_\xi\{U(g)yU(g)^*\}\xi) \\ &= (\psi,\, yj_\Psi(y)\Psi) \geqq 0 \end{aligned}$$

where $y = g^{-1}(x)$, $J_\xi = J_\Psi$, $[U(g), J_\Psi] = 0$, $U(g)^*\xi = \Psi$. This implies

$$U(g)\psi \in V_\xi' = V_\xi = V_\Psi\,.$$

Hence $U(g)V_\Psi \subset V_\Psi$.

By (8.4), we have for $\psi = U(g)\sigma_\Psi(\rho)$ and $\rho \in R_*^+$

$$\omega_\psi(gx) = \rho(x)\,.$$

By $U(g)\sigma_\Psi(\rho) \in V_\Psi$, we have (8.5).

From (8.5), we have

$$U_\Psi(g_1)U_\Psi(g_2)\psi = U_\Psi(g_1g_2)\psi$$

for $\psi \in \sigma_\Psi(R_*^+) = V_\Psi$. Since V_Ψ linearly span H, we have

$$U_\Psi(g_1)U_\Psi(g_2) = U_\Psi(g_1g_2) .$$

For two cyclic and separating vectors Ψ and Φ, there exists a unitary $u' \in R'$ such that $u'\sigma_\Psi(\omega_\Phi) = \Phi$, which automatically satisfies $u'\sigma_\Psi(\rho) = \sigma_\Phi(\rho)$ for all $\rho \in R_*^+$. Then

$$\begin{aligned} u'U_\Psi(g)\sigma_\Psi(g^*\rho) &= u'\sigma_\Psi(\rho) = \sigma_\Phi(\rho) = U_\Phi(g)\sigma_\Phi(g^*\rho) \\ &= U_\Phi(g)u'\sigma_\Psi(g^*\rho) . \end{aligned}$$

Since $\sigma_\Psi(g^*\rho)$, $\rho \in R_*^+$, is total, we have (8.6).

REMARK. The weak, strong and $*$-strong topologies coincide on unitaries and they induce a topology τ_U on Aut(R) through $U_\Psi(g)$. Since the multiplication of unitaries is continuous relative to strong topology, (Aut(R), τ_U) is a topological group. On Aut(R) there is a topology τ by the norm convergence of $g^*\rho$ for every $\rho \in R_*^+$. The two topologies τ and τ_U coincide which can be seen as follows:

The strong convergence of $U_\Psi(g)$ is equivalent to the strong convergence of $U_\Psi(g)^*$.

Since V_Ψ span H, the strong convergence of $U_\Psi(g)^*$ is equivalent to the strong convergence of $U_\Psi(g^{-1})\sigma_\Psi(\rho) = \sigma_\Psi(g^*\rho)$ for each $\rho \in R_*^+$.

Since σ_Ψ is a homeomorphism, the strong convergence of $\sigma_\Psi(g^*\rho)$ is equivalent to the norm convergence of $g^*\rho$ for each $\rho \in R_*^+$.

9. Radon-Nikodym derivative satisfying a chain rule.

THEOREM 12. *Let $\rho, \mu \in R_*^+$ and Ψ be a cyclic and separating vector.*

(1) *The following two conditions are equivalent.*

(α) *$l\rho \geqq \mu$ for some l.*

(β) *There exists $A = A(\mu/\rho) \in R$ such that*

$$\mu(x) = \rho(A^*xA) , \qquad A\sigma_\Psi(\rho) = \sigma_\Psi(\mu) , \tag{9.1}$$

$$s(\rho) \geqq s(A^*A) . \tag{9.2}$$

The operator $A \in R$ satisfying (9.1) *and* (9.2) *is unique.*

(2) *If* (α) *or* (β) *holds, then*

$$\|A(\mu/\rho)\|^2 = \inf\{l; l\rho \geqq \mu\} , \tag{9.3}$$

$$\|A(\mu/\rho)\|\,\sigma_\Psi(\rho) \geqq \sigma_\Psi(\mu) \tag{9.4}$$

where $x \geqq y$ *denotes* $x - y \in V_\Psi$.

(3) *If* $l_1\mu_1 \geqq \mu_2$, $l_2\mu_2 \geqq \mu_3$, *then*

$$A(\mu_3/\mu_1) = A(\mu_3/\mu_2)A(\mu_2/\mu_1)\ . \tag{9.5}$$

(4) $A(\mu/\mu) = s(\mu)$.

(5) $A(\mu/\rho)$ *does not depend on* Ψ.

Proof. (1) First assume (β). Noting $J_\Psi\sigma_\Psi(\rho) = \sigma_\Psi(\rho)$, we have

$$\sigma_\Psi(\mu) = J_\Psi\sigma_\Psi(\mu) = J_\Psi A\sigma_\Psi(\rho) = j_\Psi(A)\sigma_\Psi(\rho)\ .$$

Hence

$$\begin{aligned}\mu(Q) &= (Q^{1/2}\sigma_\Psi(\rho),\ j_\Psi(A^*A)Q^{1/2}\sigma_\Psi(\rho)) \\ &\leqq \|\, j_\Psi(A^*A)\,\|\ \rho(Q)\end{aligned} \tag{9.6}$$

for $Q \geqq 0$, $Q \in R$. Hence (β) implies (α).

Next assume (α). Then there exists $t' \in R'$ such that

$$\sigma_\Psi(\mu) = t'\sigma_\Psi(\rho)\ , \qquad \|\, t'\,\| \leqq l\ . \tag{9.7}$$

Since $J_\Psi\sigma_\Psi(\mu) = \sigma_\Psi(\mu)$ and $J_\Psi\sigma_\Psi(\rho) = \sigma_\Psi(\rho)$, we have

$$\begin{aligned}\sigma_\Psi(\mu) = J_\Psi\sigma_\Psi(\mu) &= J_\Psi\{t's^{R'}(\sigma_\Psi(\rho))\}\sigma_\Psi(\rho) \\ &= j_\Psi\{t's^{R'}(\sigma_\Psi(\rho))\}\sigma_\Psi(\rho)\ .\end{aligned}$$

Hence we have (9.1) with

$$A = j_\Psi\{t's^{R'}(\sigma_\Psi(\rho))\}\ .$$

Since $j_\Psi\{s^{R'}(\sigma_\Psi(\rho))\} = s^R(\sigma_\Psi(\rho)) = s(\rho)$ due to $J_\Psi\sigma_\Psi(\rho) = \sigma_\Psi(\rho)$, we have

$$s^R(A^*A) \leqq j_\Psi\{s^{R'}(\sigma_\Psi(\rho))\} = s(\rho)\ .$$

If $A_1\sigma_\Psi(\rho) = A_2\sigma_\Psi(\rho) = \sigma_\Psi(\mu)$, then $(A_1 - A_2)\sigma_\Psi(\rho) = 0$. Hence $(A_1 - A_2)s^R(\sigma_\Psi(\rho)) = 0$. By (9.2), $A_ks^R(\sigma_\Psi(\rho)) = A_k$ and hence $A_1 = A_2$.

(2) From (9.6), we have

$$l_0 \equiv \inf\{l;\ l\rho \geqq \mu\} \leqq \|\, A^*A\,\| = \|\, A\,\|^2\ .$$

From (9.7), we have

$$\|\, A\,\|^2 \leqq \|\, t'\,\|^2 \leqq l$$

for any l satisfying $l\rho \geqq \mu$. Hence we have (9.3).

To prove (9.4), we first show that

$$s(AA^*) = s(\mu)\ . \tag{9.8}$$

For $e \in R$, $e \geqq 0$, $\mu(e) = 0$ is equivalent to $eA\sigma_\Psi(\rho) = 0$, which is equivalent to $eA=0$ due to $s(A^*A)\leqq s(\rho)$. Hence (9.8) holds. We now consider restriction of R and H by $s(\rho)j_\Psi\{s(\rho)\}$. Let $M=s(\rho)Rs(\rho)\mid K$, $K=s(\rho)j_\Psi\{s(\rho)\}H$. ξ_ρ is cyclic and separating and

$$s(\rho)j_\Psi\{s(\rho)\}A\xi_\rho = s(\rho)A\xi_\rho = A\xi_\rho$$

where $s(\rho)\geqq s(\mu)$ due to $l\rho\geqq\mu$, which implies $s(\rho)A=A$, and $j_\Psi\{s(\rho)\}\xi_\rho=\xi_\rho$. Thus $A\xi_\rho\in V_{\xi_\rho}=s(\rho)j_\Psi\{s(\rho)\}V_\Psi$.

By Theorem 3 (9), we have (9.4).

(3) follows from the uniqueness.

(4) $s(\mu)$ satisfies (9.1) and (9.2) with $\rho=\mu$.

(5) follows from Theorem 7 (6).

REMARK. If R is commutative, $A(\mu/\rho)$ is the same as the positive square root of the Radon-Nikodym derivative in measure theoretical sense. The following theorem gives a condition that $A(\mu/\rho)$ coincides with Sakai's noncommutative Radon-Nikodym derivative. Because of the chain rule, it also coincides with the condition $A_1(\mu/\rho)=A_2(\mu/\rho)$ when $l_1\mu\geqq\rho$ and $l_2\rho\geqq\mu$, where $A_k(\mu/\nu)$, $k=1, 2$, are defined in [3].

THEOREM 13. *If $l\rho\geqq\mu$, the following conditions are equivalent.*

(a) $A(\mu/\rho)^*=A(\mu/\rho)$,

(b) $A(\mu/\rho)\geqq 0$,

(c) $\tau_\rho(t)A(\mu/\rho)=A(\mu/\rho)$ *where $\tau_\rho(t)$ is the modular automorphism for the state ρ of the reduced algebra $s(\rho)Rs(\rho)$.*

(d) *μ commutes with ρ.*

Proof. If (c) holds, then $A(\mu/\rho)\xi_\rho=\xi_\mu\in V_{\xi_\rho}$ implies

$$0\leqq\tau_\rho(i/4)A(\mu/\rho)=A(\mu/\rho)\,.$$

Hence (c) implies (b). (b) trivially implies (a).

Assume (a). For any $Q\in R$ and $A=A(\mu/\rho)$, we have

$$\begin{aligned}(\xi_\rho, QA\xi_\rho) &= (\xi_\rho, QJ_{\xi_\rho}A\xi_\rho) = (\xi_\rho, Qj_{\xi_\rho}(A)\xi_\rho)\\ &= (j_{\xi_\rho}(A)\xi_\rho, Q\xi_\rho) = (J_{\xi_\rho}A\xi_\rho, Q\xi_\rho)\\ &= (A\xi_\rho, Q\xi_\rho) = (\xi_\rho, AQ\xi_\rho)\,.\end{aligned}$$

Such A is known to be invariant under $\tau_\rho(t)$. ([9])

The equivalence of (c) and (d) is known. ([9])

10. **Ψ-bounded operators.** We shall call $Q\in R$ Ψ-bounded if

$$\omega_{Q\Psi}\leqq l\omega_\Psi\,.$$

for some $l\geqq 0$. We shall call $Q\in R$ Ψ-symmetric if

$$J_\Psi Q\Psi = Q\Psi .$$

We shall call $Q \in R$ Ψ-positive if

$$Q\Psi \in V_\Psi .$$

THEOREM 14.

(1) *Q is Ψ-bounded if and only if there exists a $Q^\Psi \in R$ such that*

$$\Delta_\Psi^{1/2} Q^* \Psi = Q^\Psi \Psi . \tag{10.1}$$

(2) *Any Ψ-bounded Q can be decomposed as $Q = Q_r + iQ_i$ where Q_r, $Q_i \in R$ and both are Ψ-symmetric.*

(3) *Any Ψ-symmetric $Q \in R$ is Ψ-bounded and $Q^\Psi = Q$. It has a decomposition*

$$Q = Q_1 - Q_2 \tag{10.2}$$

where Q_1, $Q_2 \in R$, both Q_1 and Q_2 are Ψ-positive, $\|Q_1\| \leqq \|Q\|$, $\|Q_2\| \leqq \|Q\|$ and

$$s(Q_1 Q_1^*) \perp s(Q_2 Q_2^*) . \tag{10.3}$$

(4) *Any $Q \in R$ has a unique decomposition*

$$Q = u\,|Q|_\Psi \tag{10.4}$$

where u is a partial isometry in R such that

$$u^* u = s(|Q|_\Psi |Q|_\Psi^*) \tag{10.5}$$

and $|Q|_\Psi$ is Ψ-positive.

(5) *$Q \in R$ is Ψ-positive if and only if Q is Ψ-symmetric and $\tau_\Psi(i/4)Q$ is positive.*

Proof (1). If $\omega_{Q\Psi} \leqq l\omega_\Psi$, then there exists $Q' \in R'$, $0 \leqq Q' \leqq l^{1/2}$ such that $\omega_{Q\Psi} = \omega_{Q'\Psi}$. Then there exists a partial isometry $u' \in R'$ such that $Q\Psi = u'Q'\Psi$. Let $Q^\Psi \equiv j_\Psi(u'Q')$. We have

$$Q^\Psi \Psi = J_\Psi u'Q'\Psi = J_\Psi Q\Psi = \Delta_\Psi^{1/2} Q^* \Psi .$$

Conversely, if (10.1) holds, then

$$Q\Psi = J_\Psi \Delta_\Psi^{1/2} Q^* \Psi = J_\Psi Q^\Psi \Psi = j_\Psi(Q^\Psi)\Psi .$$

Hence $\omega_{Q\Psi} \leqq \|j_\Psi(Q^\Psi)\|^2 \omega_\Psi$.

(2) Define $Q_r = (Q + Q^\Psi)/2$, $Q_i = (Q - Q^\Psi)/(2i)$. Then both are Ψ-symmetric and $Q = Q_r + iQ_i$.

(3) Let $Q\Psi = \Phi_1 - \Phi_2$, $\Phi_1 \in V_\Psi$, $\Phi_2 \in V_\Psi$, $s^{R'}(\Phi_1) \perp s^{R'}(\Phi_2)$, $s^R(\Phi_1) \perp$

$S^R(\Phi_2)$ be the decomposition given by Theorem 4 (6). Denote $s' = s^{R'}(\Phi_1)$. We have $\Phi_1 = s'Q\Psi$. Hence $\omega_{\Phi_1} = w_{s'Q\Psi} \leqq \omega_{Q\Psi}$. Since $Q\Psi = J_\Psi Q\Psi = j_\Psi(Q)\Psi$, $\omega_{Q\Psi} \leqq \|j_\Psi(Q)\|^2\omega_\Psi = \|Q\|^2\omega_\Psi$. Hence by Theorem 3 (8), there exists a Ψ-positive $Q_1 \in R$ such that $\Phi_1 = Q_1\Psi$ and $\|Q_1\| \leqq \|Q\|$. Similarly there exists Ψ-positive $Q_2 \in R$ such that $\Phi_2 = Q_2\Psi$ and $\|Q_2\| \leqq \|Q\|$. Since Ψ is separating, (10.2) holds.

Since Ψ is separating for R, we have $s^R(Q_k\Psi) = s(Q_kQ_k^*)$, $k = 1, 2$. Since $s^R(\Phi_1) \perp s^R(\Phi_2)$, we have (10.3).

(4) Let $\rho = \omega_{J_\Psi Q\Psi}$. Then $\rho \leqq \|j_\Psi(Q)\|^2\omega_\Psi = \|Q\|^2\omega_\Psi$. Hence there exists a Ψ-positive $Q_1 \in R$ such that $\sigma_\Psi\rho = Q_1\Psi$. Since $\omega_{J_\Psi Q\Psi} = \omega_{Q_1\Psi}$, there exists a partial isometry $u' \in R'$ such that $J_\Psi Q\Psi = u'Q_1\Psi$ and $u'^*u' = s^{R'}(Q_1\Psi) = j_\Psi\{s^R(Q_1\Psi)\} = j_\Psi\{s(Q_1Q_1^*)\}$ where we have used the property $J_\Psi Q_1\Psi = Q_1\Psi$.

We now have $Q\Psi = J_\Psi u'Q_1\Psi = j_\Psi(u')J_\Psi Q_1\Psi = uQ_1\Psi$ where $u \equiv j_\Psi(u')$. Since Ψ is separating for R, $Q = uQ_1$. We have $u^*u = j_\Psi(u'^*u') = s(Q_1Q_1^*)$. Hence $Q_1 = |Q|_\Psi$ and u satisfy (10.4) and (10.5).

Conversely, assume that $Q = u_kQ_k$, Q_k is Ψ-positive, u_k is partially isometric, u_k, $Q_k \in R$, $u_k^*u_k = s(Q_kQ_k^*)$, $k = 1, 2$. Then $\omega_{J_\Psi Q\Psi} = \omega_{Q_k\Psi}$ where we have used $J_\Psi Q_k\Psi = Q_k\Psi$. Since $Q_k\Psi \in V_\Psi$, such $Q_k\Psi$ is unique by Theorem 7 (1) and we have $Q_1 = Q_2$.

Since $u_1Q_1 = u_2Q_2 = u_2Q_1$, we have $(u_1 - u_2)s(Q_1Q_1^*) = 0$. Since $u_1^*u_1 = s(Q_1Q_1^*) = s(Q_2Q_2^*) = u_2^*u_2$, we have $u_ks(Q_1Q_1^*) = u_k$, $k = 1, 2$, and hence $u_1 = u_2$.

(5) Q is Ψ-symmetric if Q is Ψ-positive by (5.2). By Theorem 3 (7) with $\alpha = 1/4$, $\tau_\Psi(i/4)Q \geqq 0$ if $Q\Psi \in V_\Psi$. If Q is Ψ-symmetric, then $JQ\Psi = Q\Psi$. Hence $\Delta_\Psi^{1/2}Q^*\Psi = Q\Psi$, which implies $\Delta_\Psi^{-1/2}Q\Psi = Q^*\Psi$. Hence $\tau_{\Psi(z)}Q \in R$ can be defined by Lemma 6 for $\mathrm{Im}\, z \in [0, 1/2]$. Hence $(\Delta_\Psi^{-1/4}Q\Delta_\Psi^{1/4})^-$ is in R. If it is positive, then $Q\Psi \in V_\Psi$ by Theorem 3 (7).

THEOREM 15. *If $\rho \leqq l\omega_\Psi$, there exists $Q \in R$, $0 \leqq Q \leqq l^{1/4}$ such that $\sigma_\Psi\rho = Qj_\Psi(Q)\Psi$.*

Proof. Let $\rho_1(A) = (\sigma_\Psi\rho, \Delta_\Psi^{1/4}A\Psi)$ for $A \in R$. Then $\rho_1 \in R_*^+$. Since $\rho \leqq l\omega_\Psi$, there exists $Q_1 \in R$ such that $Q_1\Psi = \sigma_\Psi\rho$, $\|Q_1\|^2 \leqq l$. Then $Q_1\Psi = j_\Psi(Q_1)\Psi$ and

$$\Delta_\Psi^{1/4}j_\Psi(Q_1)\Psi = j_\Psi(\tau_\Psi(i/4)Q_1)\Psi ,$$

where

$$0 \leqq \tau_\Psi(i/4)Q_1 \equiv Q_2 \leqq \|Q_1\| \leqq l^{1/2}$$

by Theorem 3 (7).

Set $Q_2' = j_\Psi(Q_2)$. We have

$$\rho_1(A) = (Q_2'\Psi, A\Psi) = (Q_2'^{1/2}\Psi, AQ_2'^{1/2}\Psi) .$$

Hence $\rho_1 \leqq \| Q_2' \| \omega_\Psi \leqq l^{1/2}\omega_\Psi$. By Theorem 7 (1), there exists a Ψ-positive $Q_3 \in R$ such that $\sigma_\Psi \rho_1 = Q_3\Psi$, $\| Q_3 \| \leqq l^{1/4}$. Let $Q = \tau_\Psi(i/4)Q_3$. By Theorem 3 (7), $\| Q \| \leqq \| Q_3 \| \leqq l^{1/4}$ and $Q \geqq 0$. We have $Q_3\Psi = J_\Psi Q_3 \Psi = j_\Psi(Q_3)\Psi$ and hence

$$\begin{aligned}(\sigma_\Psi \rho, \Delta_\Psi^{1/4} A\Psi) &= (Q_3\Psi, AQ_3\Psi) = (Q_3\Psi, Aj_\Psi(Q_3)\Psi) \\ &= (Q_3 j_\Psi(Q_3^*)\Psi, A\Psi) = (\{\tau_\Psi(i/4)Q_3\} j_\Psi(\{\tau_\Psi(i/4)Q_3\}^*)\Psi, \Delta_\Psi^{1/4} A\Psi) .\end{aligned}$$

Since $\Delta_\Psi^{1/4} A\Psi$, $A \in R$, is dense, we have

$$\sigma_\Psi \rho = Qj_\Psi(Q)\Psi .$$

11. Additional remarks. In this paper, we have assumed that R has a faithful normal state. This assumption is not essential in defining $d'(\rho_1, \rho_2)$ and $(\sigma_\Psi\rho_1, \sigma_\Psi\rho_2)$. They can be defined relative to sRs where $s = s(\rho_1) \vee s(\rho_2)$. With such definition, Theorem 9 holds.

The cone W_Ψ has been introduced as the weakly closed convex hull of $Qj_\Psi(Q)$, $Q \in R$. It is a weakly closed selfadjoint convex cone which form a semigroup under multiplication. It is total in $W \equiv (R \cup R')''$.

If $\rho \in W_*$ is of the form $\rho = \sum_j \omega_{x_j y_j}$ with $x_j, y_j \in V_\Psi$, then $\rho(w) \geqq 0$ for all $w \in W_\Psi$. If $\rho \in W_*$, $\rho = \omega_x$ and $\rho(w) \geqq 0$ for all $w \in W_\Psi$, then $\rho = \omega_y$ for $y \in V_\Psi$ by Theorem 3. It is of interest to determine the dual of W_Ψ in W_*. If R is a type I factor, the dual of W_Ψ consists of $\rho = \sum_j \omega_{x_j y_j}$, $x_j, y_j \in V_\Psi$.

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Appendix. The result that V_Ψ is selfdual can be proved directly as follows:

We define V_Ψ first as the closed convex hull of $\{Qj_\Psi(Q)\Psi; Q \in R\}$. Then (5.1)~(5.4) are immediate. In particular (5.4) shows $V_\Psi \subset V_\Psi'$. Let $\Phi \in V_\Psi'$.

By noncommutative Radon-Nikodym theorem, there exists a positive selfadjoint A_2 affiliated with R and a partial isometry $u' \in R'$ such that $\Phi = u'A_2\Psi$. If $A_2 = \int \lambda dE_\lambda$, we set $A_2^L = A_2E_L$ and

$$\Phi^L \equiv E_L j_\Psi(E_L)\Phi = j_\Psi(E_L)u'A_2^L\Psi .$$

Then $\lim \Phi^L = \Phi$ and $\Phi^L \in V_\Psi'$. Since $\omega_\Phi L \leqq \omega_{A_2^L\Psi}$, there exists $t \in R$, $0 \leqq t \leqq 1$ and a partial isometry $w \in R'$ such that

$$\Phi^L = wtA_2^L\Psi,\ w^*\Phi^L = tA_2^L\Psi,\ s^{R'}(\Phi^L) = ww^* .$$

Set $\Phi' = A_3\Psi$, $A_3 \equiv j_\Psi(w^*)tA_2^L \in R$. Then $\Phi' = w^* j_\Psi(w^*)\Phi^L \in V'_\Psi$. Since $\Phi^L \in V'_\Psi$, we have $(x, J_\Psi \Phi^L) = (\Phi^L, J_\Psi x) = (\Phi^L, x) = (x, \Phi^L) \geqq 0$ for $x \in V_\Psi$. Since V_Ψ is total, we have $J_\Psi \Phi^L = \Phi^L$ and hence $j_\Psi(w)w\Phi' = j_\Psi(ww^*)\Phi^L = \Phi^L$. Hence it is enough to show $\Phi' \in V_\Psi$. Let $\Phi'_\beta = A_4\Psi$, $A_4 = A_3(f_\beta^G)$ defined by (3.7) and (3.11). Then $\Phi'_\beta \in V'_\Psi$ and $\lim_{\beta \to +0} \Phi'_\beta = \Phi'$.

Let $A_5 = \tau(i/4)A_4$. Since $A_4\Psi \in V'_\Psi$, we have

$$\begin{aligned}(\Delta_\Psi^{1/4}\Phi\Psi, A_5\Delta_\Psi^{1/4}Q\Psi) &= (Q\Psi, A_4\Delta_\Psi^{1/2}Q\Psi) = (Q\Psi, A_4 j_\Psi(Q^*)\Psi) \\ &= (j_\Psi(Q)Q\Psi, A_4\Psi) \geqq 0 \,.\end{aligned}$$

Since $\Delta_\Psi^{1/4}R\Psi$ is dense, we have $A_5 \geqq 0$. Let $B = A_5^{1/2}$, $B_\gamma = B(f_\gamma^G)$. Then $\lim B_\gamma^2 = A_5$ and

$$\begin{aligned}\| \Delta_\Psi^{1/4}(B_\gamma^2 - A_5)\Psi \|^2 &= (\{B_\gamma^2 - A_5\}\Psi, \Delta_\Psi^{1/2}\{B_\gamma^2 - A_5\}\Psi) \\ &= (\{B_\gamma^2 - A_5\}\Psi, J_\Psi\{B_\gamma^2 - A_5\}\Psi) \to 0\end{aligned}$$

as $\gamma \to +0$. Therefore,

$$\lim \Delta_\Psi^{1/4}B_\gamma^2\Psi = \Delta_\Psi^{1/4}A_5\Psi = \{\tau_\Psi(-i/4)A_5\}\Psi = A_4\Psi = \Phi'_\beta \,.$$

We also have

$$\Delta_\Psi^{1/4}B_\gamma^2\Psi = Cj_\Psi(C)\Psi$$

for $C = \tau_\Psi(-i/4)B_\gamma$ due to $J_\Psi C\Psi = C\Psi$. Hence $\Delta_\Psi^{1/4}B_\gamma^2\Psi \in V_\Psi$. This completes the proof.

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