

# Pacific Journal of Mathematics

**WHITEHEAD GROUPS OF TWISTED FREE ASSOCIATIVE  
ALGEBRAS**

K. G. CHOO

## WHITEHEAD GROUPS OF TWISTED FREE ASSOCIATIVE ALGEBRAS

KOO-GUAN CHOO

**Let  $R$  be an associative ring with identity and  $X$  a set of noncommuting variables  $\{x_\lambda\}_{\lambda \in A}$ . Let  $R\{X\}$  be the free associative algebra on  $X$  over  $R$ . Then S. Gersten has shown that if  $K_1R \rightarrow K_1R[t]$  is an isomorphism, where  $R[t]$  is the polynomial extension of  $R$ , then  $K_1R \rightarrow K_1R\{X\}$  is an isomorphism.**

**The purpose of this paper is to extend the result of Gersten to twisted free associative algebras.**

Let  $R$  be an associative ring with identity and  $X$  a set of noncommuting variables  $\{x_\lambda\}_{\lambda \in A}$  and  $\alpha = \{\alpha_\lambda\}_{\lambda \in A}$  a set of automorphisms  $\alpha_\lambda$  of  $R$ . The  $\alpha$ -twisted free associative algebra on  $X$  over  $R$ , denoted by  $R_\alpha\{X\}$ , is defined as follows:

Additively,  $R_\alpha\{X\} = R\{X\}$  so that its elements are finite linear combinations of words  $w(x_\lambda)$  in  $x_\lambda$  with coefficients in  $R$ .

If  $w(x_\lambda) = x_{\lambda_1} \cdots x_{\lambda_k}$  is a word in  $x_\lambda$ , we denote the automorphism  $\alpha_{\lambda_1} \cdots \alpha_{\lambda_k}$  by  $w(\alpha_\lambda)$ .

Multiplication in  $R_\alpha\{X\}$  is given by:

$$(rw(x_\lambda))(r'w'(x_\lambda)) = rw(\alpha_\lambda)^{-1}(r')w'(x_\lambda)w'(x_\lambda),$$

for any  $rw(x_\lambda), r'w'(x_\lambda) \in R_\alpha\{X\}$ .

In particular, if  $X = \{t\}$  and  $\alpha = \{\alpha\}$ , then  $R_\alpha\{X\}$  is just the  $\alpha$ -twisted polynomial ring  $R_\alpha[t]$ .

We shall consider  $R_\alpha\{X\}$  as an  $R$ -ring with augmentation  $\varepsilon_X: R_\alpha\{X\} \rightarrow R$  defined by  $\varepsilon_X(x_\lambda) = 0$  for each  $x_\lambda \in X$ . Denoted by  $\bar{K}_1R_\alpha\{X\}$  the cokernel of the homomorphism  $i_*: K_1R \rightarrow K_1R_\alpha\{X\}$  induced by the inclusion  $i: R \rightarrow R_\alpha\{X\}$ . Note that the augmentation  $\varepsilon_X$  induces a homomorphism  $\varepsilon_{X*}: K_1R_\alpha\{X\} \rightarrow K_1R$  which splits  $i_*$ .

Let  $W(X)$  be the set of all the words  $w(x_\lambda)$  in  $x_\lambda$ . For each  $w(x_\lambda)$  in  $W(X)$ , let  $\beta_w$  be the automorphism  $w(\alpha_\lambda)$ ,  $h_{\beta_w}$  the homomorphism of  $R_{\beta_w}[t]$  into  $R_\alpha\{X\}$  defined by  $h_{\beta_w}(t) = w(x_\lambda)$  and  $\bar{h}_{\beta_w}$  the homomorphism of  $\bar{K}_1R_{\beta_w}[t]$  into  $\bar{K}_1R_\alpha\{X\}$  induced by  $h_{\beta_w}$ . Then our main result is:

**THEOREM 1.** *The group  $\bar{K}_1R_\alpha\{X\}$  is generated by the homomorphic images of  $\bar{K}_1R_{\beta_w}[t]$  under  $\bar{h}_{\beta_w}$  and  $w(x_\lambda)$  runs over  $W(X)$ .*

As a consequence, we have:

**THEOREM 2.** *(Twisted Case of Gersten's Theorem). If  $K_1R \rightarrow K_1R_{\beta_w}[t]$  is an isomorphism for each  $\beta_w$ , then  $K_1R \rightarrow K_1R_\alpha\{X\}$  is an*

*isomorphism.*

Now, let  $A$  be an invertible matrix over  $R_n\{X\}$ . By Higman's trick (cf. [4]), we can make  $A$  equivalent in  $K_1R_n\{X\}$  to

$$B = B_0 + B_1x_1 + \cdots + B_nx_n,$$

where  $x_1, \dots, x_n$  are distinct elements of  $X$  and  $B_i (i = 0, 1, \dots, n)$  are  $m \times m$  matrices over  $R$  for some integer  $m$ . By applying the homomorphism  $\varepsilon_{X^*}$  to  $B$ , we deduce that  $B_0$  is invertible. Hence  $A$  can be made equivalent in  $\bar{K}_1R_n\{X\}$  to

$$(1) \quad N = I + N_1x_1 + \cdots + N_nx_n,$$

where  $N = B_0^{-1}B$  and  $N_i = B_0^{-1}B_i (i = 1, \dots, n)$ .

The inverse of this matrix  $N$  can be written explicitly in the ring of formal power series. Since this inverse exists in  $R_n\{X\}$ , all but a finite number of the power series coefficients are zero. That is, if

$$M = M_0 + M_1x_1 + \cdots + M_nx_n + \sum_{i,j=1}^n M_{i,j}x_ix_j + \cdots$$

is a matrix over  $R_n\{X\}$ , where all  $M_i, M_{i,j}, \dots$  are matrices over  $R$ , such that  $MN = NM = I$ , then there is an integer  $K > 0$  such that  $M_{i_1, i_2, \dots, i_k} = 0$  for all  $k > K$ , where  $i_1, i_2, \dots, i_k$  run over  $1, \dots, n$  respectively. From  $NM = I$ , we get, by equating coefficients of monomials in the  $x$ 's, the following relations:

$$\begin{aligned} M_0 &= I; \\ M_i &= -N_i && (i = 1, \dots, n); \\ M_{i,j} &= N_i\alpha_i^{-1}(N_j) && (i, j = 1, \dots, n); \\ &\vdots \\ M_{i_1, i_2, \dots, i_l} &= (-1)^l N_{i_1}\alpha_{i_1}^{-1}(N_{i_2}) \cdots (\alpha_{i_1}^{-1}\alpha_{i_2}^{-1} \cdots \alpha_{i_{l-1}}^{-1})(N_{i_l}) \\ &&& (i_1, i_2, \dots, i_l = 1, \dots, n). \end{aligned}$$

Hence, for all  $k > K$ ,

$$(2) \quad N_{i_1}\alpha_{i_1}^{-1}(N_{i_2}) \cdots (\alpha_{i_1}^{-1}\alpha_{i_2}^{-1} \cdots \alpha_{i_{k-1}}^{-1})(N_{i_k}) = 0.$$

Let us call a matrix  $P$  over  $R$   $\beta$ -twisted nilpotent ( $\beta$  is any automorphism of  $R$ ) if there exists an integer  $k > 0$  such that

$$P\beta^{-1}(P) \cdots \beta^{-(k-1)}(P) = 0.$$

Hence, it follows from (2) that each  $N_i (i = 1, \dots, n)$  in (1) is  $\alpha_i$ -twisted nilpotent.

Our next lemma is the key to the main result:

LEMMA 3. *The matrix  $N$  in (1) is a product of matrices of the form  $I + Pw(x_1, \dots, x_n)$ , where  $P$  is an  $w(\alpha_1, \dots, \alpha_n)$ -twisted nilpotent matrix over  $R$ . ( $w(x_1, \dots, x_n)$  denotes a word in  $x_1, \dots, x_n$ .)*

*Proof.* Recall from (1) and (2) that each  $N_i (i = 1, \dots, n)$  in (1) is  $\alpha_i$ -twisted nilpotent. Consider

$$I + Q = (I - N_1x_1) \cdots (I - N_nx_n)N .$$

Then  $Q$  is of the form  $\sum_j Q_j s_j$ , where  $s_j$  is a monomial of degree at least two in the  $x_1, \dots, x_n$ . In fact, if  $s_j = x_{i_1}x_{i_2} \cdots x_{i_l} (l \geq 2)$ , then

$$(3) \quad Q_j = \pm N_{i_1}\alpha_{i_1}^{-1}(N_{i_2}) \cdots (\alpha_{i_1}^{-1} \cdots \alpha_{i_{l-1}}^{-1})(N_{i_l}) .$$

Hence, for  $k > K/2$ ,

$$Q_j\beta^{-1}(Q_j) \cdots \beta^{-(k-1)}(Q_j) = 0 ,$$

for each  $j$ , where  $\beta$  is an automorphism obtained by replacing the  $x_i$  in  $s_j$  by  $\alpha_i$  respectively. That is,  $Q_j$  is  $s_j(\alpha_1, \dots, \alpha_n)$ -twisted nilpotent for each  $j$ . Now, consider

$$I + Q' = \prod_j (I - Q_j s_j)(I + Q) .$$

Then  $Q'$  is of the form  $\sum_\sigma Q'_\sigma y_\sigma$ , where each  $y_\sigma$  is a monomial of degree at least four in the  $x_1, \dots, x_n$  and for  $l \geq 4$ ,  $Q'_\sigma$  is of the form as given on the right hand side of (3). Thus, for  $k > K/4$ ,

$$Q'_\sigma\gamma^{-1}(Q'_\sigma) \cdots \gamma^{-(k-1)}(Q'_\sigma) = 0 ,$$

for each  $\sigma$ , where  $\gamma$  is an automorphism obtained by replacing the  $x_i$  in  $y_\sigma$  by  $\alpha_i$  respectively. That is,  $Q'_\sigma$  is  $y_\sigma(\alpha_1, \dots, \alpha_n)$ -twisted nilpotent for each  $\sigma$ .

Left multiplying  $I + Q'$  by  $\prod_\sigma (I - Q'_\sigma y_\sigma)$ , and repeating the above argument, we will finally arrive, after a finite steps (because of the finite bound  $K$  and condition (2)), at the conclusion that

$$\prod (I + Pw(x_1, \dots, x_n)) \cdot N = I ,$$

where  $P$  is an  $w(\alpha_1, \dots, \alpha_n)$ -twisted nilpotent matrix over  $R$  and  $w(x_1, \dots, x_n)$  is a word in  $x_1, \dots, x_n$ .

This completes the proof.

The above discussions are modifications of those given in [3] and ([1], p. 647) for (untwisted) free associative algebras; and the following result is already contained in the above proof (also cf. [2]).

LEMMA 4. *For any automorphism  $\beta$  of  $R$ ,  $\bar{K}_1 R_\beta[t]$  is generated by the elements of the form  $I + Pt$ , where  $P$  is an  $\beta$ -twisted nilpotent matrix over  $R$ .*

*Proof of Theorem 1.* It follows immediately from Lemmas 3 and 4.

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