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Let R be an associative ring with identity and X a set of noncommuting variables $\{x_\lambda\}_{\lambda\in\Lambda}$. Let $R\{X\}$ be the free associative algebra on X over R. Then S. Gersten has shown that if $K_1R \to K_1R[t]$ is an isomorphism, where $R[t]$ is the polynomial extension of R, then $K_1R \to K_1R\{X\}$ is an isomorphism. The purpose of this paper is to extend the result of Gersten to twisted free associative algebras.

Let R be an associative ring with identity and X a set of noncommuting variables $\{x_\lambda\}_{\lambda\in\Lambda}$ and $\mathfrak{a} = \{\alpha_\lambda\}_{\lambda\in\Lambda}$ a set of automorphisms α_λ of R. The $\mathfrak{a}$-twisted free associative algebra on X over R, denoted by $R_\mathfrak{a}\{X\}$, is defined as follows:

Additively, $R_\mathfrak{a}\{X\} = R\{X\}$ so that its elements are finite linear combinations of words $w(x_\lambda)$ in x_λ with coefficients in R.

If $w(x_\lambda) = x_{\lambda_1} \cdots x_{\lambda_k}$ is a word in x_λ, we denote the automorphism $\alpha_{\lambda_1} \cdots \alpha_{\lambda_k}$ by $w(\alpha_\lambda)$.

Multiplication in $R_\mathfrak{a}\{X\}$ is given by:

$$(rw(x_\lambda))(r'w'(x_\lambda)) = rw(\alpha_\lambda)^{-1}(r')w(x_\lambda)w'(x_\lambda)\ ,$$

for any $rw(x_\lambda),\ r'w'(x_\lambda) \in R_\mathfrak{a}\{X\}$.

In particular, if $X = \{t\}$ and $\mathfrak{a} = \{\alpha\}$, then $R_\mathfrak{a}\{X\}$ is just the α-twisted polynomial ring $R_\alpha[t]$.

We shall consider $R_\mathfrak{a}\{X\}$ as an R-ring with augmentation ε_X: $R_\mathfrak{a}\{X\} \to R$ defined by $\varepsilon_X(x_\lambda) = 0$ for each $x_\lambda \in X$. Denoted by $\bar{K}_1R_\mathfrak{a}\{X\}$ the cokernel of the homomorphism $i_*: K_1R \to K_1R_\mathfrak{a}\{X\}$ induced by the inclusion $i: R \to R_\mathfrak{a}\{X\}$. Note that the augmentation ε_X induces a homomorphism $\varepsilon_{X*}: K_1R_\mathfrak{a}\{X\} \to K_1R$ which splits i_*.

Let $W(X)$ be the set of all the words $w(x_\lambda)$ in x_λ. For each $w(x_\lambda)$ in $W(X)$, let β_w be the automorphism $w(\alpha_\lambda)$, h_{β_w} the homomorphism of $R_{\beta_w}[t]$ into $R_\mathfrak{a}\{X\}$ defined by $h_{\beta_w}(t) = w(x_\lambda)$ and $\bar{h}_{\beta_w}$ the homomorphism of $\bar{K}_1R_{\beta_w}[t]$ into $\bar{K}_1R_\mathfrak{a}\{X\}$ induced by h_{β_w}. Then our main result is:

THEOREM 1. *The group $\bar{K}_1R_\mathfrak{a}\{X\}$ is generated by the homomorphic images of $\bar{K}_1R_{\beta_w}[t]$ under $\bar{h}_{\beta_w}$ and $w(x_\lambda)$ runs over $W(X)$.*

As a consequence, we have:

THEOREM 2. (*Twisted Case of Gersten's Theorem*). *If $K_1R \to K_1R_{\beta_w}[t]$ is an isomorphism for each β_w, then $K_1R \to K_1R_\mathfrak{a}\{X\}$ is an* *isomorphism.*

Now, let A be an invertible matrix over $R_\alpha\{X\}$. By Higman's trick (cf. [4]), we can make A equivalent in $K_1R_\alpha\{X\}$ to

$$B = B_0 + B_1x_1 + \cdots + B_nx_n ,$$

where $x_1, \cdots, x_n$ are distinct elements of X and $B_i (i = 0, 1, \cdots, n)$ are $m \times m$ matrices over R for some integer m. By applying the homomorphism ε_{X^*} to B, we deduce that B_0 is invertible. Hence A can be made equivalent in $\bar{K}_1R_\alpha\{X\}$ to

$$N = I + N_1x_1 + \cdots + N_nx_n , \tag{1}$$

where $N = B_0^{-1}B$ and $N_i = B_0^{-1}B_i (i = 1, \cdots, n)$.

The inverse of this matrix N can be written explicitly in the ring of formal power series. Since this inverse exists in $R_\alpha\{X\}$, all but a finite number of the power series coefficients are zero. That is, if

$$M = M_0 + M_1x_1 + \cdots + M_nx_n + \sum_{i,j=1}^{n} M_{i,j}x_ix_j + \cdots$$

is a matrix over $R_\alpha\{X\}$, where all $M_i, M_{i,j}, \cdots$ are matrices over R, such that $MN = NM = I$, then there is an integer $K > 0$ such that $M_{i_1,i_2,\cdots,i_k} = 0$ for all $k > K$, where $i_1, i_2, \cdots, i_k$ run over $1, \cdots, n$ respectively. From $NM = I$, we get, by equating coefficients of monomials in the x's, the following relations:

$$\begin{aligned}
&M_0 = I; \\
&M_i = - N_i && (i = 1, \cdots, n) ; \\
&M_{i,j} = N_i\alpha_i^{-1}(N_j) && (i, j = 1, \cdots, n) ; \\
&\vdots \\
&M_{i_1,i_2,\cdots,i_l} = (-1)^l N_{i_1}\alpha_{i_1}^{-1}(N_{i_2}) \cdots (\alpha_{i_1}^{-1}\alpha_{i_2}^{-1} \cdots \alpha_{i_{l-1}}^{-1})(N_{i_l}) \\
&\qquad (i_1, i_2, \cdots, i_l = 1, \cdots, n) .
\end{aligned}$$

Hence, for all $k > K$,

$$N_{i_1}\alpha_{i_1}^{-1}(N_{i_1}) \cdots (\alpha_{i_1}^{-1}\alpha_{i_2}^{-1} \cdots \alpha_{i_{k-1}}^{-1})(N_{i_k}) = 0 . \tag{2}$$

Let us call a matrix P over R β-twisted nilpotent (β is any automorphism of R) if there exists an integer $k > 0$ such that

$$P\beta^{-1}(P) \cdots \beta^{-(k-1)}(P) = 0 .$$

Hence, it follows from (2) that each $N_i (i = 1, \cdots, n)$ in (1) is α_i-twisted nilpotent.

Our next lemma is the key to the main result:

LEMMA 3. *The matrix N in* (1) *is a product of matrices of the form $I + Pw(x_1, \cdots, x_n)$, where P is an $w(\alpha_1, \cdots, \alpha_n)$-twisted nilpotent matrix over R.* ($w(x_1, \cdots, x_n)$ *denotes a word in* $x_1, \cdots, x_n$.)

Proof. Recall from (1) and (2) that each $N_i (i = 1, \cdots, n)$ in (1) is α_i-twisted nilpotent. Consider

$$I + Q = (I - N_1 x_1) \cdots (I - N_n x_n) N .$$

Then Q is of the form $\sum_j Q_j s_j$, where s_j is a monomial of degree at least two in the $x_1, \cdots, x_n$. In fact, if $s_j = x_{i_1} x_{i_2} \cdots x_{i_l} (l \geqq 2)$, then

$$Q_j = \pm N_{i_1} \alpha_{i_1}^{-1}(N_{i_2}) \cdots (\alpha_{i_1}^{-1} \cdots \alpha_{i_{l-1}}^{-1})(N_{i_l}) . \tag{3}$$

Hence, for $k > K/2$,

$$Q_j \beta^{-1}(Q_j) \cdots \beta^{-(k-1)}(Q_j) = 0 ,$$

for each j, where β is an automorphism obtained by replacing the x_i in s_j by α_i respectively. That is, Q_j is $s_j(\alpha_1, \cdots, \alpha_n)$-twisted nilpotent for each j. Now, consider

$$I + Q' = \prod_j (I - Q_j s_j)(I + Q) .$$

Then Q' is of the form $\sum_\sigma Q'_\sigma y_\sigma$, where each y_σ is a monomial of degree at least four in the $x_1, \cdots, x_n$ and for $l \geqq 4$, Q'_σ is of the form as given on the right hand side of (3). Thus, for $k > K/4$,

$$Q'_\sigma \gamma^{-1}(Q'_\sigma) \cdots \gamma^{-(k-1)}(Q'_\sigma) = 0 ,$$

for each σ, where γ is an automorphism obtained by replacing the x_i in y_σ by α_i respectively. That is, Q'_σ is $y_\sigma(\alpha_1, \cdots, \alpha_n)$-twisted nilpotent for each σ.

Left multiplying $I + Q'$ by $\prod_\sigma (I - Q'_\sigma y_\sigma)$, and repeating the above argument, we will finally arrive, after a finite steps (because of the finite bound K and condition (2)), at the conclusion that

$$\prod (I + Pw(x_1, \cdots, x_n)) \cdot N = I ,$$

where P is an $w(\alpha_1, \cdots, \alpha_n)$-twisted nilpotent matrix over R and $w(x_1, \cdots, x_n)$ is a word in $x_1, \cdots, x_n$.

This completes the proof.

The above discussions are modifications of those given in [3] and ([1], p. 647) for (untwisted) free associative algebras; and the following result is already contained in the above proof (also cf. [2]).

LEMMA 4. *For any automorphism β of R, $\bar{K}_1R_\beta[t]$ is generated by the elements of the form $I + Pt$, where P is an β-twisted nilpotent matrix over R.*

Proof of Theorem 1. It follows immediately from Lemmas 3 and 4.

REFERENCES

1. H. Bass, *Algebraic K-Theory*, W. A. Benjamin, Inc., New York, 1968.
2. H. Bass, A. Heller, and R. G. Swan, *The Whitehead group of a polynomial extension*, Publ. I. H. E. S. No. 22, (1964), 61–79.
3. S. Gersten, *Whitehead groups of free associative algebras*, Bull. Amer. Math. Soc., **71** (1965), 157–159.
4. G. Higman, *The units of group rings*, Proc. London Math. Soc., **46** (1940), 231-248.

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