

Pacific Journal of Mathematics

**WHITEHEAD GROUPS OF TWISTED FREE ASSOCIATIVE
ALGEBRAS**

K. G. CHOO

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Let R be an associative ring with identity and X a set of noncommuting variables $\{x_\lambda\}_{\lambda \in A}$. Let $R\{X\}$ be the free associative algebra on X over R . Then S. Gersten has shown that if $K_1R \rightarrow K_1R[t]$ is an isomorphism, where $R[t]$ is the polynomial extension of R , then $K_1R \rightarrow K_1R\{X\}$ is an isomorphism.

The purpose of this paper is to extend the result of Gersten to twisted free associative algebras.

Let R be an associative ring with identity and X a set of noncommuting variables $\{x_\lambda\}_{\lambda \in A}$ and $\alpha = \{\alpha_\lambda\}_{\lambda \in A}$ a set of automorphisms α_λ of R . The α -twisted free associative algebra on X over R , denoted by $R_\alpha\{X\}$, is defined as follows:

Additively, $R_\alpha\{X\} = R\{X\}$ so that its elements are finite linear combinations of words $w(x_\lambda)$ in x_λ with coefficients in R .

If $w(x_\lambda) = x_{\lambda_1} \cdots x_{\lambda_k}$ is a word in x_λ , we denote the automorphism $\alpha_{\lambda_1} \cdots \alpha_{\lambda_k}$ by $w(\alpha_\lambda)$.

Multiplication in $R_\alpha\{X\}$ is given by:

$$(rw(x_\lambda))(r'w'(x_\lambda)) = rw(\alpha_\lambda)^{-1}(r')w(x_\lambda)w'(x_\lambda),$$

for any $rw(x_\lambda), r'w'(x_\lambda) \in R_\alpha\{X\}$.

In particular, if $X = \{t\}$ and $\alpha = \{\alpha\}$, then $R_\alpha\{X\}$ is just the α -twisted polynomial ring $R_\alpha[t]$.

We shall consider $R_\alpha\{X\}$ as an R -ring with augmentation $\varepsilon_X: R_\alpha\{X\} \rightarrow R$ defined by $\varepsilon_X(x_\lambda) = 0$ for each $x_\lambda \in X$. Denoted by $\bar{K}_1R_\alpha\{X\}$ the cokernel of the homomorphism $i_*: K_1R \rightarrow K_1R_\alpha\{X\}$ induced by the inclusion $i: R \rightarrow R_\alpha\{X\}$. Note that the augmentation ε_X induces a homomorphism $\varepsilon_{X*}: K_1R_\alpha\{X\} \rightarrow K_1R$ which splits i_* .

Let $W(X)$ be the set of all the words $w(x_\lambda)$ in x_λ . For each $w(x_\lambda)$ in $W(X)$, let β_w be the automorphism $w(\alpha_\lambda)$, h_{β_w} the homomorphism of $R_{\beta_w}[t]$ into $R_\alpha\{X\}$ defined by $h_{\beta_w}(t) = w(x_\lambda)$ and \bar{h}_{β_w} the homomorphism of $\bar{K}_1R_{\beta_w}[t]$ into $\bar{K}_1R_\alpha\{X\}$ induced by h_{β_w} . Then our main result is:

THEOREM 1. *The group $\bar{K}_1R_\alpha\{X\}$ is generated by the homomorphic images of $\bar{K}_1R_{\beta_w}[t]$ under \bar{h}_{β_w} and $w(x_\lambda)$ runs over $W(X)$.*

As a consequence, we have:

THEOREM 2. *(Twisted Case of Gersten's Theorem). If $K_1R \rightarrow K_1R_{\beta_w}[t]$ is an isomorphism for each β_w , then $K_1R \rightarrow K_1R_\alpha\{X\}$ is an*

isomorphism.

Now, let A be an invertible matrix over $R_\alpha\{X\}$. By Higman's trick (cf. [4]), we can make A equivalent in $K_1R_\alpha\{X\}$ to

$$B = B_0 + B_1x_1 + \cdots + B_nx_n,$$

where x_1, \dots, x_n are distinct elements of X and $B_i (i = 0, 1, \dots, n)$ are $m \times m$ matrices over R for some integer m . By applying the homomorphism ε_{X^*} to B , we deduce that B_0 is invertible. Hence A can be made equivalent in $\bar{K}_1R_\alpha\{X\}$ to

$$(1) \quad N = I + N_1x_1 + \cdots + N_nx_n,$$

where $N = B_0^{-1}B$ and $N_i = B_0^{-1}B_i (i = 1, \dots, n)$.

The inverse of this matrix N can be written explicitly in the ring of formal power series. Since this inverse exists in $R_\alpha\{X\}$, all but a finite number of the power series coefficients are zero. That is, if

$$M = M_0 + M_1x_1 + \cdots + M_nx_n + \sum_{i,j=1}^n M_{i,j}x_ix_j + \cdots$$

is a matrix over $R_\alpha\{X\}$, where all $M_i, M_{i,j}, \dots$ are matrices over R , such that $MN = NM = I$, then there is an integer $K > 0$ such that $M_{i_1, i_2, \dots, i_k} = 0$ for all $k > K$, where i_1, i_2, \dots, i_k run over $1, \dots, n$ respectively. From $NM = I$, we get, by equating coefficients of monomials in the x 's, the following relations:

$$\begin{aligned} M_0 &= I; \\ M_i &= -N_i && (i = 1, \dots, n); \\ M_{i,j} &= N_i\alpha_i^{-1}(N_j) && (i, j = 1, \dots, n); \\ &\vdots \\ M_{i_1, i_2, \dots, i_l} &= (-1)^l N_{i_1}\alpha_{i_1}^{-1}(N_{i_2}) \cdots (\alpha_{i_1}^{-1}\alpha_{i_2}^{-1} \cdots \alpha_{i_{l-1}}^{-1})(N_{i_l}) \\ &&& (i_1, i_2, \dots, i_l = 1, \dots, n). \end{aligned}$$

Hence, for all $k > K$,

$$(2) \quad N_{i_1}\alpha_{i_1}^{-1}(N_{i_1}) \cdots (\alpha_{i_1}^{-1}\alpha_{i_2}^{-1} \cdots \alpha_{i_{k-1}}^{-1})(N_{i_k}) = 0.$$

Let us call a matrix P over R β -twisted nilpotent (β is any automorphism of R) if there exists an integer $k > 0$ such that

$$P\beta^{-1}(P) \cdots \beta^{-(k-1)}(P) = 0.$$

Hence, it follows from (2) that each $N_i (i = 1, \dots, n)$ in (1) is α_i -twisted nilpotent.

Our next lemma is the key to the main result:

LEMMA 3. *The matrix N in (1) is a product of matrices of the form $I + Pw(x_1, \dots, x_n)$, where P is an $w(\alpha_1, \dots, \alpha_n)$ -twisted nilpotent matrix over R . ($w(x_1, \dots, x_n)$ denotes a word in x_1, \dots, x_n .)*

Proof. Recall from (1) and (2) that each $N_i (i = 1, \dots, n)$ in (1) is α_i -twisted nilpotent. Consider

$$I + Q = (I - N_1x_1) \cdots (I - N_nx_n)N.$$

Then Q is of the form $\sum_j Q_j s_j$, where s_j is a monomial of degree at least two in the x_1, \dots, x_n . In fact, if $s_j = x_{i_1}x_{i_2} \cdots x_{i_l} (l \geq 2)$, then

$$(3) \quad Q_j = \pm N_{i_1} \alpha_{i_1}^{-1}(N_{i_2}) \cdots (\alpha_{i_1}^{-1} \cdots \alpha_{i_{l-1}}^{-1})(N_{i_l}).$$

Hence, for $k > K/2$,

$$Q_j \beta^{-1}(Q_j) \cdots \beta^{-(k-1)}(Q_j) = 0,$$

for each j , where β is an automorphism obtained by replacing the x_i in s_j by α_i respectively. That is, Q_j is $s_j(\alpha_1, \dots, \alpha_n)$ -twisted nilpotent for each j . Now, consider

$$I + Q' = \prod_j (I - Q_j s_j)(I + Q).$$

Then Q' is of the form $\sum_\sigma Q'_\sigma y_\sigma$, where each y_σ is a monomial of degree at least four in the x_1, \dots, x_n and for $l \geq 4$, Q'_σ is of the form as given on the right hand side of (3). Thus, for $k > K/4$,

$$Q'_\sigma \gamma^{-1}(Q'_\sigma) \cdots \gamma^{-(k-1)}(Q'_\sigma) = 0,$$

for each σ , where γ is an automorphism obtained by replacing the x_i in y_σ by α_i respectively. That is, Q'_σ is $y_\sigma(\alpha_1, \dots, \alpha_n)$ -twisted nilpotent for each σ .

Left multiplying $I + Q'$ by $\prod_\sigma (I - Q'_\sigma y_\sigma)$, and repeating the above argument, we will finally arrive, after a finite steps (because of the finite bound K and condition (2)), at the conclusion that

$$\prod (I + Pw(x_1, \dots, x_n)) \cdot N = I,$$

where P is an $w(\alpha_1, \dots, \alpha_n)$ -twisted nilpotent matrix over R and $w(x_1, \dots, x_n)$ is a word in x_1, \dots, x_n .

This completes the proof.

The above discussions are modifications of those given in [3] and ([1], p. 647) for (untwisted) free associative algebras; and the following result is already contained in the above proof (also cf. [2]).

LEMMA 4. For any automorphism β of R , $\bar{K}_1 R_\beta[t]$ is generated by the elements of the form $I + Pt$, where P is an β -twisted nilpotent matrix over R .

Proof of Theorem 1. It follows immediately from Lemmas 3 and 4.

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